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# A PROPERTY OF MANIFOLDS COMPACTLY EQUIVALENT TO COMPACT MANIFOLDS

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**In this paper it is shown that there is a countable collection $\mathscr{G}=\{G_k\}_{k=1}^{\infty}$ of connected $n$-manifolds such that any manifold $M$ which is compactly equivalent to a compact manifold is an open monotone union of some $G_{\alpha(M)} \in \mathscr{G}$.**

In [4] it is shown that if $\mathscr{F}$ is the class consisting of all open 2-manifolds of finite genus, then there is a countable collection $\mathscr{D} = \{D_k\}_{k=1}^{\infty}$ of open 2-manifolds with the property that given $M \in \mathscr{F}$, there exists some $D_j \in \mathscr{D}$ such that $M$ is an open monotone union of $D_j$. By appropriately extending the concept of genus to higher dimensions, one can obtain similar results for a larger class of manifolds.

**1. Preliminaries.** Unless otherwise specified, all manifolds will be assumed to be connected and $\operatorname{bd} M$ and $\operatorname{int} M$ will denote the boundary and interior respectively of a manifold $M$. Let $M$ and $N$ be $n$-manifolds. $M$ and $N$ are compactly equivalent, denoted by $M \sim_c N$, if given any proper compact set $K \subset M$ there is an embedding $i$ of the pair $(K, K \cap \operatorname{bd} M)$ into $(N, \operatorname{bd} N)$ such that $i(K \cap \operatorname{bd} M) = i(K) \cap \operatorname{bd} N$ and given any proper compact set $L \subset N$ there is an embedding $j$ of $(L, L \cap \operatorname{bd} N)$ into $(M, \operatorname{bd} M)$ such that $j(L \cap \operatorname{bd} N) = j(L) \cap \operatorname{bd} M$. Clearly compact equivalence is an equivalence relation on the class of all $n$-manifolds. Note that a 2-manifold $M$ without boundary has finite genus if and only if $M \sim_c Q$ where $Q$ is some closed 2-manifold.

Let $\mathscr{L}$ be the class consisting of all non-compact $n$-manifolds $M$, $n \geqq 2$ and $n \neq 4$, such that $M \in \mathscr{L}$ if and only if $M \sim_c N$, $N$ a compact manifold. The principal result of this paper is the following:

THEOREM 1.1. *There is a countable collection $\mathscr{G} = \{G_k\}_{k=1}^{\infty}$ of manifolds such that given $M \in \mathscr{L}$ there is some positive integer $\alpha(M)$ such that $M$ is an open monotone union of $G_{\alpha(M)}$.*

As usual an $n$-manifold $M$ is called an open monotone union of an $n$-manifold $H$ if $M = \bigcup_{i=1}^{\infty} H_i$ where for all $i$, $H_i$ is open in $M$, $H_i \subset H_{i+1}$ and $H_i \equiv H$ ($\equiv$ denotes topological equivalence).

**2. Proof of the theorem.** If $M$ is an $n$-manifold, let $I(M)$ rel $\operatorname{bd} M = \{f \mid f$ is a homeomorphism of $M$ onto itself such that $f$ is isotopic to the identity relative to $\operatorname{bd} M\}$.

The following lemma gives the existence of a complicated domain which is the basic tool used in the construction of the collection $\mathscr{G}$ mentioned in Theorem 1.1.

LEMMA 2.1. *Let $E$ be an $n$-cell, $n \geqq 2$. There exists a proper domain (open connected set) $G$ of $E$,* $\operatorname{bd} E \subset G$, *such that if $U$ is open in $E$ and $K$ is a proper continuum,* $\operatorname{bd} E \subset K \subset U$, *then there exists a $g \in I(E)$* rel bd $E$ *such that $K \subset g(G) \subset U$.*

*Proof.* This follows immediately from Lemma 3.8 of [5].

LEMMA 2.2. *Let $Q$ be a compact $n$-manifold, $n \geqq 2$. There is a proper domain $D$ of $Q$ such that if $U$ is open in $Q$ and contains a residual set $R$ of $Q$, and $K$ is proper continuum in $Q$, $R \subset K \subset U$, then there exists $h \in I(Q)$* rel bd Q *such that $K \subset h(D) \subset U$.*

*Proof.* Let $E$ be a bicollared $n$-cell, $E \subset \operatorname{int} G$, and let $G$ be a proper domain $G$ of $E$ which satisfies the conditions of Lemma 2.1. We will show that $D = (Q - E) \cup G$ is the required domain. Without loss of generality, we may assume that $U$ is connected. Since $U$ contains a residual set $R$ (see [3] for appropriate definition) there is a bicollared $n$-cell $E'$ and $\alpha \in I(Q)$ rel bd such that $R \subset Q - \operatorname{int} E' \subset U$ and $\alpha(E') = E$. Note that $E$ and $\alpha$ can be obtained as follows: one easily constructs $\gamma_1$, $\gamma_2$, and $\gamma_3 \in I(Q)$ rel bd $Q$ such that $\gamma_1$ only moves points inside $E \cup$ (collar of bd $E$) and shrinks $E$ to a very small set, $\gamma_2$ moves $\gamma_1(E)$ into the open $n$-cell $Q - R$, and $\gamma_3$ moves only points inside $Q - R$ and expands $\gamma_2(\gamma_1(E))$ so that $Q - U \subset \gamma_3(\gamma_2(\gamma_1(\operatorname{int} E)) \subset Q - R$. Thus we can set $\alpha^{-1} = \gamma_3\gamma_2\gamma_1$ and $E' = \alpha^{-1}(E)$. Let $R \subset K \subset U$, $K$ a proper continuum. Without loss of generality, we may assume that $K \cap E'$ is a proper continuum in $E'$ and $\operatorname{bd} E' \subset K \cap E'$. Then $K'' = \alpha(K \cap E') = \alpha(K) \cap E$ is a proper continuum in $E$, $U'' = \alpha(U) \cap E = \alpha(U \cap E')$ is open in $E$ and $\operatorname{bd} E \subset K'' \subset U''$. Therefore it follows from Lemma 2.1 that there is a homeomorphism $h \in I(E)$ rel bd $E$ such that $K'' \subset h(G) \subset U''$. Now extend $h$ to all of $Q$ by defining $h(x) = x$, $x \in Q - E$. Then $\alpha(K) \subset h(D) \subset \alpha(U)$ and so $g = \alpha^{-1}h$ is the required homeomorphism.

Since there are only a countable number of topologically distinct compact manifolds [1], Theorem 1.1 follows immediately from the following theorem.

THEOREM 2.3. *Let $Q$ be a compact $n$-manifold, $n > 1$ and $n \neq 4$. There is a domain $D$ of $Q$ such that if $M$ is a non-compact $n$-manifold and $M \sim_c Q$, then $M$ is an open monotone union of $D$.*

*Proof.* Let $D$ be a domain of $Q$ which satisfies Lemma 2.2. and let $L = Q - \operatorname{int} E$, $E$ a bicollared $n$-cell contained in $\operatorname{int} Q$. Let $M$ be a non-compact $n$-manifold such that $M \sim_c Q$. It is easily seen that $\operatorname{bd} M = \operatorname{bd} Q$ and that there is an embedding $f$ of $(L, \operatorname{bd} Q)$ into $(M, \operatorname{bd} M)$ such that $f(\operatorname{bd} E)$ (note that $\operatorname{bd} E = L - \operatorname{int}_Q L$ where $\operatorname{int}_Q L$ denotes the point set interior of $L$ relative to $Q$) is a bicollared $(n-1)$-sphere in $\operatorname{int} M$. Since $M$ is an $n$-manifold, there exists a sequence $\{C_i\}_{i=1}^{\infty}$ of continua in $M$ such that $M = \bigcup_{i=1}^{\infty} C_i$ and for all $i \geqq 1$, $f(L) \subset \operatorname{int}_M C_i \subset C_i \subset \operatorname{int}_M C_{i+1}$. Since $M$ is not compact and $M \sim_c Q$, for each $i \geqq 1$ there is an embedding $h_{i+1}$ of $(C_{i+1}, \operatorname{bd} M)$ into $(Q, \operatorname{bd} Q)$ such that $\operatorname{bd} Q \subset h_{i+1}(f(L)) \subset h_{i+1}(C_i) \subset h_{i+1}(\operatorname{int}_M C_{i+1})$, where $K_i = h_{i+1}(C_i)$ is a proper continuum in $Q$ and $U_i = h_{i+1}(\operatorname{int}_M C_{i+1})$ is open in $Q$. Since $n \neq 4$, it follows from [2] that $Q - h_{i+1}(f(\operatorname{int}_Q L))$ is a bicollared $n$-cell and therefore there is a residual set $R$ of $Q$ such that $R \subset K_i \subset U_i$. It follows from Lemma 2.2 that there exists $\alpha_i \in I(Q)$ rel bd $Q$ such that $K_i \subset \alpha_i(D) \subset U_i$. Define $\beta_i \colon D \to M$ by $\beta_i(x) = h_{i+1}^{-1}(\alpha_i(x))$. Then $\beta_i$ is an embedding of $(D, \operatorname{bd} Q)$ into $(M, \operatorname{bd} M)$ and $C_i \subset \beta_i(D) \subset \operatorname{int}_M C_{i+1}$. Therefore $M = \bigcup_{i=1}^{\infty} \beta_i(D)$, where $\beta_i(D)$ is open and $\beta_i(D) \subset \beta_{i+1}(D)$ for all $i \geqq 1$. Therefore $M$ is an open monotone union of $D$.

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