GENERALIZED QUASICENTER AND HYPERQUASICENTER
OF A FINITE GROUP

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The notion of quasicentral element is generalized to \( p \)-quasicentral element and the \( p \)-quasicenter and the \( p \)-hyperquasicenter are defined. It is shown that the \( p \)-quasicenter is \( p \)-supersolvable and the \( p \)-hyperquasicenter is \( p \)-solvable.

The quasicenter \( Q(G) \) of a group \( G \) is the subgroup of \( G \) generated by all quasicentral elements of \( G \), where an element \( x \) of \( G \) is called a quasicentral element (QC-element) when the cyclic subgroup \( \langle x \rangle \) generated by \( x \) satisfies \( \langle x \rangle <\!\!\!\langle y \rangle = \langle y \rangle <\!\!\!\langle x \rangle \) for all elements \( y \) of \( G \). The hyperquasicenter \( Q^*(G) \) of a group \( G \) is the terminal member of the upper quasicentral series \( 1 = Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n = Q_{n+1} = Q^*(G) \) of \( G \), where \( Q_{i+1} \) is defined by \( Q_{i+1} / Q_i = Q(G / Q_i) \). Mukherjee has shown [3, 4] that the quasicenter of a group is nilpotent and the hyperquasicenter is the largest supersolvably immersed subgroup of a group. The proofs of these structure theorems rely on the fact that the powers of QC-elements are again QC-elements.

In this paper we generalize the notion of a quasicentral element in a way which allows the results about the quasicenter and the hyperquasicenter [3, 4] to be extended. All groups mentioned are assumed to be finite.

For a given group \( G \) and a fixed prime \( p \), the definition of QC-element might suggest that an element \( x \) of \( G \) be called a \( p \)-quasicentral element provided \( \langle x \rangle <\!\!\!\langle y \rangle = \langle y \rangle <\!\!\!\langle x \rangle \) holds for all \( p \)-elements \( y \) of \( G \). An apparent difficulty with this definition is that the powers of \( p \)-quasicentral elements need not again be \( p \)-quasicentral elements. For example, consider the group of order 18 defined by \( G = \langle a, b, x \mid a^3 = b^3 = 1 = x^2, [a, b] = 1 = [a, x], [b, x] = a \rangle \). A simple calculation shows that \( ax \) is 3-quasicentral while \( x = (ax)^3 \) is not 3-quasicentral—otherwise \( \langle x \rangle <\!\!\!\langle b \rangle = \langle b \rangle <\!\!\!\langle x \rangle \) shall imply that \( x \) normalizes \( \langle b \rangle \), which is not the case however. Because of this example we choose to generalize the notion of a QC-element as follows.

**DEFINITION 1.** Let \( G \) be a given group and \( p \) a fixed prime. Suppose \( x \) is an element of \( G \) and let the order of \( x \) be written as \( |x| = p^r m \) where \( (p, m) = 1 \). Then \( x \) is called a \( p \)-quasicentral \( (p\text{-QC}) \) element of \( G \) provided \( \langle x^m \rangle <\!\!\!\langle y \rangle = \langle y \rangle <\!\!\!\langle x^m \rangle \) and \( \langle x^{p^r} \rangle <\!\!\!\langle y \rangle = \langle y \rangle <\!\!\!\langle x^{p^r} \rangle \) hold for all \( p \)-elements \( y \) of \( G \). (It should be noted that every element of a \( p' \)-group is \( p \)-\( \text{QC} \).)
THEOREM 1. If \( x \) is a p-QC element of a group \( G \) and \( k \) is a fixed integer, then \( x^k \) is also a p-QC element of \( G \).

Proof. Suppose \( |x| = p^b \cdot m \) where \((p, m) = 1\). Since \( |x^m| = p^b \), \( |x^p| = m \) and \( x^p \) commutes with \( x^m \), \( \langle x \rangle = \langle x^p \rangle \langle x^m \rangle = \langle x^m \rangle \langle x^p \rangle \).

If \( |x^p| = p^c \cdot n \) where \((p, c) = 1\), then \( (x^p)^{p^c} \) is a \( p' \)-element of \( \langle x \rangle \) and \( (x^p)^{p^c} \) is a \( p \)-element of \( \langle x \rangle \). It follows that \( (x^p)^{p^c} \) is some power of \( x^m \) and \( (x^p)^n \) is some power of \( x^m \). To show that \( x^k \) is a p-QC element of \( G \), it will suffice to show that \( \langle (x^m)^i \rangle \langle y \rangle = \langle y \rangle \langle (x^m)^i \rangle \) and \( \langle (x^p)^i \rangle \langle y \rangle = \langle y \rangle \langle (x^p)^i \rangle \) hold for all integers \( i \) and all \( p \)-elements \( y \) of \( G \).

Let \( y \) be any \( p \)-element in \( G \). Since \( x \) is a p-QC element of \( G \), \( \langle x^m \rangle \langle y \rangle = \langle y \rangle \langle x^m \rangle \). Therefore \( \langle x^m \rangle \langle y \rangle \) is some subgroup \( H \) of \( G \) whose order divides \( |x^m| |y| \). Since \( x^m \) is then a p-QC element of the \( p \)-group \( H \), \( x^m \) is a QC-element of \( H \). It follows \([3, 4]\) that every power of \( x^m \) is a QC-element of \( H \). In particular, \( \langle (x^m)^i \rangle \langle y \rangle = \langle y \rangle \langle (x^m)^i \rangle \) holds for every integer \( i \).

Now proceed by induction on the order of \( G \) to show that \( \langle (x^p)^i \rangle \langle y \rangle = \langle y \rangle \langle (x^p)^i \rangle \) holds for every integer \( i \) and every \( p \)-element \( y \). Let \( y \) be a fixed \( p \)-element of \( G \) of order \( p^r \). If \( \langle x^p \rangle \langle y \rangle = \langle y \rangle \langle x^p \rangle \) is a proper subgroup of \( G \), induction completes the argument. Assume therefore that \( G = \langle x^p \rangle \langle y \rangle = \langle y \rangle \langle (x^p)^i \rangle \). Then \( G \) is a supersolvable group (Theorem 13.3.1, \([5]\)).

Let \( \pi \) denote the set of prime divisors of \( |x^p| = m \) which are larger than \( p \). Since \( G \) is supersolvable with order \( |G| = p^r m \), \( G \) has a normal Hall \( \pi \)-subgroup \( K \). Distinguish two cases.

Case 1. \( \pi \) is empty. Then \( p \) is the largest prime dividing \( |G| \). Since \( \langle y \rangle \) is a Sylow \( p \)-subgroup of \( G \), \( \langle y \rangle \) must be normal in \( G \). Clearly \( \langle (x^p)^i \rangle \langle y \rangle = \langle y \rangle \langle (x^p)^i \rangle \) holds for all integers \( i \) in this case.

Case 2. \( \pi \) is nonempty. Let \( s \) and \( t \) denote integers such that \( x_1 = (x^p)^s \) is a \( \pi \)-element, \( x_2 = (x^p)^t \) is a \( \pi' \)-element and \( x^p = x_1 x_2 = x_2 x_1 \) (Theorem 4, \([2]\), p. 23). Then \( \langle x_i \rangle \) is a Hall \( \pi \)-subgroup of \( G \). Since \( G \) is supersolvable, \( \langle x_i \rangle \leq G \). It follows that \( \langle x_i^i \rangle \langle y \rangle = \langle y \rangle \langle x_i^i \rangle \) holds for every integer \( i \). Since \( \langle (x^p)^i \rangle = \langle x_i^i \rangle \langle x_i^i \rangle \) for all integers \( i \), the argument will be complete if we show \( \langle x_i^i \rangle \langle y \rangle = \langle y \rangle \langle x_i^i \rangle \) holds for all \( i \). Since \( \langle x_i \rangle \) is a normal Hall \( \pi \)-subgroup of \( G \), the Schur-Zassenhaus theorem shows that \( G \) possesses a \( \pi \)-complement \( R \). Since \( y \) is a \( \pi' \)-element of \( G \), we may choose \( R \) so that \( y \in R \). Then \( \langle y \rangle \) is a Sylow \( p \)-subgroup of \( R \). Since \( R \) is supersolvable and \( p \) is the
largest prime dividing $|R|$. We now use the fact that $x_i$ is a $\pi'$-element. Since $R$ is a Hall $\pi'$-subgroup of the solvable group $G$, some conjugate $x_i^\gamma$ of $x_i$ lies in $R$. It now follows from $G = \langle x_i^u \rangle \langle y \rangle$ that $x_i \in R$, since every element $g$ in $G$ can be written as $(x_i^u)^i y^v$ for some integers $u, v$. Therefore $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x_i^\gamma \rangle$ holds for every integer $i$. This completes the proof of the theorem.

**Lemma 1.** Let $\theta$ be a homomorphism from a group $G$ onto a group $\bar{G}$. If $x$ is a $p$-QC element of $G$, the image $x^\theta$ of $x$ is a $p$-QC element of $\bar{G}$.

**Proof.** Let $|x| = p^b m$ where $(p, m) = 1$ and let $|x^\theta| = p^c n$ where $(p, n) = 1$. It follows that $\langle x \rangle = \langle x_i^u \rangle \langle x_i^m \rangle$ and $\langle x^\theta \rangle = \langle (x_i^u)^p \rangle \langle (x_i^m)^n \rangle$. Now $\langle x^\theta \rangle = \langle x \rangle^\theta$ implies $\langle x_i^u \rangle^p = \langle (x_i^m)^n \rangle$ and $\langle x_i^m \rangle^n = \langle (x_i^u)^p \rangle$.

Let $\bar{u}$ be any $p$-element of $\bar{G}$. Then there is a $p$-element $y$ of $G$ with $y^p = \bar{u}$. Since $x$ is a $p$-QC element of $G$, $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x_i^u \rangle$ and $\langle x_i^m \rangle \langle y \rangle = \langle y \rangle \langle x_i^m \rangle$. This shows $\langle x_i^u \rangle^p \langle y \rangle^p = \langle y \rangle^p \langle x_i^u \rangle^p$ and $\langle x_i^m \rangle^n \langle y \rangle^p = \langle y \rangle^p \langle x_i^m \rangle^n$. Now $\langle y \rangle^p = \langle y \rangle^p = \langle \bar{u} \rangle$ implies $\langle (x_i^u)^p \rangle \langle \bar{u} \rangle = \langle \bar{u} \rangle \langle (x_i^m)^n \rangle$ and $\langle (x_i^m)^n \rangle \langle \bar{u} \rangle = \langle \bar{u} \rangle \langle (x_i^u)^p \rangle$. The proof of the lemma is therefore complete.

**Definition 2.** Let $G$ be a given group and $p$ a fixed prime. The $p$-quasicenter $Q_p(G)$ is the subgroup of $G$ generated by all $p$-QC elements of $G$.

We mention a few simple consequences of the definition of the $p$-quasicenter. For any group $G$ and any prime $p$, the quasicenter of $G$ is contained in the $p$-quasicenter of $G$. The $p$-quasicenter of a group is always a characteristic subgroup of the group. It should be noted that if a prime $p$ does not divide the order of a group $G$ then $Q_p(G) = G$.

**Theorem 2.** For any group $G$ and every prime $p$, the $p$-quasicenter $Q_p(G)$ is $p$-supersolvable.

**Proof.** First we notice that $Q_p(G) = G$ is $p$-supersolvable if $p$ does not divide $|G|$. Consequently we assume that $p$ divides $|G|$. The proof is by induction on $|G|$. It suffices to show that $G$ contains a nontrivial normal subgroup $N$ of order $p$ or of order prime to $p$. For, by induction, $Q_p(G/N)$ is then $p$-supersolvable. Since Lemma 1 shows $Q_p(G)N/N \subseteq Q_p(G/N)$ it will follow that $Q_p(G)$ is $p$-supersolvable. (This is because of the fact that normal subgroups of $p$-supersolvable groups are $p$-supersolvable and $N$ being of order $p$ or prime to $p$, the $p$-supersolvability of $Q_p(G)N/N$ implies $Q_p(G)N$ is $p$-supersolvable.) Since $Q_p(Q_p(G)) = Q_p(G)$, induction lets us assume that $Q_p(G) = G$. Thus $G$ is generated by $p$-QC
First we show that $G$ contains a proper normal subgroup. Distinguish two cases.

**Case 1:** Some $x_i$ has order divisible by $p$. Assume $p$ divides the order of $x_i$. Then there is an integer $d$ such that $|x_i^d| = p$. Since $x_i^d$ is a $p$-QC element of $G$, $\langle x_i^d \rangle$ permutes with each Sylow $p$-subgroup of $G$. Therefore $\langle x_i^d \rangle$ lies in the maximum normal $p$-subgroup $O_p(G)$ of $G$. Therefore $O_p(G)$ is a proper normal subgroup of $G$ or $O_p(G) = G$ and $G$ is a $p$-group. If $G$ is a $p$-group, the theorem is trivially true.

**Case 2:** No $x_i$ has order divisible by $p$. Then $x_1, x_2, \ldots, x_n$ are $p$-QC elements of $G$ with $p'$-orders. Since $|G|$ is divisible by $p$, $G$ must contain nonidentity $p$-elements. Let $T$ denote the subgroup of $G$ generated by all the $p$-elements of $G$. Since $T \leq G$, we can assume $T = G$. Therefore $G$ contains nonidentity $p$-elements $y_1, y_2, \ldots, y_m$ with $\langle y_1, y_2, \ldots, y_m \rangle = G$. Let $q$ be the largest prime dividing the product $|x_1| \cdot |x_2| \cdot \ldots \cdot |x_n|$. First suppose $p > q$. Since $x_i$ is a $p$-QC element and $y_i$ is a $p$-element, $\langle x_i, y_i \rangle = \langle x_i \rangle \langle y_i \rangle$ holds for all $i = 1, 2, \ldots, n$. It follows (theorem 13.3.1, [5]) that $\langle x_i, y_i \rangle$ is supersolvable of order $|x_i| \cdot |y_i|$ for $i = 1, 2, \ldots, n$. Since $x_i$ is a $p'$-element and $p > q$, $\langle y_i \rangle$ is a normal Sylow $p$-subgroup of each group $\langle x_i, y_i \rangle$. Then $x_1, x_2, \ldots, x_n$ normalize $\langle y_1 \rangle$ and $\langle y_i \rangle$ is a normal subgroup of $G = \langle x_1, x_2, \ldots, x_n \rangle$. Now suppose $p < q$ and let $|x_i|$ be divisible by $q$. Let $s$ be an integer such that $\langle x_i^s \rangle$ is a Sylow $q$-subgroup of $\langle x_i \rangle$. Since $\langle x_i, y_i \rangle = \langle y_i \rangle \langle x_i \rangle$, $\langle x_i \rangle$ is a supersolvable group and $q$ is the largest prime dividing $|y_i| \cdot |x_i|$, $y_i$ normalizes $\langle x_i^s \rangle$ for $j = 1, 2, \ldots, m$. Therefore $\langle x_i^s \rangle \leq G = \langle y_1, y_2, \ldots, y_m \rangle$. This shows that in every case $G$ contains a proper normal subgroup $M$. If $M$ has order prime to $p$, we are finished. Assume now that $M$ is a minimal normal subgroup of $G$ and $p$ divides $|M|$. We will show that $|M| = p$.

Since $Q_p(G) = G$, $G$ is generated by $p$-QC elements $x_1, x_2, \ldots, x_n$ of $G$. For each $i$, $1 \leq i \leq n$, $\langle x_i \rangle = \langle v_i \rangle \langle v_2 \rangle \cdots \langle v_d \rangle$ where $v_1, v_2, \ldots, v_d$ are powers of $x_i$, $v_i$ is a $p$-element, and $v_2, v_3, \ldots, v_d$ are $p'$-elements of prime power orders. Since powers of $p$-QC elements are also $p$-QC elements, it follows that $G$ can be written as $G = \langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k \rangle$ where each $a_i$ is a $p$-QC $p$-element of $G$ and each $b_j$ is a $p$-QC $p'$-element of $G$ having prime power order.

Let $P$ denote the subgroup of $G$ generated by all $p$-QC $p$-elements of $G$. Clearly $P$ is a characteristic $p$-subgroup of $G$ with $\langle a_1, a_2, \ldots, a_n \rangle \leq P$. Since $M$ is a minimal normal subgroup of $G$, $P \cap M = 1$ or $P \cap M = M$. First suppose that $P \cap M = 1$. Then $[P, M] \leq P \cap M = 1$ and $P$ centralizes $M$. Let $w \in M$ with $|w| = p$. Clearly $a_i$ normalizes $\langle w \rangle$ for $i = 1, 2, \ldots, h$. Since each $b_j$ is a $p$-QC element of $G$, $\langle b_j \rangle \langle w \rangle = \langle w \rangle \langle b_j \rangle$ holds for $j = 1, 2, \ldots, k$. It follows that each group $\langle b_j \rangle \langle w \rangle$ is supersolvable of order $|b_j| \cdot |w|$. 


Since \(|b_j|\) is a power of a prime other than \(p\), \(\langle b_j \rangle\) is a Sylow subgroup of \(\langle b_j \rangle \langle w \rangle\). Hence \(b_j\) normalizes \(\langle w \rangle\) or \(w\) normalizes \(\langle b_j \rangle\) for each \(j = 1, 2, \ldots, k\). Since \(\langle b_j \rangle \cap M = 1\) implies \(b_j\) normalizes \(\langle w \rangle\), \(\langle w \rangle \trianglelefteq G\) unless \(\langle b_j \rangle \cap M \neq 1\) for some \(j\). Assume that \(\langle b_d \rangle \cap M \neq 1\) for some integer \(d, 1 \leq d \leq k\). This implies that some prime different from \(p\) divides the order of \(M\). Since every power of \(b_d\) is a \(p\)-QC element of \(G\), \(Q_p(M) \neq 1\). From the minimality of \(M\) it follows that \(Q_p(M) = M\), since \(Q_p(M)\) is characteristic in \(M\) and \(M\) is normal in \(G\). Induction applied to \(M\) then shows that \(M\) is \(p\)-supersolvable.

If \(N\) is a minimal normal subgroup of \(M\) then \(|N|\) is either \(p\) or is prime to \(p\). Then \(T = \langle N^g | g \in G \rangle\) is a normal subgroup of \(G\) contained in \(M\) and \(T = N^{g_1} \cdots N^{g_t}\) where \(g_1, \ldots, g_t\) are elements of \(G\). But \(M\) being minimal normal in \(G\) it follows that \(T = M\). Therefore \(M\) is either a \(p\)-group or a \(p'\)-group, since \(T\) is so. But \(p\) divides the order of \(M\) and therefore \(M\) must be a \(p\)-group. This however contradicts the assumption that \(\langle b_d \rangle \cap M \neq 1\). Thus \(\langle w \rangle \trianglelefteq G\). Since \(\langle w \rangle \subseteq M, M = \langle w \rangle\) and \(M\) has order \(|w| = p\). Now suppose \(P \cap M = M\). Then \(M\) is a normal subgroup of the \(p\)-group \(P\) and \(M \cap Z(P) \neq 1\). Let \(z\) be a nonidentity element of \(M \cap Z(P)\) with \(|z| = p\). Since \(z \in Z(P)\), surely \(\langle a_1, a_2, \ldots, a_k \rangle\) normalizes \(\langle z \rangle\). On the other hand, \(M\) being a \(p\)-group it is evident that \(\langle b_j \rangle \cap M = 1\) for each \(j = 1, 2, \ldots, k\). As before, \(\langle z \rangle \trianglelefteq G\) unless \(\langle b_j \rangle \cap M \neq 1\) for some \(j\). Therefore \(\langle z \rangle \trianglelefteq G\). Since \(1 \neq \langle z \rangle \subseteq M\), the minimality of \(M\) shows \(M = \langle z \rangle\). Therefore \(M\) has order \(|z| = p\) and the proof is complete.

Since the quasicenter of a group is nilpotent it is natural to ask if the \(p\)-quasicenter of a group must be \(p\)-nilpotent. We give an example to show that this need not be the case. Let \(S_3\) denote the symmetric group of degree 3. The \(3\)-quasicenter of \(S_3\) is \(S_3\) itself. Clearly \(Q_3(S_3) = S_3\) is not \(3\)-nilpotent.

**Definition 3.** Let \(G\) be a given group and \(p\) a fixed prime. The upper \(p\)-quasicentral series \(1 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = H_{n+1}\) of \(G\) is the characteristic series where \(H_{i+1}\) is defined by \(H_{i+1}/H_i = Q_p(G/H_i)\). The number of distinct nontrivial terms in the upper \(p\)-quasicentral series of \(G\) is called the \(p\)-quasicentral length of \(G\). The terminal member of the upper \(p\)-quasicentral series of \(G\) is called the \(p\)-hyperquasicenter of \(G\). We denote this characteristic subgroup of \(G\) by \(Q_p^*(G)\).

**Theorem 3.** In any group \(G\), the \(p\)-hyperquasicenter \(Q_p^*(G)\) is the intersection of all normal subgroups \(N\) with \(Q_p(G/N) = N/N\).

**Proof.** Let \(S = \bigcap \{N \mid N \trianglelefteq G \text{ and } Q_p(G/N) = N/N\}\). Clearly
We now show that $Q^*(G)$ is included in every normal subgroup $N$ for which $Q_p(G/N) = N/N$. Let $1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_m = Q^*_p(G)$ be the upper $p$-quasicentral series of $G$. Trivially $H_0 \subseteq N$. Assume that $H_i \subseteq N$ and $H_{i+1} \nsubseteq N$. Then for some $p$-QC element $yH_i$ of $G/H_i$, $y \notin N$. This implies that under the natural homomorphism of $G/H_i$ to $G/N$, the $p$-QC element $yH_i$ is mapped onto the $p$-QC element $yN$ of $G/N$. Therefore $Q_p(G/N)$ is nontrivial, a contradiction. Hence $H_{i+1} \subseteq N$ and $Q^*_p(G) \subseteq N$ follows by induction.

We shall now investigate the structure of the $p$-hyperquasicenter $Q^*_p(G)$.

**Lemma 2.** Let $G$ be a group and $p$ a fixed prime. If $N \subseteq G$ and $N \subseteq Q^*_p(G)$ then $Q^*_p(G/N) = Q^*_p(G)/N$.

**Proof.** Let $1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k = Q^*_p(G)$ be the upper $p$-quasicentral series of $G$ and let $N/N = L_0 \subset L_1 \subset \cdots \subset L_k = Q^*_p(G/N)$ be the upper $p$-quasicentral series of $G/N$. By Lemma 1, $H_iN/N = Q_p(G)N/N \subseteq Q^*_p(G/N) = L_i/N$. Thus $H_i \subseteq L_i \subseteq L_k$. Now assume $H_i \subseteq L_k$ and deduce $H_{i+1} \subseteq L_k$. Since $H_i \subseteq L_k$, $G/L_k$ is a homomorphic image of $G/H_i$. Let $\theta$ be the natural homomorphism described by $(xH_i)^\theta = xL_k$. Then Lemma 1 shows that $(Q_p(G/H_i))^\theta \subseteq Q_p(G/L_k) = L_k/L_k$. Since $Q_p(G/H_i) = H_{i+1}/H_i$, $(Q_p(G/H_i))^\theta = H_{i+1}/L_k/L_k \subseteq L_k/L_k$. Therefore $H_{i+1} \subseteq L_k$ and by induction $H_n \subseteq L_k$. We complete, the proof by showing $L_i \subseteq H_n = Q^*_p(G)$ for each $i = 1, 2, \cdots, k$. By hypothesis $L_0 = N \subseteq Q_p(G)$. Now assume $L_i \subseteq Q_p(G)$ and deduce $L_{i+1} \subseteq Q^*_p(G)$. Since $L_i \subseteq Q^*_p(G), G/Q^*_p(G)$ is a homomorphic image of $G/L_i$. The argument above can be repeated to obtain $L_{i+1} \subseteq Q^*_p(G)$.

**Theorem 4.** For any group $G$ and any prime $p$, $Q^*_p(G)$ is $p$-solvable.

**Proof.** If $Q^*_p(G) = Q_p(G)$, $Q^*_p(G)$ is $p$-supersolvable and the theorem is proved. Assume now that $Q_p(G) \subseteq Q^*_p(G)$. Let $N$ denote any minimal normal subgroup of $Q_p(G)$. Since $Q_p(G)$ is $p$-supersolvable, $N$ has $p'$-order or $|N| = p$. Set $S = \langle N^g | g \in G \rangle$. Since $N \subseteq Q_p(G) \subseteq G$, $N^g \subseteq Q_p(G)$ for each $g \in G$. It follows that $S$ has order prime to $p$ or order a power of $p$. Since $S \subseteq G$ and $S \subseteq Q_p(G) \subseteq Q^*_p(G)$ induction shows that $Q^*_p(G/S) = Q^*_p(G)/S$ is $p$-solvable. Therefore $Q^*_p(G)$ is $p$-solvable.

It is possible to characterize the $p$-hyperquasicenter in terms of the normal subgroups included in it. We begin with the following definition.

**Definition 4.** Let $G$ be a group and $p$ a fixed prime. A normal subgroup $N$ of $G$ is called $p$-hyperquasicentral ($p$-HQ) if $N/M \cap
$Q_p^*(G/M) \neq M/M$ holds for each normal subgroup $M$ of $G$ which is properly contained in $N$.

The lemmas proved next will be useful for the proof of Theorem 5.

**Lemma 3.** Let $G$ be any group and $p$ a fixed prime. If $N \unlhd G$ then $Q_p^*(G)N/N \subseteq Q_p^*(G/N)$.

**Proof.** Let $1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = Q_p^*(G)$ be the upper $p$-quasicentral series of $G$. By Lemma 1, $H_iN/N = Q_p^*(G)N/N \subseteq Q_p^*(G/N) = L/N$. Thus $H_iN \subseteq L$. Now assume $H_iN \subseteq L$ and deduce that $H_{i+1}N \subseteq L$. Since $H_i \subseteq H_iN$, $G/H_iN$ is a homomorphic image of $G/H_i$. Let $\phi$ be the natural homomorphism of $G/H_i$ onto $G/H_iN$ described by $(xH_i)^\phi = xH_iN$. Then Lemma 1 shows $(Q_p^*(G/H_i))^\phi \subseteq Q_p^*(G/H_iN)$. Since $Q_p^*(G/H_i) = H_{i+1}/H_i$, $H_{i+1}N/H_iN = (H_{i+1}/H_i)^\phi \subseteq Q_p^*(G/H_iN)$. Next let $\theta$ be the natural homomorphism of $G/H_iN$ onto $G/L$ given by $(xH_iN)^\theta = xL$. By Lemma 1, $(Q_p^*(G/H_iN))^\theta \subseteq Q_p^*(G/L) = L/L$. Since $H_{i+1}N/H_iN \subseteq Q_p^*(G/H_iN)$, $(H_{i+1}N/H_iN)^\theta = H_{i+1}N/L \subseteq L/L$. Therefore $H_{i+1}N \subseteq L$ and the assertion follows.

**Lemma 4.** If any two groups $G_1$ and $G_2$ are isomorphic under a map $\theta$ then $(Q_p^*(G_1))^\theta = Q_p^*(G_2)$.

**Lemma 5.** For any group $G$ and any prime $p$, the product of $p$-HQ subgroups of $G$ is a $p$-HQ subgroup of $G$.

**Proof.** It suffices to show that for any $p$-HQ subgroups $A$ and $B$ of $G$, the product $AB$ is a $p$-HQ subgroup of $G$. Let $M$ be any normal subgroup of $G$ with $M \subseteq AB$. If $M \subseteq A$ or $M \subseteq B$ then $AB/M \cap Q_p^*(G/M) \neq M/M$. Now suppose $M$ is not a proper subgroup of either $A$ or $B$. Since $A \cap M = A$ and $B \cap M = B$ together imply $AB \subseteq M$, we may assume $AB \subseteq M \subseteq A$. Since $A$ is $p$-HQ, $A/R \cap Q_p^*(G/R) \neq R/R$. Let $yR$ be any nonidentity element of $A/R \cap Q_p^*(G/R)$. Then $y \in A$ and $y \in R$ show $y \in M$. Since $M/R \subseteq G/R$, Lemma 3 shows $Q_p^*(G/R) \cdot M/R/M/R \subseteq Q_p^*(G/R/M/R)$. It now follows from the isomorphism of $G/R/M/R$ and $G/M$ that $yM$ is a nonidentity element of $Q_p^*(G/M)$. Therefore $AB/M \cap Q_p^*(G/M) \neq M/M$ and the assertion is proved.

**Theorem 5.** For any group $G$ and any prime $p$, $Q_p^*(G)$ is the product of all $p$-HQ subgroups of $G$.

**Proof.** Let $S$ denote the product of all $p$-HQ subgroups of $G$. From Lemma 2 and the definition of $p$-HQ subgroup it is easily seen that $Q_p^*(G)$ is a $p$-HQ subgroup of $G$. Therefore $Q_p^*(G) \subseteq S$. 
Assume for the sake of contradiction that $Q_p^*(G) \subseteq S$. Since $S$ is a $p$-HQ subgroup of $G$ (Lemma 5) $S/Q_p^*(G) \cap Q_p^*(G/Q_p^*(G)) \neq Q_p^*(G)/Q_p^*(G)$. Since $Q_p^*(G/Q_p^*(G)) = Q_p^*(G)/Q_p^*(G)$, this is the desired contradiction.

It should be remarked that for a set of primes $\pi$, $\pi$-quasicentrality can be defined in a manner analogous to $p$-quasicentrality. The $p$-quasicenter and $p$-hyperquasicenter can be extended in the natural way to obtain the notions of $\pi$-quasicenter and $\pi$-hyperquasicenter. It is easily checked that the results about the $p$-quasicenter and the $p$-hyperquasicenter of a group remain valid when $p$ is replaced by $\pi$.

REFERENCES


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