NORMPRESERVING EXTENSIONS IN SUBSPACES OF $C(X)$

Eggert Briem and Murali Rao
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OF C(X)

EGGERT BRIEM AND MURALI RAO

If $B$ is a subspace of $C(X)$ and $F$ is a closed subset of $X$, this note gives sufficient conditions in order that every function in the restriction subspace $B|_F$ has an extension in $B$ with no increase in norm.

Introduction. Let $X$ be a compact Hausdorff space, $C(X)$ the Banach algebra of all continuous complex-valued functions on $X$ and let $B$ be a closed linear subspace of $C(X)$ separating the points of $X$ and containing the constants. A closed subset $F$ of $X$ is said to have the normpreserving extension property w.r.t. $B$ if any function $b_0$ in the restriction subspace $B|_F$ has an extension $b \in B$ (i.e. $b|_F = b_0$) such that $\|b\| = \|b_0\|_F$ (resp. $\|b\|_F$) denotes the supremum norm on $X$ (resp. $F$). The main result is the following:

Let $F$ be a closed subset of $X$ and suppose there is a map $T$ (not necessarily linear) from $M(X)$ into $M(X)$ satisfying the following conditions

(i) $m - Tm \in B^\perp$ for all $m \in M(X)$
(ii) $T\lambda$ is a probability measure when $\lambda$ is
(iii) If $s_i \in C$ and $m_i \in M(X)$, $i = 1, \cdots, n$ and $\sum_{i=1}^n s_i m_i \in k(F)^\perp$ then $\sum_{i=1}^n s_i (Tm_i)|_F \in B^\perp$.

Then $F$ has the normpreserving extension property.

$M(X)$ denotes the set of regular Borel measures on $X$, and if $A$ is a subset of $B$ then $A^\perp$ is the set of those measures in $M(X)$ which annihilate $A$. $k(F)$ consists of those functions in $B$ which are identically 0 on $F$. Also if $G$ is a Borel subset of $X$ and $m \in M(X)$ then $m|_G$ is the measure $\chi_G m$ where $\chi_G$ is the characteristic function for $G$.

Two conditions, either of which is known to imply that a closed subset $F$ of $X$ has the normpreserving extension property are the following:

Condition 1. For all $\sigma \in B^\perp$, $\sigma|_F \in B^\perp$.

Condition 2. $F$ is a compact subset of the Choquet boundary $\Sigma_B$ for $B$ and for all $\sigma \in M(\Sigma_B) \cap B^\perp$, $\sigma|_F \in B^\perp$.

($M(\Sigma_B)$ denotes the set of those $\sigma \in M(X)$ for which the total variation $|\sigma|$ is maximal in Choquet’s ordering for positive measures (see [1])
In Chapter 2 of this note we show that when either Condition 1 or Condition 2 is satisfied there exists a map $T$ with the above properties. Actually, when Condition 1 or Condition 2 is satisfied stronger extension properties than the norm-preserving one hold. (In the case of Condition 1 see [4] Theorem 3 and [5] Theorem 4.8 in the case of Condition 2 see [2] Theorem 4.5 and [3] Theorem 2). But as we show in Chapter 2 these stronger extension properties are corollaries to theorems based on the existence of a map $T$ described above. Thus we are able to deal simultaneously with Conditions 1 and 2.

1. A condition for the norm-preserving extension property.

Throughout this chapter $F$ is a fixed closed subset of $X$ and $T$ is a map from $M(X)$ into $M(X)$ satisfying

(i) $m - Tm \in B^1$ for all $m \in M(X)$

(ii) $T\lambda$ is a probability measure when $\lambda$ is.

(iii) If $s_i \in C$ and $m_i \in M(X)$ and $\sum_{i=1}^n s_i m_i \in k(F)^1$ then

$$\sum_{i=1}^n s_i |m_i|_{X \setminus F} \in B^1.$$

**Remark 1.1.** It follows from conditions (i) and (iii) that if $\sum s_i \sigma_i \in B^1$ then $\sum s_i (T\sigma_i) |_{X \setminus F} \in B^1$. Also if $\lambda$ is a probability measure and $\lambda = \lambda|_{X \setminus F}$ then $T\lambda = (T\lambda)|_{X \setminus F}$, because $\lambda \in k(F)^1$ hence by (iii) $(T\lambda)|_{X \setminus F} \in B^1$. Since $B$ contains the constants and $T\lambda$ is a positive measure $(T\lambda)|_{X \setminus F} = 0$.

We let $S_B$ denote the state space of $B$ s.e. $S_B = \{p \in B^*: \|p\| = p(1) = 1\}$. $S_B$ is a convex set which is compact in the $w^*$-topology and the natural map of $X$ into $S_B$ is a homeomorphism. We shall frequently think of $X$ as embedded in $S_B$. A representing measure for $p \in S_B$ is a probability measure $v_p$ on $X$ such that $p(f) = \int f dv_p$ for all $f \in B$.

**Definition 1.2.** For each $b_0 \in B|_{X \setminus F}$ we define a function $\tilde{b}_0$ on $S_B$ as follows. If $p \in S_B$ put

$$\tilde{b}_0(p) = \int_{X \setminus F} b_0 dTv_p$$

where $v_p$ is any representing measure for $p$ on $X$.

**Remark 1.3.** The above definition is meaningful because if $v'_p$ is another representing measure for $p$ on $X$ then $v_p - v'_p \in B^1$; hence by Remark 1.1 $(Tv_p)|_{X \setminus F} = (Tv'_p)|_{X \setminus F} \in B^1$.

**Lemma 1.4.** $\tilde{b}_0$ has the following properties:
(1) \( \bar{b}_0 \) is an affine function

(2) \( |\bar{b}_0(p)| \leq \|b_0\|_F \) for all \( p \in S_B \)

(3) \( \bar{b}_0(p) = b_0(p) \) if \( p \in F \)

(4) \( \bar{b}_0 \) is a linear combination of upper semicontinuous affine functions.

(5) \( \int \bar{b}_0 d\sigma = 0 \) for all \( \sigma \in B^\perp \).

Proof 1. follows from the definition of \( \bar{b}_0 \) and remark 1.1. (2) is trivial: To prove (3) observe that if \( x \in F \) then by remark 1.1 \( T\delta_x = (T\delta_x)|_F \) (\( \delta_x \) is point mass at \( x \)). But \( T\delta_x \) is a representing measure for \( x \). (4) Observe that if \( b_0 \in B|_F \) and \( f_0 = Re b_0 \), we can define \( \bar{f}_0 \) in exactly the same way as we defined \( \bar{b}_0 \). Then \( \bar{f}_0 \) is affine on \( S_B \) and \( \bar{f}_0 = Re \bar{b}_0 \). First assume that \( f_0 \geq 0 \). We want to show that \( \bar{f}_0 \) is upper semi-continuous. For each \( t \geq 0 \) put \( K_t = \{ p \in S_B : \bar{f}_0(p) \geq t \} \) we must show that \( K_t \) is closed. Let \( \{p_\alpha\} \) be a net from \( K_t \) with limit point \( p_0 \), and \( v_\alpha \) a representing measure for \( p_\alpha \) on \( X \) for each \( \alpha \). Write \( T v_\alpha = u_\alpha + w_\alpha \) where \( u_\alpha = (Tv_\alpha)|_F \). Let \( u_0 \) be a \( w^* \)-clusterpoint for \( \{u_\alpha\} \) and let \( \{w_\beta\} \) be a subnet from \( \{u_\alpha\} \) converging to \( u_0 \). Also let \( w_0 \) be a clusterpoint for \( \{w_\beta\} \). Then \( v_0 = u_0 + w_0 \) is a representing measure for \( p_0 \) and since

\[
  u_0 = u_0|_F, \quad T\left(\frac{u_0}{\|u_0\|}\right) = T\left(\frac{u_0}{\|u_0\|}\right)|_F.
\]

(Remark 1.1). Using this and Remark 1.1 once more we get:

\[
  \bar{f}_0(p_0) = \int_F f_0 dT v_0 = \|u_0\| \int_F f_0 dT \left(\frac{u_0}{\|u_0\|}\right) + \|w_0\| \cdot \int_F f_0 dT \left(\frac{w_0}{\|w_0\|}\right)
\]

\[
  \geq \|u_0\| \int_F f_0 dT \left(\frac{u_0}{\|u_0\|}\right) = \int_F f_0 d u_0 \geq t. \quad \text{Hence } p_0 \in K_t.
\]

In general take a positive number \( k \) such that \( f_0 + k \geq 0 \). Then

\[
  \bar{f}_0 = \bar{f}_0 + k - k
\]

is the difference of upper semi-continuous functions. Since this holds for any \( f_0 \in ReB|_F \) (4) is proved.

Since \( \bar{b}_0 \) is a linear combination of real valued affine upper semi-continuous functions it satisfies the barycenter formula i.e. if \( p \in S_B \) and \( v_\alpha \) is a representing measure for \( p \) then

\[
  \int \bar{b}_0 d v_\alpha = \bar{b}_0(p)
\]

(See [1] Cor. I 1.4)

Now we consider a measure \( \sigma \in B^\perp \) with a decomposition \( \sigma = \sum_{i=1}^n t_i \sigma_i \) into probability measures \( \sigma_i \) representing points \( p_i \in S_B \) for
By axiom (i) the measure $T\sigma_i$ also represent $p_i$ for $i = 1, 2, 3, 4$. Applying the above result together with the definition of $\delta_o$ and axiom (iii), we obtain:

$$\int b_o d\sigma = \sum_{i=1}^{4} t_i \int b_o d\sigma_i = \sum_{i=1}^{4} t_i \delta_o(p_i) = \sum_{i=1}^{4} t_i \left( \int F b_o(T\sigma_i) = 0 \right).$$

This completes the proof of (5).

**Proposition 1.5.** $B|_F$ is closed in $C(F)$

**Proof.** Let $\sigma \in B^+$, and consider a $b_o \in B|_F$ such that $\|b_o\|_F \leq 1$. By statement (5) of Lemma 1.4:

$$0 = \int b_o d\sigma = \int_F b_o d\sigma + \int_{X \setminus F} b_o d\sigma.$$

Hence

$$\left| \int_F b_o d\sigma \right| = \left| \int_{X \setminus F} b_o d\sigma \right| \leq \|\sigma\|_{X \setminus F},$$

and so $\|\sigma\|_F \leq \|\sigma\|_{X \setminus F}$.

By a result of Gamelin [4] and Glicksberg [5] (see also [3, Prop. 1]) this implies that $B|_F$ is almost normpreserving, or what is equivalent, that $B|_{k(F)}$ is isometric to $B|_F$. Hence $B|_F$ is complete in uniform norm, and we are done.

**Proposition 1.6.** Let $b_o \in B|_F$ and let $\psi$ be a strictly positive lower semi-continuous function on $X$ such that $\psi(x) > |b_o(x)|$ for all $x \in X$. Then there is a function $b \in B$ such that $b|_F = b_o$ and $|b(x)| < \psi(x)$ for all $x \in X$.

**Proof.** Apply Theorem 2.2 of [2].

**Theorem 1.7.** Let $F$ and $T$ be as in the beginning of this chapter and let $b_o \in B|_F$ with $\|b_o\|_F \leq 1$ and let $\psi$ be a strictly positive lower semi-continuous function such that $\psi(x) > |b_o(x)|$ for all $x \in X$. Then there is a function $b \in B$ such that $b|_F = b_o$, $\|b\| = \|b_o\|_F$ and $|b(x)| < \psi(x)$ for all $x \in X$.

**Proof.** The proof is exactly the same as proof of [3] Theorem 2 after replacing the function $A$ from [3] by $\delta_o$ and Lemma 1 of [3] by Proposition 1.6 of this note.
COROLLARY 1.8. $F$ and $T$ as before. Then $F$ has the normpreserving extension property w.r.t. $B$.

**Theorem 1.9.** Let $F$ and $T$ be as before let $b_0 \in B_p$ and let $\psi$ be a strictly positive lower semi-continuous function such that $\psi(x) \geq |b_0(x)|$ for all $x \in X$. Suppose furthermore that $\psi(x) \geq \int \psi d T\lambda_x$ for all $x \in X \setminus F$ for which $b_0(x) \neq 0$ ($\lambda_x$ is a representing measure for $x$). Then there is a function $b \in B$ such that

$$b|_F = b_0 \text{ and } |b(x)| \leq \psi(x) \text{ for all } x \in X.$$  

**Proof.** The proof is the same as the proof of [2] Theorem 4.5 replacing in the proof of Theorem 2.1 of [2] by Proposition 1.6 of this note.

2. Relations to conditions 1 and 2. We start by showing the equivalence of condition 1 to a condition involving $k(F)^\perp$.

**Proposition 2.1.** Let $F$ be a closed subset of $X$. Then the following conditions are equivalent:

1. For all $\sigma \in B^1$, $\sigma|_F \in B^\perp$.

1'. For all $\sigma \in k(F)^\perp$, $\sigma|_{X \setminus F} \in B^\perp$.

**Proof.** Condition 1' trivially implies 1. Suppose Condition 1 is satisfied and let $\sigma \in k(F)^\perp$. Let $b_0 \in B|_F$ and let $b \in B$ be any extension of $b_0$. Since $\sigma \in k(F)^\perp$ the quantity $\int bd\sigma$ is independent of the choice of the extension $b$. Thus $b_0 \rightarrow \int bd\sigma$ is a well defined linear functional on $B|_F$. By [4] Theorem 1, $B|_F$ is closed in $C(F)$. It then follows from the open mapping theorem that $b_0 \rightarrow bd\sigma$ is a continuous linear functional. Thus we can find a measure $\sigma_1 = \sigma|_F$ such that $\sigma_1 - \sigma \in B^\perp$. But then $\sigma|_{X \setminus F} = (\sigma_1 - \sigma)|_{X \setminus F} \in B^\perp$.

Let again $F$ be a closed subset of $X$ and suppose that Condition 1 is satisfied. Let $T$ be the identity map from $M(X)$ to $M(X)$. By the above proposition $T$ satisfies requirements (i) (ii) and (iii) from the beginning of Chapter 1. In this case if $b \in B|_F$, $\tilde{b}_0(x) = 0$ for all $x \in X \setminus F$. From Theorem 1.9 we can then deduce the following well known theorem.

**Theorem 2.2.** Let $F$ be a closed subset of $X$ and suppose that $\mu|_F \in B^1$ for all $\mu \in B^1$. If $b_0 \in B|_F$ and $\psi$ is a strictly positive lower semi-continuous function with $\psi(x) \geq |b_0(x)|$ for all $x \in F$ then there is function $b \in B$ such that
We now look at Condition 2. Let $F$ be a compact subset of the Choquet boundary $\Sigma_B$ and suppose Condition 2 is satisfied i.e. for all $\sigma \in B^1 \cap M(\Sigma_B), \sigma_{\mid F} \in B^1$. We need the following lemma

**Lemma 2.3.** Under the above hypotheses $B_{\mid F}$ is closed in $C(F)$. 

**Proof.** By [5] Theorem 3.1 we must show the existence of a constant $c \geq 1$ such that $\|\mu - (B_{\mid F})\| \leq c \|\mu - B\|$ for all $\mu \in M(F)$. Let $\mu \in M(F)$ and $\sigma \in B^1$. We write $\sigma = \sigma_{\mid F} + \sigma_{\mid X \setminus F}$ and further write $\sigma_{\mid X \setminus F} = t_1\lambda_1 - t_2\lambda_2 + it(t_3\lambda_3 - t_4\lambda_4)$ where the $t_i$'s are positive numbers and the $\lambda$'s are probability measures such that $\lambda_1$ and $\lambda_2$ (resp. $\lambda_3$ and $\lambda_4$) live on disjoint subsets of $X$. For $i = 1, \cdots, 4$ let $v_i$ be a maximal measure such that $\lambda_i - v_i \in B^1$. Put $w = t_1v_1 - t_2v_2 + it(t_3v_3 - t_4v_4)$. Then $\sigma_{\mid X \setminus F} - w \in B^1$ and $\|w\| \leq \sum_{i=1}^{4} t_i \|v_i\| = \sum_{i=1}^{4} t_i \|\lambda_i\| \leq 2\|\sigma_{\mid X \setminus F}\|$. Now $\sigma_{\mid F} + w \in B^1 \cap M(\Sigma_B)$ so that $\sigma_{\mid F} + w_{\mid F} \in B^1$. Hence $\|\mu - (A \sigma_{\mid F})\| \leq \|\mu - (\sigma_{\mid F} + w_{\mid F})\| \leq \|\mu - \sigma_{\mid F}\| + 2\|\sigma_{\mid X \setminus F}\| \leq 2\|\mu - \sigma\|$. Thus we can take $c = 2$ and the lemma is proved.

As above let $F$ be a compact subset of $\Sigma_B$ and suppose that for all $\sigma \in M(\Sigma_B) \cap B^1, \sigma_{\mid F} \in B^1$. We define a map $T$ from $M(X)$ to $M(X)$ as follows. If $\lambda$ is a probability measure on $X$ pick a maximal measure $v$ with $\lambda - v \in B^1$ and put $T\lambda = v$. If $\lambda$ is already maximal put $T\lambda = \lambda$. If $\sigma \in M(X)$ write $\sigma = t_1\lambda_1 - t_2\lambda_2 + it(t_3\lambda_3 - t_4\lambda_4)$ where the $t_i$'s are positive numbers and where $\lambda_1$ and $\lambda_2$ (resp. $\lambda_3$ and $\lambda_4$) are probability measures living on disjoint subsets of $X$. Then put $T\sigma = t_1T\lambda_1 - t_2T\lambda_2 + it(t_3T\lambda_3 - t_4T\lambda_4)$. The map $T$ from $M(X)$ to $M(X)$ we get in this way obviously has properties (i) and (ii) from the beginning of Chapter 1. Observe that $T\sigma = \sigma$ if $\sigma = \sigma_{\mid F}$ since $F \subset \Sigma_B$. To see that $T$ also has property (iii) let $\Sigma_B, \sigma_{\mid F} \in k(F)^1$. By Lemma 2.3 $B_{\mid F}$ is closed in $C(F)$. Just as in the proof of Proposition 2.1 we can find a measure $\mu = \mu_{\mid F}$ such that $\mu - \Sigma_B, \sigma_{\mid F} \in B^1$. Then $\mu - \Sigma_B, T\sigma_{\mid F} \in B^1 \cap M(\Sigma_B)$ so that $\mu - \Sigma_B, (T\sigma_{\mid F})_{\mid F} \in B^1$, but then $\Sigma_B, (T\sigma_{\mid F})_{\mid X \setminus F} \in B^1$. We can then using Theorems 1.7 and 1.9 deduce the same interpolation theorems as in [2] and [3]. In particular we get from Theorem 1.7:

**Theorem 2.4.** Let $F$ be a compact subset of the Choquet boundary $\Sigma_B$ and suppose that for all $\sigma \in B^1 \cap M(\Sigma_B), \sigma_{\mid F} \in B^1$. Then $F$ has the normpreserving extension property w.r.t. $B$. 

\[ b \mid_F = b_0 \text{ and } |b(x)| \leq \psi(x) \text{ for all } x \in X. \]
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