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This paper is about linear transformations of the k-fold symmetric tensor product of an n-dimensional vector space V which carry nonzero decomposable tensors to nonzero decomposable tensors. The main theorem shows that every such transformation is induced by a nonsingular transformation of V provided both

(i) the field has characteristic either 0 or a prime greater than k and every polynomial over the field with degree at n is a product of linear factors.

(ii) $n > k + 1$.

Condition (i) includes the important special case where the field is algebraically closed with characteristic 0.

The linear transformations which preserve decomposable tensors in the skew-symmetric case have been studied in two papers by Westwick [6, 8]. In [6] he showed that if the field is algebraically closed then the transformation is induced by a linear transformation of V except, possibly, when the dimension of V is $2k$. In the latter case the transformation may be the composition of one induced by a linear transformation of V and one induced by a correlation of the k-dimensional subspaces of V. A series of papers [3, 4, 7, 2] has been devoted to linear transformations which preserve decomposable tensors in the case of the full tensor product.

Our result partially answers a question first raised by Marcus and Newman in [5]. They asked for necessary and sufficient conditions in order that every decomposable mapping of the space of k-fold symmetric tensors be induced.

1. Preliminaries. Let V^k denote the k-fold Cartesian product of V where $k > 1$. A *k-fold symmetric tensor space* (or rank k symmetric tensor space) is a vector space denoted by $\bigvee_k V$ together with a fixed multilinear symmetric mapping σ: $V^k \to \bigvee_k V$ which is universal for multilinear and symmetric mappings of $\bigvee_k V$. We assume that $\bigvee_k V$ is generated by the image of σ. Thus, if W is any vector space and g: $V^k \to W$ is both multilinear and symmetric then g has a unique extension h: $\bigvee_k V \to W$ such that

$$(1.1) \qquad \begin{array}{ccc} & W & \\ {}^{g}\nearrow & & \nwarrow^{h} \\ V^k & \xrightarrow[\sigma]{} & \bigvee_k V \end{array}$$

is commutative and $\bigvee_k V$ is isomorphic to any other vector space with this property. In particular, if $A\colon V \to V$ is linear then the assignment

$$(x_1, \cdots, x_k) \longmapsto Ax_1 \vee \cdots \vee Ax_k$$

is a multilinear and symmetric mapping of V^k. We will denote its unique linear extension to $\bigvee_k V$ by $\bigvee_k A$.

The *decomposable symmetric tensors* or "symmetric products" are images under σ of k-tuples in V^k. For convenience we denote $\sigma(x_1, \cdots, x_k)$ by $x_1 \vee \cdots \vee x_k$. A subspace s of $\bigvee_k V$ is decomposable if $S \subseteqq \sigma(V^k)$. *Trivial decomposable subspaces* are the zero subspace and the 1-dimensional subspaces whose elements are scalar multiples of a single nonzero decomposable symmetric tensor. If V and F satisfy (i) and (ii) the maximal decomposable subspaces of $\bigvee_k V$ were determined in [1].

A symmetric product is zero if and only if at least one of its factors is zero. More generally, if

$$x_1 \vee \cdots \vee x_k = y_1 \vee \cdots \vee y_k \neq 0$$

then there are scalars $\lambda_1, \cdots, \lambda_k$ such that $\lambda_1 \cdots \lambda_k = 1$ and

$$x_i = \lambda_i y_{\pi(i)} \qquad i = 1, \cdots, k\,. \tag{1.2}$$

Here $\pi \in S_k$, the symmetric group on $\{1, \cdots, k\}$.

A linear transformation $f\colon \bigvee_k V \to \bigvee_k V$ is *decomposable* if

$$f(\sigma(V^k)) \subseteqq \sigma(V^k)$$

and

$$\ker f \cap \sigma(V^k) = 0\,. \tag{1.3}$$

If V is an n-dimensional vector space then the dimension of $\bigvee_k V$ is $\binom{n+k-1}{k}$.

2. Type 1 subspaces and associate mappings.

Subspaces in $\bigvee_k V$ of the form

$$M = x_1 \vee \cdots \vee x_{k-1} \vee V \qquad k > 1 \tag{2.1}$$

where $x_1, \cdots, x_{k-1}$ are fixed nonzero vectors in V are always decomposable because of the multilinearity of the mapping σ. It is convenient to call these *type* 1 *subspaces*. The 1-dimensional subspaces $\langle x_1 \rangle, \cdots, \langle x_{k-1} \rangle$ are called the factors of M.

PROPOSITION 1. *If F is a field whose characteristic* (*if any*) *is* *not less than k then*

$$x_1 \vee \cdots \vee x_{k-1} \vee V = x'_1 \vee \cdots \vee x'_{k-1} \vee V \qquad \text{in } \bigvee_k V \tag{2.2}$$

implies

$$\langle x_1 \vee \cdots \vee x_{k-1} \rangle = \langle x'_1 \vee \cdots \vee x'_{k-1} \rangle \qquad \text{in } \bigvee_{k-1} V .$$

Proof. This proof requires the choice of a vector not in the set-theoretic union

$$\langle x_1 \rangle \cup \cdots \cup \langle x_{k-1} \rangle . \tag{2.3}$$

By Lemma 12 of [1, p. 73] we know that if V were the union (2.3) then the cardinality of F could not exceed the finite integer $k - 1$. This would mean that the characteristic of F exceeds the cardinality of F. Accordingly we may choose v in V not in the union (2.3) and (2.2) implies the existence of a u in V satisfying

$$x_1 \vee \cdots \vee x_{k-1} \vee u = x'_1 \vee \cdots \vee x'_{k-1} \vee v .$$

By the choice of v and (1.2) there is a nonzero scalar λ for which $u = \lambda v$ and

$$x_i = \lambda_i x'_{\pi(i)} \qquad i - 1, \cdots, k - 1$$

where $\pi \in S_{k-1}$ and $1 = \lambda \Pi \lambda_i$. Therefore,

$$\lambda x_1 \vee \cdots \vee x_{k-1} = x'_1 \vee \cdots \vee x'_{k-1}$$

in $\bigvee_{k-1} V$.

Hereafter we will assume that F satisfies the hypothesis of Proposition 1.

A *type* 1 *mapping* is a decomposable mapping of $\bigvee_k V$ for which the image of every type 1 subspace is again a type 1 subspace. If f is a type 1 mapping and M is the type 1 subspace (2.1) then we may choose nonzero vectors $y_1, \cdots, y_{k-1}$ in V such that

$$f(M) = y_1 \vee \cdots \vee y_{k-1} \vee V . \tag{2.4}$$

We obtain a well-defined linear mapping A of V by setting $Au = v$ if

$$f(x_1 \vee \cdots \vee x_{k-1} \vee u) = y_1 \vee \cdots \vee y_{k-1} \vee v . \tag{2.5}$$

The mapping A will be called an *associate mapping* of f with respect to M. In general, the associate map defined by (2.5) depends not only on M and f but the choice of the vectors $y_1, \cdots, y_{k-1}$ as well.

PROPOSITION 2. *Any two associate mappings of a type* 1 *mapping with respect to the same type* 1 *subspace are multiples.*

Proof. This follows easily from Proposition 1 and (1.1).

PROPOSITION 3. *Every associate of a type* 1 *mapping is non-singular.*

Proof. Let A be an associate of a type 1 mapping f with respect to (2.1) and suppose $A(u) = A(u')$ for some vectors u, u' in V. From (2.5) we have

$$f(x_1 \vee \cdots \vee x_{k-1} \vee u) = f(x_1 \vee \cdots \vee x_{k-1} \vee u') .$$

Since f is linear and decomposable we have

$$x_1 \vee \cdots \vee x_{k-1} \vee (u - u') = 0$$

which implies $u = u'$.

Two type 1 subspaces will be called *adjacent* if they have exactly $k - 2$ common factors (counting multiplicity). Accordingly a typical pair of adjacent subspaces may be written in the form

$$M_i = x_1 \vee \cdots \vee x_{k-1} \vee z_i \vee V \qquad i = 1, 2 \tag{2.6}$$

where z_1, z_2 are two independent vectors of V and $x_1, \cdots, x_{k-1}$ are arbitrary nonzero vectors.

Two arbitrary type 1 subspaces are always connected by a chain of adjacent subspaces; explicitly, if

$$M = x_1 \vee \cdots \vee x_{k-1} \vee V \tag{2.7}$$

and

$$N = y_1 \vee \cdots \vee y_{k-1} \vee V$$

then M_p is adjacent to M_{p+1} where

$$M_p = x_1 \vee \cdots \vee x_{k-p-1} \vee y_1 \vee \cdots \vee y_p \vee V \quad p = 1, \cdots, k - 2 \tag{2.8}$$

and we take $M = M_0$ and $N = M_{k-1}$.

PROPOSITION 4. *Two type* 1 *subspaces M and N are adjacent if and only if* $\dim M \cap N = 1$. *Otherwise* $M \cap N = 0$ *whenever M and N are distinct.*

Proof. Consider the adjacent type 1 subspaces (2.6). If $t \in M_1 \cap M_2$ then there exist vectors u and v in V such that

(2.9) $$t = x_1 \vee \cdots \vee x_{k-2} \vee z_1 \vee u = x_1 \vee \cdots x_{k-2} \vee z_2 \vee v \,.$$

Now the multilinear and symmetric mapping $g_p(x)$: $V^p \to \bigvee_{p+1} V$ defined for each $p = 2, \cdots, k-1$ by

(2.10) $$(v_1, \cdots, v_p) \longmapsto x \vee v_1 \vee \cdots \vee v_p$$

extends as in (1.1) to a linear mapping $h_p(x)$: $\bigvee_p V \to \bigvee_{p+1} V$. If the vector x in (2.10) is nonzero then each $h_p(x)$ is injective and so is the composite

$$h = h_{k-1}(x_1) \cdots h_{k-i}(x_i) \cdots h_2(x_{k-2}) \,.$$

Thus (2.9) is just

$$h(z_1 \vee u) = h(z_2 \vee v)$$

and so

$$z_1 \vee u = z_2 \vee v \,.$$

Since z_1 and z_2 are independent (1.2) implies that u is a scalar multiple of z_2. Therefore

(2.11) $$M_1 \cap M_2 = \langle x_1 \vee \cdots \vee x_{k-2} \vee z_1 \vee z_2 \rangle \,.$$

Now consider an arbitrary pair of type 1 subspaces (2.7) and suppose they have nonzero intersection. Let

$$t = x_1 \vee \cdots \vee x_{k-1} \vee u = y_1 \vee \cdots \vee y_{k-1} \vee v$$

be a nonzero element of the intersection. If $\langle u \rangle = \langle v \rangle$ then by (1.2) we have $M_1 = M_2$ and otherwise M_1 and M_2 must have exactly $k-2$ common factors.

PROPOSITION 5. *The images of adjacent type* 1 *subspaces under type* 1 *mappings are adjacent provided the underlying field satisfies* (i).

Proof. Consider the adjacent type 1 subspaces (2.6). We know from Proposition 4 that

$$M_1 \cap M_2 = \langle x_1 \vee \cdots \vee x_{k-2} \vee z_1 \vee z_2 \rangle \,.$$

If f is a type 1 mapping then $f(M_1) \cap f(M_2)$ is nonzero and Proposition 4 yields the desired conclusion provided $f(M_1)$ and $f(M_2)$ are distinct. We complete the proof by showing that the images of adjacent subspaces are always distinct.

Consider the two linear mappings A_i: $V \to \bigvee_k V$ defined by

$$A_i(v) = f(x_1 \vee \cdots \vee x_{k-1} \vee z_i \vee v) \qquad i = 1, 2 .$$

It follows that they are injective because f is linear and decomposable. Suppose range A_1 = range A_2 and let A_2^{-1}: range $A_2 \to V$ be the inverse of A_2. Then $A_2^{-1}A_1$ is a well-defined linear transformation of V. Because of (i), $A_2^{-1}A_1$ has at least one characteristic value, say λ. If u is a corresponding characteristic vector then $A_1 u = \lambda A_2 u$. That is,

$$f(x_1 \vee \cdots \vee x_{k-1} \vee z_1 \vee u) = \lambda f(x_1 \vee \cdots \vee x_{k-1} \vee z_2 \vee u) .$$

Since f is linear and decomposable we obtain $z_1 = \lambda z_2$, contradicting the assumption that M_1 and M_2 are adjacent.

Any collection of two or more type 1 subspaces in $\bigvee_k V\ (k > 2)$ will be called an *adjacent family* if there are vectors $x_1, \cdots, x_{k-2}$ in V such that any subspace in the collection can be written as

$$x_1 \vee \cdots \vee x_{k-2} \vee u \vee V$$

for some vector $u \in V$. When $k = 2$ any collection containing at least two distinct type 1 subspaces will be called an adjacent family. Of course every pair of adjacent type 1 subspaces constitutes an adjacent family, but a collection of three or more need not be, as is easily seen by example.

PROPOSITION 6. *Any collection of more than k pair-wise adjacent type 1 subspaces in $\bigvee_k V$ is an adjacent family.*

Proof. We assign to each type 1 subspace (2.1) the set

$$\{(\langle x_i \rangle, i) \mid i = 1, \cdots, k-1\}$$

which always contains $k - 1$ distinct elements even if (2.1) does not have distinct factors.

The proposition now follows from the combinatorial result that a collection of more than k finite sets each containing $k - 1$ elements which intersect pair-wise in $k - 2$ elements always intersect in the same set of $k - 2$ elements:

If $k = 2$ there is nothing to prove. If $k > 2$ let X and Y be any two sets of the collection. There are elements a and b such that

$$X = (X \cap Y) \cup \{a\}$$

and

$$Y = (X \cap Y) \cup \{b\} .$$

Because any two sets in the collection intersect in $k-2$ elements, any set of the collection not containing $X \cap Y$ must contain both a and b and intersect $X \cap Y$ in exactly $k-3$ elements. But there are at most $k-2 = \binom{k-2}{k-3}$ distinct such sets. Therefore, the collection must contain at least one set Z distinct from X and Y but which contains $X \cap Y$. Let

$$Z = X \cap Y \cup \{c\}$$

and suppose there exists a set W in the collection not containing $X \cap Y$. Then $\{a, b, c\} \subseteqq W$, contradicting the hypothesis that $X \cap W$ has $k-2$ elements.

3. Main results. A collection of vectors in an n-dimentional vector space is said to be in *general position* when any n vectors chosen from the collection form a basis of V. The following well known lemma about vectors in general position will be used in showing that any two associate mappings of a type 1 mapping are multiples whenever $n > 2$ and the underlying field is infinite.

LEMMA 1. *If $m \geqq n$ then an n-dimensional vector space over an infinite field always contains m vectors in general position.*

LEMMA 2. *Let $z_1, \cdots, z_m$ be any finite set of vectors in an n-dimensional vector space over an infinite field. If $A: V \rightarrow V$ is non-singular and B is any other linear mapping of V satisfying*

$$\langle A(x) \rangle = \langle B(x) \rangle \tag{3.1}$$

for all vectors x not in $S = \langle z_1 \rangle \cup \cdots \cup \langle z_m \rangle$ then there is a scalar λ such that $B = \lambda A$.

Proof. Since F is infinite Lemma 12 of [1] and induction show the existence of a basis of V disjoint from the set S. If $b_1, \cdots, b_n$ is such a basis let $\lambda_1, \cdots, \lambda_n$ be scalars such that

$$B(b_i) = \lambda_i A(b_i) \qquad i = 1, \cdots, n . \tag{3.2}$$

Since F is infinite we may choose a vector $v = \Sigma \alpha_i b_i$ not in S but all of whose coordinates with respect to $b_1, \cdots, b_n$ are non-zero. Then (3.1) and (3.2) imply the existence of a scalar λ such that

$$\Sigma \alpha_i \lambda_i A(b_i) = \Sigma \lambda \alpha_i A(b_i) .$$

Since A is nonsingular we have $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$.

REMARK. In (i) we assume that every polynomial of degree at most n splits completely over the underlying field. This means that the field is necessarily infinite since the polynomial ring over a finite field has irreducible elements of every degree. Thus Lemmas 1 and 2 are immediately applicable in the following theorems.

THEOREM 1. *The associate mappings of a type* 1 *mapping of* $\bigvee kV$ *are a* 1*-dimensional subspace of the linear mappings of* V, *provided* $\dim V > 2$ *and* F *satisfies* (i).

Proof. We show first that an associate map of a type 1 mapping f with respect to one of type 1 subspaces (2.6) is always a scalar multiple of every associate mapping of the other. By Lemma 1 we complete the vectors z_1, z_2 to a set $z_1, \cdots, z_m$ in general position where $m = \mathrm{Max}\{k, \dim V\}$. As in the proof of the Proposition 1 we may choose a vector z_{m+1} not in the set-theoretic union $\langle z_1\rangle \cup \cdots \cup \langle z_m\rangle$. Then the subspaces

$$M_i = x_1 \vee \cdots \vee x_{k-2} \vee z_i \vee V \quad i = 1, \cdots, m+1$$

are an adjacent family. The images of these subspaces form a family of pair-wise adjacent subspaces by Proposition 5. They form an adjacent family by Proposition 6 and the choice of m. Thus we may choose vectors $y_1, \cdots, y_{k-2}; w_1, \cdots, w_{m+1}$ in V such that

$$f(M_i) = y_1 \vee \cdots \vee y_{k-2} \vee w_i \vee V \quad i = 1, \cdots, m+1 . \tag{3.3}$$

We proceed to examine the effect of f on the intersections $M_i \cap M_{m+1}$; $i = 1, 2$. By (3.3)

$$\begin{aligned} f(x_1 \vee \cdots \vee x_{k-2} \vee z_i \vee z_{m+1}) &= y_1 \vee \cdots \vee y_{k-2} \vee w_i \vee A_i(z_{m+1}) \\ &= y_1 \vee \cdots \vee y_{k-2} \vee w_{m+1} \vee A_{m+1}(z_i) \\ & \qquad i = 1, 2 . \end{aligned}$$

where A_i denotes any associate map of M_i under f and A_{m+1} is an associate of M_{m+1}. It follows that $\langle w_{m+1}\rangle = \langle A_i(z_{m+1})\rangle$ for $i = 1, 2$ because w_{m+1} is not in $\langle w_1\rangle \cup \langle w_2\rangle$. Since z_{m+1} is restricted only by its exclusion from $\langle z_1\rangle \cup \cdots \cup \langle z_m\rangle$ Lemma 2 applies and yields a scalar γ such that $A_1 = \gamma A_2$.

To complete the proof we need only consider an arbitrary pair of type 1 subspaces (2.7) and a chain (2.8) of adjacent subspaces between them. If A_p is an associate map of M_p then we have just shown the existence of a scalar γ_p such that

$$A_p = \gamma_p A_{p+1} \qquad p = 0, \cdots, k-2 .$$

Therefore, $A_0 = \gamma_0 \cdots \gamma_{k-2} A_{k-1}$.

REMARK. If $\dim V = 1$ then $\bigvee_k V = 1$ and $L(\bigvee_k V, \bigvee_k V) \cong F$. Hence $L(\bigvee_k V, \bigvee_k V)$ consists of induced mappings if and only if every polynomial of the form $x^k - a$ has a root in F.

THEOREM 2. *Every type* 1 *mapping of* $\bigvee_k V$ *is induced by an associate mapping, provided* $\dim V > 2$ *and* F *satisfies* (i).

Proof. Let $x = x_1 \vee \cdots \vee x_k$ be any nonzero product of $\bigvee_k V$. The trivial subspace $\langle x \rangle$ is the intersection of the k type 1 subspaces

$$T_i = x_1 \vee \cdots \vee \hat{x}_i \vee \cdots \vee x_k \vee V \qquad i = 1, \cdots, k \,. \tag{3.4}$$

By Theorem 1 the associate mappings of a type 1 mapping f with respect to the subspaces (3.4) are scalar multiples of one another. If A is any one of them then Theorem 1 and definition (2.5) show then that Ax_i must be a factor of $f(x)$ for each $i = 1, \cdots, k$. Thus, if x has distinct factors it follows from (1.2) and Proposition 3 that

$$f(x) = \lambda_x Ax_1 \vee \cdots \vee Ax_k \tag{3.5}$$

for some scalar λ_x and

$$f(T_i) = Ax_1 \vee \cdots \vee \widehat{Ax_i} \vee \cdots \vee Ax_k \vee V \quad i = 1, \cdots, k \,. \tag{3.6}$$

We next verify (3.6) when the factors $\langle x_1 \rangle, \cdots, \langle x_k \rangle$ are not necessarily distinct. To this end consider a chain of adjacent subspaces (2.8) where we suppose M_{k-1} has arbitrary factors and take the factors of M_0 as distinct and distinct from the factors of M_{k-1}. This we may always do since any field satisfying (i) must be infinite. (See the remark following Lemma 2.) Thus (3.6) may be applied to M_0 which contains $z_1 = x_1 \vee \cdots \vee x_{k-1} \vee y_1$. By Theorem 1 there is a scalar λ for which

$$f(z_1) = \lambda Ax_1 \vee \cdots \vee Ax_{k-1} \vee Ay_1 \,. \tag{3.7}$$

Therefore the $k-1$ factors of $f(M_1)$ must be among the factors of (3.7). Now $\langle Ay_1 \rangle$ could not be excluded because then M_0 and M_1 would have the same type 1 subspace as image, contradicting Proposition 5. If, say, Ax_1 were excluded then

$$f(M_1) = Ax_2 \vee \cdots \vee Ax_{k-1} \vee Ay_1 \vee V$$

and Theorem 1 yields

$$f(z_1) = \lambda_1 Ax_2 \vee \cdots \vee Ay_1 \vee Ax_{k-1} \tag{3.8}$$

for some scalar λ_1.

Comparison of (3.7) and (3.8) shows that Ax_{k-1} would be a scalar multiple of either Ay_1 or some Ax_i with $1 \leqq i < k - 1$. Hence

$$f(M_1) = Ax_1 \vee \cdots \vee Ax_{k-2} \vee Ay_1 \vee V .$$

Suppose it has been shown that

$$(3.9) \qquad f(M_p) = Ax_1 \vee \cdots \vee Ax_{k-p-1} \vee Ay_1 \vee \cdots \vee Ay_p \vee V$$

for some p, $1 < p \leqq k - 2$. Since

$$M_p \cap M_{p+1} = \langle x_1 \vee \cdots \vee x_{k-p-1} \vee y_1 \vee \cdots \vee y_{p+1} \rangle$$

(3.9) implies that $f(M_{p+1})$ contains

$$(3.10) \qquad Ax_1 \vee \cdots \vee Ax_{k-p-1} \vee Ay_1 \vee \cdots \vee Ay_{p+1}$$

and so the $k - 1$ factors of $f(M_{p+1})$ are among the factors of (3.10). Arguing as before we see that Ay_{p+1} must be a factor of $f(M_{p+1})$ since otherwise the images of $f(M_p)$ and $f(M_{p+1})$ would coincide. If, say, Ax_1 were not a factor then

$$f(M_{p+1}) = Ax_2 \vee \cdots \vee Ax_{k-p-1} \vee Ay_1 \vee \cdots \vee Ay_{p+1} \vee V$$

and by Theorem 1 there is a scalar μ for which

$$(3.11) \qquad f(x_1 \vee \cdots \vee x_{k-p-1} \vee y_1 \vee \cdots \vee y_{p+1}) \\ = \mu Ax_2 \vee \cdots \vee Ax_{k-p-1} \vee Ay_1 \vee \cdots \vee Ay_{p+1} \vee Ax_{k-p-1} .$$

Comparison of (3.10) and (3.11) shows that Ax_{k-p-1} would be either a multiple of some Ay_i, $1 \leqq i \leqq p + 1$, or some Ax_j, $1 \leqq j < k - p - 1$, contradicting the assumption that the factors of M_0 are distinct and distinct from the factors of M_{k-1}.

Since any product x is in some type 1 subspace we have shown that $f(x) = \lambda_x(\bigvee_k A)(x)$ for some scalar λ_x. If x and y are products in the same type 1 subspace a simple comparison argument shows that $\lambda_x = \lambda_y$. Denote the common value by λ. When x and y are arbitrary products we obtain the same result by considering type 1 subspaces containing them and a chain (2.8) between the subspaces since any two of the latter have 1-dimensional intersections. Because the field always contains a root of $x^k - \lambda = 0$ by (i), we have shown that f is induced by $\lambda^{1/k} A$.

THEOREM 3. *Every decomposable mapping of* $\bigvee_k V$ *is induced by a nonsingular mapping of* V*, provided* V *is a finite dimensional vector space satisfying* (i) *and* (ii).

Proof. Because of the previous theorem we need only show with the additional hypothesis that every decomposable mapping of

$\bigvee_k V$ is type 1. If M is any type 1 subspace and f decomposable then $f(M)$ is a decomposable subspace and hence contained in a maximal decomposable subspace of $\bigvee_k V$. In [1] the maximal decomposable subspaces of $\bigvee_k V$ were determined for the case when V satisfies the hypothesis of this theorem. The subspaces are

(a) type 1 subspaces

(b) type r subspaces which are of the form

$$x_1 \vee \cdots \vee x_{k-r} \vee S \vee \cdots \vee S$$

where $1 < r \leqq k$ and S is a 2-dimensional subspace of V.

Those subspaces of type $r > 1$ have dimension $r + 1$. If the maximal decomposable subspace containing $f(M)$ was one of these types then $\dim V \leqq r + 1 \leqq k + 1$ by (1.3) because every type 1 subspace has the same dimension as V. The hypothesis $\dim V > k + 1$ thus implies that the maximal decomposable subspace containing $f(M)$ is type 1 and therefore f is type 1.

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