

# Pacific Journal of Mathematics

**COHOMOLOGY OF FINITELY PRESENTED GROUPS**

PETER MICHAEL CURRAN

## COHOMOLOGY OF FINITELY PRESENTED GROUPS

P. M. CURRAN

Let  $G$  be a finitely presented group,  $G'$  a finite quotient of  $G$  and  $K$  a field. Let  $G$  act on the group algebra  $V = K[G']$  in the natural way. For a suitable choice of  $G'$  we obtain estimates on the dimension of  $H^1(G, V)$  in terms of the presentation and then use these estimates to derive information about  $G$ .

If  $G$  is generated by  $n$  elements, of which  $m$  have finite orders  $k_1, \dots, k_m$ , resp., and  $G$  has the presentation

$$\langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q} \rangle,$$

then, in particular, we show that (a) the minimum number of generators of  $G$  is  $\geq n - q - \sum 1/k_i$ ; (b) if this lower bound is actually attained, then  $G$  is free, of this rank, and (c)  $G$  is infinite if  $\sum 1/k_i \leq n - q - 1$ . The latter, together with a result of R. Fox, yields an algebraic proof that the group

$$\langle a_1, \dots, a_m; a_1^{k_1}, \dots, a_m^{k_m}, a_1 \cdots a_m \rangle$$

is infinite if  $\sum 1/k_i \leq m - 2$ .

1. An exact sequence. Let  $G$  be a group with the presentation  $\langle a_1, \dots, a_n; r_1, r_2, \dots \rangle$ , i.e.,  $G = F/N$ , where  $F$  is the free group on  $\{a_1, \dots, a_n\}$  and  $N$  is the normal subgroup generated by  $\mathcal{R} = \{r_1, r_2, \dots\}$ . We denote by  $\varphi$  the homomorphism of group rings  $Z[F] \rightarrow Z[G]$  which extends the natural map  $F \rightarrow F/N$ , and by  $A_i$  the element  $\varphi a_i$  of  $G$ .

Let  $\rho$  be a representation of  $G$  in  $\text{Aut}(V)$ , where  $V$  is a finite-dimensional vector space over a field  $K$ . We shall be concerned with the first cohomology group  $H^1(G, V)$ , which is also a vector space over  $K$  in an obvious way. One knows that an arbitrary map  $f: \{A_1, \dots, A_n\} \rightarrow V$  extends to a 1-cocycle of  $G$  in  $V$  if and only if the 1-cocycle of  $F$  determined by  $a_i \mapsto f(A_i)$  vanishes on the relators. More precisely, the following sequence is exact:

$$(*) \quad 0 \longrightarrow Z^1(G, V) \xrightarrow{E} V^n \xrightarrow{D} V_1 \oplus V_2 \oplus \dots$$

Here,  $Z^1(G, V)$  is the space of 1-cocycles,  $V^n$  is the direct sum of  $n$  copies of  $V$ ,  $V_i = V$  for each  $i$ ,  $E$  is the map  $f \mapsto (f(A_1), \dots, f(A_n))$ ,  $D$  is the map

$$(u_1, \dots, u_n) \longmapsto \left( \sum_j (\partial r_1 / \partial a_j) u_j, \sum_j (\partial r_2 / \partial a_j) u_j, \dots \right),$$

and in the last term of the sequence there is one copy of  $V$  for each

member of  $\mathcal{R}$ .  $\partial r/\partial a_j$  is the Fox derivative of  $r$  with respect to  $a_j$  [2, Chap. VII, § 2].

Now suppose  $r_i = a_i^{k_i}$  for  $i = 1, \dots, m$ , and that the characteristic of  $K$  does not divide any of the  $k_i$ . Then for  $i = 1, \dots, m$ ,

$$\sum_j (\partial r_i/\partial a_j)u_j = (1 + T_i + \dots + T_i^{k_i-1})u_i,$$

where  $T_i = \rho\varphi a_i$ , so, using the fact that

$$\text{Ker}(1 + T_i + \dots + T_i^{k_i-1}) = \text{Im}(1 - T_i),$$

we may replace (\*) by

$$\begin{aligned} (**) \quad 0 &\longrightarrow Z^1(G, V) \xrightarrow{E} \text{Im}(1 - T_1) \oplus \dots \oplus \text{Im}(1 - T_m) \\ &\oplus V^{n-m} \xrightarrow{D'} V_{m+1} \oplus V_{m+2} \oplus \dots \end{aligned}$$

where  $D'$  is given by

$$(u_1, \dots, u_n) \longmapsto \left( \sum_j (\partial r_{m+1}/\partial a_j)u_j, \sum_j (\partial r_{m+2}/\partial a_j)u_j, \dots \right).$$

2. Conditions for  $G$  to be a free product. The following lemma will be needed for the applications in the next section. In what follows,  $[x, y, \dots]$  denotes the subgroup of  $G$  generated by  $\{x, y, \dots\}$ ,  $|x|$  is the order of  $x$ ,  $A = \varphi a$  and  $G_1 * G_2$  is the free product of  $G_1$  and  $G_2$ . Otherwise the notation is that of § 1.

LEMMA. Let  $G = \langle a = a_1, a_2, \dots, a_n; \mathcal{R} \rangle$ . Then

(a) The following statements are equivalent.

(1)  $\varphi(\partial r/\partial a(1 - a)) = 0$  for all  $r \in \mathcal{R}$  (and therefore for all  $r \in N$ ).

(2)  $G = [A] * [A_2, \dots, A_n]$ .

(b) If (1) is replaced by the stronger condition  $\varphi(\partial r/\partial a) = 0$  for all  $r \in \mathcal{R}$ , then the condition  $|A| = \infty$  may be added to (2).

Proof. (a) If  $A = 1$ , then (1) and (2) are trivially true, so we may assume that  $A \neq 1$  from now on.

(2)  $\Rightarrow$  (1): Given  $G = [A] * [A_2, \dots, A_n]$ , let  $\langle a_2, \dots, a_n; \mathcal{S} \rangle$  be a presentation for  $[A_2, \dots, A_n]$ . Then  $\varphi(\partial s/\partial a(1 - a)) = 0$  for all  $s \in \mathcal{S}$ , and if  $|A| = k$ ,  $\varphi(\partial a^k/\partial a(1 - a)) = \varphi(1 - a^k) = 0$ . Thus  $\varphi(\partial r/\partial a(1 - a)) = 0$  for all  $r$  in a system of defining relations for  $G$ . It follows easily that the same is true for all  $r \in N$ , hence, in particular, for all members of  $\mathcal{R}$ .

(1)  $\Rightarrow$  (2): Suppose  $\varphi(\partial r/\partial a(1 - a)) = 0$  for all  $r \in \mathcal{R}$ . We may assume that no proper part of any member of  $\mathcal{R}$  is in  $N$ , and if  $|A| = k < \infty$ , that  $a^k \in \mathcal{R}$ . Let  $\mathcal{R}_1 = \mathcal{R} - \{a^k\}$  if  $a^k \in \mathcal{R}$ ; otherwise, let  $\mathcal{R}_1 = \mathcal{R}$ . We claim that all members of  $\mathcal{R}_1$  are free of  $a$  and  $a^{-1}$ . This will complete the proof.

Suppose some  $r \in \mathcal{R}_1$  involves  $a$  or  $a^{-1}$ . We may assume that  $r$  has the form  $r = aw_1a^{\pm 1}w_2 \dots a^{\pm 1}w_r$ , where  $w_1$  is not the empty word. Applying condition (1) to  $r$  and multiplying the resulting equation on the left by  $\varphi(a^{-1})$ , we obtain

$$\begin{aligned} \varphi(a^{-1}) \pm \varphi(w_1(a^{-1})) \pm \dots \pm \varphi(w_1 \dots w_{r-1}(a^{-1})) \\ = 1 \pm \varphi(w_1(a)) \pm \dots \pm \varphi(w_1 \dots w_{r-1}(a)), \end{aligned}$$

where the parenthetical  $a^{-1}$  in the left hand member occurs precisely when the term has a minus sign and the parenthetical  $a$  on the right goes with the plus sign. But all terms except the first term on each side are images of proper parts of  $r$ , hence  $\neq 1$ , and  $\varphi(a^{-1}) \neq 1$  by hypothesis, so the last equation is impossible in  $Z[G]$ . This contradiction completes the proof of (a).

As for (b), if  $G = [A]*[A_2, \dots, A_n]$  and  $|A| = \infty$ , then  $G$  has a presentation in which no relator involves  $a$ , so  $\partial r/\partial a = 0$  for all  $r$  in  $N$ . Conversely, if  $|A| = k < \infty$ , then  $\varphi(\partial a^k/\partial a) = 1 + A + \dots + A^{k-1} \neq 0$ .

**COROLLARY.** Let  $G = \langle a_1, \dots, a_n; \mathcal{R} \rangle$ . Suppose that

(1) for  $j = 1, \dots, m$ ,  $\varphi(\partial r/\partial a_j(1 - a_j)) = 0$ , all  $r \in \mathcal{R}$ , but there exists  $r_j \in N$  such that  $\varphi(\partial r_j/\partial a_j) \neq 0$ , and

(2) for  $j = m + 1, \dots, m + p$ ,  $\varphi(\partial r/\partial a_j) = 0$  for all  $r \in \mathcal{R}$ . Then

(a)  $G = [A_1]* \dots [A_{m+p}]*[A_{m+p+1}, \dots, A_n]$  and

(b)  $|A_j| < \infty, j = 1, \dots, m$  and  $|A_j| = \infty, j = m + 1, \dots, m + p$ .

**3. The main theorem.** We recall that a group  $G$  is *residually finite* if given  $1 \neq g \in G$ , there exists a finite quotient of  $G$  in which the image of  $g$  is  $\neq 1$ . By a theorem of Mal'cev [5], all finitely generated linear groups over a field are residually finite.

We note for future reference some easily deduced properties of residually finite groups. ( $R$  is any ring with unity.)

**RF1.** If  $G$  is residually finite and  $\alpha_1, \dots, \alpha_r$  are nonzero elements of the group ring  $R[G]$ , there exists a finite quotient  $G'$  of  $G$  such that the images of  $\alpha_1, \dots, \alpha_r$  in  $R[G']$  are all nonzero.

**RF2.** Let  $g_i$  have finite order  $k_i, i = 1, \dots, m$ , in a residually finite group  $G$ . Then there exists a finite quotient of  $G$  in which the image of  $g_i$  has order  $k_i$  for each  $i$ .

Now suppose  $G$  is a group,  $G'$  a finite quotient of  $G$  and  $K$  a field. Let an action of  $G$  on the group algebra  $V = K[G']$  be defined

as follows: If  $g \in G$  and  $v \in V$ ,  $gv$  is defined to be the product  $g'v$  in  $K[G']$  where  $g'$  is the image of  $g$  in  $G'$ . Then it is easy to show that  $V^G = \{v \in V: gv = v, \text{ all } g \in G\}$  is the one-dimensional subspace generated by  $s = \sum_{g' \in G'} g'$ . We shall also need to know the "fixed point" space of an element  $g \in G$ , i.e.  $\{v \in V: gv = v\}$ . Let  $G' = \{g'_1, \dots, g'_d\}$ . If  $\pi$  is the permutation of  $\{1, \dots, d\}$  such that  $g'g'_i = g'_{\pi(i)}$  and  $g'$  has order  $k$ , then  $\pi$  is the product of  $d/k$  disjoint cycles:  $\pi = (i_1, \dots, i_k)(i_{k+1}, \dots, i_{2k}) \dots$ . It follows easily that the fixed point space of  $g$  is the  $d/k$ -dimensional subspace of  $V$  generated by the elements

$$\sum_{j=1}^k g'_{i_j}, \sum_{j=k+1}^{2k} g'_{i_j}, \dots .$$

The main results are consequences of the following theorem. The notation is that of § 2.

**THEOREM.** *Let  $G$  be a residually finite group with the presentation*

$$\langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, r_{m+2}, \dots \rangle$$

*and let  $K$  be a field of characteristic 0. (We assume the  $k_i > 1$ .) Then there exists a finite quotient  $G'$  of  $G$  such that if  $G$  acts on  $V = K[G']$  as above, then, letting  $d = |G'|$ ,  $\sigma = \sum_{i=1}^m 1/|A_i|$  and  $\tau = \sum_{i=1}^m 1/k_i$ , we have*

- (a)  $\dim H^1(G, V) \leq (n - \sigma - 1)d + 1 \leq (n - \tau - 1)d + 1$
- (b) *if equality holds throughout (a), then  $G = [A_1]^* \dots [A_m]^*$ ,  $|A_j| = k_j, j = 1, \dots, m$  and  $|A_j| = \infty, j = m + 1, \dots, n$ .*
- (c) *if the set of defining relations is finite, say*

$$\mathcal{R} = \{a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q}\},$$

*then  $\dim H^1(G, V) \geq (n - \sigma - q - 1)d + 1$ .*

**REMARK.** It will be clear from the proof that if a finite number of presentations of  $G$  are given,  $G'$  can be chosen so that (a) through (c) are simultaneously true for all the given presentations.

*Proof.* By RF2, choose  $G'$  so that the image of  $A_i$  in  $G'$  has order  $|A_i|$ ,  $i = 1, \dots, m$ . In the notation of (\*\*), § 1,

$$\dim \text{Im} (1 - T_i) = d - \dim \text{Ker} (1 - T_i) = d(1 - 1/|A_i|)$$

by the remarks preceding the theorem. Hence, by (\*\*)

$$(1) \quad \dim Z^1(G, V) = (n - \sigma)d - \text{rank} (D') .$$

Now the map of  $V$  onto the space  $B^1(G, V)$  of coboundaries given by  $v \mapsto f_v$ , where  $f_v(g) = gv - v$  for all  $g \in G$ , has kernel  $V^G$ , so

$\dim B^i = d - 1$ . Combining this with (1) yields the first inequality in (a). The second inequality is clear.

To prove (b), note first that if  $|A_i| < k_i$  for some  $i$ , then the second inequality in (a) is strict. Therefore it with suffice to show that if  $G \neq [A_1] * \dots * [A_n]$  or if some  $A_j$  with  $j > m$  has finite order, then  $G'$  can be chosen so that (in addition to the preservation of orders  $|A_i|$ ,  $i = 1, \dots, m$ ) we have  $D' \neq 0$ . For then, (1) implies that the first inequality in (a) is strict.

Consider the following elements of  $K[G]$ :

$$\begin{aligned} &\varphi(\partial r_i / \partial a_j), \quad i > m, j > m \\ &\varphi(\partial r_i / \partial a_j (1 - a_j)), \quad i > m, j \leq m. \end{aligned}$$

One of these must be nonzero since otherwise, by the Corollary of § 2,  $G = [A_1] * \dots * [A_n]$  and  $|A_i| = \infty$ ,  $i > m$ , contrary to hypothesis. Therefore by RF1 there exists a finite quotient  $G'$  such that the image in  $K[G']$  of this nonzero element is also nonzero. One easily sees then that  $D' \neq 0$ . This proves (b).

Given the hypothesis of (c), we have  $\text{rank } D' \leq qd$ . The conclusion then follows from (1) above.

**COROLLARY 1.** *Let  $G$  be a residually finite group with two presentations*

$$\begin{aligned} G &= \langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q} \rangle \\ &= \langle b_1, \dots, b_N; b_1^{h_1}, \dots, b_M^{h_M}, s_{M+1}, \dots \rangle. \end{aligned}$$

Then

$$n - \sum_{i=1}^m 1/|A_i| - q \leq N - \sum_{j=1}^M 1/h_j,$$

and if equality holds, then  $G = [B_1] * \dots * [B_N]$ ,  $|B_j| = h_j$  for  $j = 1, \dots, M$  and  $|B_j| = \infty$  for  $j > M$ . ( $B_k$  is the image in  $G$  of the free generator  $b_k$ .)

*Proof.* Apply part (c) to the first presentation and parts (a) and (b) to the second. (See the remark preceding the proof of the theorem.)

Note that Corollary 1 implies for residually finite groups the well-known result [4, Cor. 5.14.2] that if a group  $G$  with  $n$  generators and  $q$  defining relations can be generated by  $n - q$  elements, then  $G$  is free of rank  $n - q$ .

**COROLLARY 2.** *Let*

$$G = \langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q} \rangle.$$

Then  $G$  is infinite if

$$\sum_{i=1}^m 1/|A_i| \leq n - q - 1.$$

*Proof.* If  $|G| = d < \infty$ , we may take  $G' = G$  in the proof of the Theorem. But then  $dH^1(G, V) = 0$  [1, Chap. XII, Prop. 2.5] so  $H^1(G, V) = 0$  since  $K$  has characteristic zero. The conclusion now follows from part (c) of the Theorem.

Finally, we apply Corollary 2 to a classical case. Let

$$G = \langle a_1, \dots, a_m; a_1^{k_1}, \dots, a_m^{k_m}, a_1 \cdots a_m \rangle.$$

From geometric considerations (e.g. [7, p. 28, Satz 8]) one knows that the group is infinite if  $\sum 1/k_i \leq m - 2$ . In 1902, Miller [6] gave an algebraic proof of this fact for the case  $m = 3$ , but the argument involves consideration of many cases.

In [3] Fox shows that if  $k_1, k_2, k_3$  are integers  $> 1$ , then there exist permutations  $A$  and  $B$  of orders  $k_1$  and  $k_2$ , resp., such that  $AB$  has order  $k_3$ . It follows easily from this that  $k_i = |A_i|$  in the above group (assuming  $m > 2$ ). Hence Corollary 2, together with this result, yields an algebraic proof that  $G$  is infinite when  $\sum 1/k_i \leq m - 2$ .

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