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COHOMOLOGY OF FINITELY PRESENTED GROUPS

PETER MICHAEL CURRAN

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Let G be a finitely presented group, G' a finite quotient of G and K a field. Let G act on the group algebra $V = K[G']$ in the natural way. For a suitable choice of G' we obtain estimates on the dimension of $H^1(G, V)$ in terms of the presentation and then use these estimates to derive information about G .

If G is generated by n elements, of which m have finite orders k_1, \dots, k_m , resp., and G has the presentation

$$\langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q} \rangle,$$

then, in particular, we show that (a) the minimum number of generators of G is $\geq n - q - \sum 1/k_i$; (b) if this lower bound is actually attained, then G is free, of this rank, and (c) G is infinite if $\sum 1/k_i \leq n - q - 1$. The latter, together with a result of R. Fox, yields an algebraic proof that the group

$$\langle a_1, \dots, a_m; a_1^{k_1}, \dots, a_m^{k_m}, a_1 \cdots a_m \rangle$$

is infinite if $\sum 1/k_i \leq m - 2$.

1. An exact sequence. Let G be a group with the presentation $\langle a_1, \dots, a_n; r_1, r_2, \dots \rangle$, i.e., $G = F/N$, where F is the free group on $\{a_1, \dots, a_n\}$ and N is the normal subgroup generated by $\mathcal{R} = \{r_1, r_2, \dots\}$. We denote by φ the homomorphism of group rings $Z[F] \rightarrow Z[G]$ which extends the natural map $F \rightarrow F/N$, and by A_i the element φa_i of G .

Let ρ be a representation of G in $\text{Aut}(V)$, where V is a finite-dimensional vector space over a field K . We shall be concerned with the first cohomology group $H^1(G, V)$, which is also a vector space over K in an obvious way. One knows that an arbitrary map $f: \{A_1, \dots, A_n\} \rightarrow V$ extends to a 1-cocycle of G in V if and only if the 1-cocycle of F determined by $a_i \mapsto f(A_i)$ vanishes on the relators. More precisely, the following sequence is exact:

$$(*) \quad 0 \longrightarrow Z^1(G, V) \xrightarrow{E} V^n \xrightarrow{D} V_1 \oplus V_2 \oplus \dots$$

Here, $Z^1(G, V)$ is the space of 1-cocycles, V^n is the direct sum of n copies of V , $V_i = V$ for each i , E is the map $f \mapsto (f(A_1), \dots, f(A_n))$, D is the map

$$(u_1, \dots, u_n) \longmapsto \left(\sum_j (\partial r_1 / \partial a_j) u_j, \sum_j (\partial r_2 / \partial a_j) u_j, \dots \right),$$

and in the last term of the sequence there is one copy of V for each

member of \mathcal{R} . $\partial r/\partial a_j$ is the Fox derivative of r with respect to a_j [2, Chap. VII, § 2].

Now suppose $r_i = a_i^{k_i}$ for $i = 1, \dots, m$, and that the characteristic of K does not divide any of the k_i . Then for $i = 1, \dots, m$,

$$\sum_j (\partial r_i/\partial a_j)u_j = (1 + T_i + \dots + T_i^{k_i-1})u_i,$$

where $T_i = \rho\varphi a_i$, so, using the fact that

$$\text{Ker}(1 + T_i + \dots + T_i^{k_i-1}) = \text{Im}(1 - T_i),$$

we may replace (*) by

$$\begin{aligned}
 (**) \quad 0 &\longrightarrow Z^1(G, V) \xrightarrow{E} \text{Im}(1 - T_1) \oplus \dots \oplus \text{Im}(1 - T_m) \\
 &\oplus V^{n-m} \xrightarrow{D'} V_{m+1} \oplus V_{m+2} \oplus \dots
 \end{aligned}$$

where D' is given by

$$(u_1, \dots, u_n) \longmapsto \left(\sum_j (\partial r_{m+1}/\partial a_j)u_j, \sum_j (\partial r_{m+2}/\partial a_j)u_j, \dots \right).$$

2. Conditions for G to be a free product. The following lemma will be needed for the applications in the next section. In what follows, $[x, y, \dots]$ denotes the subgroup of G generated by $\{x, y, \dots\}$, $|x|$ is the order of x , $A = \varphi a$ and $G_1 * G_2$ is the free product of G_1 and G_2 . Otherwise the notation is that of § 1.

LEMMA. *Let $G = \langle a = a_1, a_2, \dots, a_n; \mathcal{R} \rangle$. Then*

(a) *The following statements are equivalent.*

(1) $\varphi(\partial r/\partial a(1 - a)) = 0$ for all $r \in \mathcal{R}$ (and therefore for all $r \in N$).

(2) $G = [A] * [A_2, \dots, A_n]$.

(b) *If (1) is replaced by the stronger condition $\varphi(\partial r/\partial a) = 0$ for all $r \in \mathcal{R}$, then the condition $|A| = \infty$ may be added to (2).*

Proof. (a) If $A = 1$, then (1) and (2) are trivially true, so we may assume that $A \neq 1$ from now on.

(2) \Rightarrow (1): Given $G = [A] * [A_2, \dots, A_n]$, let $\langle a_2, \dots, a_n; \mathcal{S} \rangle$ be a presentation for $[A_2, \dots, A_n]$. Then $\varphi(\partial s/\partial a(1 - a)) = 0$ for all $s \in \mathcal{S}$, and if $|A| = k$, $\varphi(\partial a^k/\partial a(1 - a)) = \varphi(1 - a^k) = 0$. Thus $\varphi(\partial r/\partial a(1 - a)) = 0$ for all r in a system of defining relations for G . It follows easily that the same is true for all $r \in N$, hence, in particular, for all members of \mathcal{R} .

(1) \Rightarrow (2): Suppose $\varphi(\partial r/\partial a(1 - a)) = 0$ for all $r \in \mathcal{R}$. We may assume that no proper part of any member of \mathcal{R} is in N , and if $|A| = k < \infty$, that $a^k \in \mathcal{R}$. Let $\mathcal{R}_1 = \mathcal{R} - \{a^k\}$ if $a^k \in \mathcal{R}$; otherwise, let $\mathcal{R}_1 = \mathcal{R}$. We claim that all members of \mathcal{R}_1 are free of a and a^{-1} . This will complete the proof.

Suppose some $r \in \mathcal{R}_1$ involves a or a^{-1} . We may assume that r has the form $r = aw_1a^{\pm 1}w_2 \cdots a^{\pm 1}w_r$, where w_1 is not the empty word. Applying condition (1) to r and multiplying the resulting equation on the left by $\varphi(a^{-1})$, we obtain

$$\begin{aligned} \varphi(a^{-1}) \pm \varphi(w_1(a^{-1})) \pm \cdots \pm \varphi(w_1 \cdots w_{r-1}(a^{-1})) \\ = 1 \pm \varphi(w_1(a)) \pm \cdots \pm \varphi(w_1 \cdots w_{r-1}(a)), \end{aligned}$$

where the parenthetical a^{-1} in the left hand member occurs precisely when the term has a minus sign and the parenthetical a on the right goes with the plus sign. But all terms except the first term on each side are images of proper parts of r , hence $\neq 1$, and $\varphi(a^{-1}) \neq 1$ by hypothesis, so the last equation is impossible in $Z[G]$. This contradiction completes the proof of (a).

As for (b), if $G = [A]*[A_2, \dots, A_n]$ and $|A| = \infty$, then G has a presentation in which no relator involves a , so $\partial r/\partial a = 0$ for all r in N . Conversely, if $|A| = k < \infty$, then $\varphi(\partial a^k/\partial a) = 1 + A + \dots + A^{k-1} \neq 0$.

COROLLARY. Let $G = \langle a_1, \dots, a_n; \mathcal{R} \rangle$. Suppose that

(1) for $j = 1, \dots, m$, $\varphi(\partial r/\partial a_j(1 - a_j)) = 0$, all $r \in \mathcal{R}$, but there exists $r_j \in N$ such that $\varphi(\partial r_j/\partial a_j) \neq 0$, and

(2) for $j = m + 1, \dots, m + p$, $\varphi(\partial r/\partial a_j) = 0$ for all $r \in \mathcal{R}$. Then

(a) $G = [A_1]*\cdots*[A_{m+p}]*[A_{m+p+1}, \dots, A_n]$ and

(b) $|A_j| < \infty, j = 1, \dots, m$ and $|A_j| = \infty, j = m + 1, \dots, m + p$.

3. The main theorem. We recall that a group G is *residually finite* if given $1 \neq g \in G$, there exists a finite quotient of G in which the image of g is $\neq 1$. By a theorem of Mal'cev [5], all finitely generated linear groups over a field are residually finite.

We note for future reference some easily deduced properties of residually finite groups. (R is any ring with unity.)

RF1. If G is residually finite and $\alpha_1, \dots, \alpha_r$ are nonzero elements of the group ring $R[G]$, there exists a finite quotient G' of G such that the images of $\alpha_1, \dots, \alpha_r$ in $R[G']$ are all nonzero.

RF2. Let g_i have finite order $k_i, i = 1, \dots, m$, in a residually finite group G . Then there exists a finite quotient of G in which the image of g_i has order k_i for each i .

Now suppose G is a group, G' a finite quotient of G and K a field. Let an action of G on the group algebra $V = K[G']$ be defined

as follows: If $g \in G$ and $v \in V$, gv is defined to be the product $g'v$ in $K[G']$ where g' is the image of g in G' . Then it is easy to show that $V^\sigma = \{v \in V: gv = v, \text{ all } g \in G\}$ is the one-dimensional subspace generated by $s = \sum_{g' \in G'} g'$. We shall also need to know the "fixed point" space of an element $g \in G$, i.e. $\{v \in V: gv = v\}$. Let $G' = \{g'_1, \dots, g'_d\}$. If π is the permutation of $\{1, \dots, d\}$ such that $g'g'_i = g'_{\pi i}$ and g' has order k , then π is the product of d/k disjoint cycles: $\pi = (i_1, \dots, i_k)(i_{k+1}, \dots, i_{2k}) \dots$. It follows easily that the fixed point space of g is the d/k -dimensional subspace of V generated by the elements

$$\sum_{j=1}^k g'_{i_j}, \sum_{j=k+1}^{2k} g'_{i_j}, \dots.$$

The main results are consequences of the following theorem. The notation is that of § 2.

THEOREM. *Let G be a residually finite group with the presentation*

$$\langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, r_{m+2}, \dots \rangle$$

and let K be a field of characteristic 0. (We assume the $k_i > 1$.) Then there exists a finite quotient G' of G such that if G acts on $V = K[G']$ as above, then, letting $d = |G'|$, $\sigma = \sum_{i=1}^m 1/|A_i|$ and $\tau = \sum_{i=1}^m 1/k_i$, we have

(a) $\dim H^1(G, V) \leq (n - \sigma - 1)d + 1 \leq (n - \tau - 1)d + 1$

(b) if equality holds throughout (a), then $G = [A_1] * \dots * [A_m]$, $|A_j| = k_j$, $j = 1, \dots, m$ and $|A_j| = \infty$, $j = m + 1, \dots, n$.

(c) if the set of defining relations is finite, say

$$\mathcal{R} = \{a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q}\},$$

then $\dim H^1(G, V) \geq (n - \sigma - q - 1)d + 1$.

REMARK. It will be clear from the proof that if a finite number of presentations of G are given, G' can be chosen so that (a) through (c) are simultaneously true for all the given presentations.

Proof. By RF2, choose G' so that the image of A_i in G' has order $|A_i|$, $i = 1, \dots, m$. In the notation of (**), § 1,

$$\dim \text{Im} (1 - T_i) = d - \dim \text{Ker} (1 - T_i) = d(1 - 1/|A_i|)$$

by the remarks preceding the theorem. Hence, by (**)

$$(1) \quad \dim Z^1(G, V) = (n - \sigma)d - \text{rank} (D').$$

Now the map of V onto the space $B^1(G, V)$ of coboundaries given by $v \mapsto f_v$, where $f_v(g) = gv - v$ for all $g \in G$, has kernel V^σ , so

$\dim B^i = d - 1$. Combining this with (1) yields the first inequality in (a). The second inequality is clear.

To prove (b), note first that if $|A_i| < k_i$ for some i , then the second inequality in (a) is strict. Therefore it with suffice to show that if $G \neq [A_1] * \dots * [A_n]$ or if some A_j with $j > m$ has finite order, then G' can be chosen so that (in addition to the preservation of orders $|A_i|$, $i = 1, \dots, m$) we have $D' \neq 0$. For then, (1) implies that the first inequality in (a) is strict.

Consider the following elements of $K[G]$:

$$\begin{aligned} &\varphi(\partial r_i / \partial a_j), \quad i > m, j > m \\ &\varphi(\partial r_i / \partial a_j (1 - a_j)), \quad i > m, j \leq m. \end{aligned}$$

One of these must be nonzero since otherwise, by the Corollary of §2, $G = [A_1] * \dots * [A_n]$ and $|A_i| = \infty$, $i > m$, contrary to hypothesis. Therefore by RF1 there exists a finite quotient G' such that the image in $K[G']$ of this nonzero element is also nonzero. One easily sees then that $D' \neq 0$. This proves (b).

Given the hypothesis of (c), we have $\text{rank } D' \leq qd$. The conclusion then follows from (1) above.

COROLLARY 1. *Let G be a residually finite group with two presentations*

$$\begin{aligned} G &= \langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q} \rangle \\ &= \langle b_1, \dots, b_N; b_1^{h_1}, \dots, b_M^{h_M}, s_{M+1}, \dots \rangle. \end{aligned}$$

Then

$$n - \sum_{i=1}^m 1/|A_i| - q \leq N - \sum_{j=1}^M 1/h_j,$$

and if equality holds, then $G = [B_1] * \dots * [B_N]$, $|B_j| = h_j$ for $j = 1, \dots, M$ and $|B_j| = \infty$ for $j > M$. (B_k is the image in G of the free generator b_k .)

Proof. Apply part (c) to the first presentation and parts (a) and (b) to the second. (See the remark preceding the proof of the theorem.)

Note that Corollary 1 implies for residually finite groups the well-known result [4, Cor. 5.14.2] that if a group G with n generators and q defining relations can be generated by $n - q$ elements, then G is free of rank $n - q$.

COROLLARY 2. *Let*

$$G = \langle a_1, \dots, a_n; a_1^{k_1}, \dots, a_m^{k_m}, r_{m+1}, \dots, r_{m+q} \rangle.$$

Then G is infinite if

$$\sum_{i=1}^m 1/|A_i| \leq n - q - 1.$$

Proof. If $|G| = d < \infty$, we may take $G' = G$ in the proof of the Theorem. But then $dH^1(G, V) = 0$ [1, Chap. XII, Prop. 2.5] so $H^1(G, V) = 0$ since K has characteristic zero. The conclusion now follows from part (c) of the Theorem.

Finally, we apply Corollary 2 to a classical case. Let

$$G = \langle a_1, \dots, a_m; a_1^{k_1}, \dots, a_m^{k_m}, a_1 \dots a_m \rangle.$$

From geometric considerations (e.g. [7, p. 28, Satz 8]) one knows that the group is infinite if $\sum 1/k_i \leq m - 2$. In 1902, Miller [6] gave an algebraic proof of this fact for the case $m = 3$, but the argument involves consideration of many cases.

In [3] Fox shows that if k_1, k_2, k_3 are integers > 1 , then there exist permutations A and B of orders k_1 and k_2 , resp., such that AB has order k_3 . It follows easily from this that $k_i = |A_i|$ in the above group (assuming $m > 2$). Hence Corollary 2, together with this result, yields an algebraic proof that G is infinite when $\sum 1/k_i \leq m - 2$.

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