

Pacific Journal of Mathematics

UNIVERSAL COSIMPLE ISOLS

ERIK MAURICE ELLENTUCK

UNIVERSAL COSIMPLE ISOLS

ERIK ELLENTUCK

Our results follow from a single priority scheme which we give in detail. They are (i) if an arbitrary $n + 1$ -ary relation over the nonnegative integers determines an n -ary function, then its canonical extension to the isols determines a function on the cosimple isols if and only if the function determined on the integers is an almost recursive combinatorial function, and (ii) every countable partially ordered set can be embedded in (a) the cosimple isols, and even more (b) the cosimple regressive isols. The remaining material generalizes and extends these results.

1. Independence. Let ω = the nonnegative integers, P = the set of all subsets of ω , and Q = the set of all finite subsets of ω . Use $X^k A$ for the k -fold direct power of A . The results of this section all follow from the priority method of [11]. Let $f: \omega \rightarrow X^2 Q$ be a sequence of pairs $f(s) = (\alpha_s, \beta_s)$ where $\alpha_s \cap \beta_s = \emptyset$, and let $g: \omega \rightarrow \omega$. The requirement $R_k = \{(\alpha_s, \beta_s) \mid g(s) = k\}$. $\xi \in P$ is said to meet the requirement R_k if $\alpha \subseteq \xi, \xi \cap \beta = \emptyset$ for some $(\alpha, \beta) \in R_k$. With every pair (f, g) as above we associate the priority sequence $\xi: \omega \rightarrow Q$ constructed in stages. Stage $s = 0, \xi_0 = \emptyset$ and for stage $s > 0, \xi_s = \xi_{s-1}$ if (1), (2), or (3) below is true. Otherwise $\xi_s = \xi_{s-1} \cup \alpha_s$.

(1) $\xi_{s-1} \cap \beta_s \neq \emptyset$.

(2) there is an $r < s, r > 0, g(r) < g(s)$ such that

$$\alpha_r \not\subseteq \xi_{r-1}, \alpha_r \subseteq \xi_r, \xi_{s-1} \cap \beta_r = \emptyset \text{ and } \alpha_s \cap \beta_r \neq \emptyset.$$

(3) there is an $r < s, r > 0, g(r) = g(s)$ such that

$$\alpha_r \not\subseteq \xi_{r-1}, \alpha_r \subseteq \xi_r \text{ and } \xi_{s-1} \cap \beta_r = \emptyset.$$

The requirement R_k is met at stage s if $s > 0, g(s) = k$ and $\alpha_s \not\subseteq \xi_{s-1}$ but $\alpha_s \subseteq \xi_s$. The requirement R_k is injured at stage s if for some $r < s, R_k$ was met at stage $r, \xi_{s-1} \cap \beta_r = \emptyset$ but $\xi_s \cap \beta_r \neq \emptyset$. The basic combinatorial content of the priority method is summarized in the fundamental

LEMMA 1. (Sacks [11]). For each k the set of stages s where R_k is either met or injured is finite.

Proof. First we need two facts, (4) and (5) below. They are

(4) if R_k is injured at stage s then for some $j < k, R_j$ is met at stage s .

This follows because if R_k is injured at stage s then there is an $r < s$ such that R_k was met at stage r , $\xi_{s-1} \cap \beta_r = \emptyset$ but $\xi_s \cap \beta_r \neq \emptyset$. Hence (1), (2), and (3) fail at stage s and $r > 0$, $g(r) = k$, $\alpha_r \not\subseteq \xi_{r-1}$ but $\alpha_r \subseteq \xi_r$. If $g(r) < g(s)$ then (2) is true at stage s and if $g(r) = g(s)$ then (3) is true at stage s . Hence $g(s) < g(r) = k$. The second fact that we need is

(5) if R_k is met at stages r and s where $r < s$ then for some u , $r < u < s$ and R_k is injured at stage u .

This follows because if R_k is met at stage r then (1) fails at stage r , so $\xi_{r-1} \cap \beta_r = \emptyset$, and $\alpha_r \not\subseteq \xi_{r-1}$ but $\alpha_r \subseteq \xi_r$. Since $\alpha_r \cap \beta_r = \emptyset$ we have $\xi_r \cap \beta_r = \emptyset$ and since R_k is met at stage s , (3) must fail at stage s . But $r > 0$ and $g(r) = g(s)$ which implies $\xi_{s-1} \cap \beta_r \neq \emptyset$. Let u be the least integer $r < u < s$ such that $\xi_u \cap \beta_r \neq \emptyset$. Clearly R_k is injured at stage u .

We prove our lemma by induction. Suppose that for each $j < k$ the set of stages s where R_j is met or injured is finite. Choose $u < \omega$ such that for no $j < k$ and $s > u$ is R_j met or injured at stage s . Then for $s > u$, R_k cannot be injured at stage s because otherwise by (4) R_j would be met at stage s for some $j < k$. Also for at most one $s > u$ is R_k met at stage s because otherwise by (5) R_k would be injured at some stage $r > u$.

With every pair (f, g) as above we associate the *priority set* $\xi = \bigcup_{s < \omega} \xi_s$, where ξ_s is the priority sequence associated with (f, g) . First note that if a requirement R_k is met at stage s and subsequently never injured then ξ meets R_k . We lift recursive notions from ω to Q via the canonical enumeration of finite sets and note that ξ_s is recursive in the pair (f, g) which implies that ξ is r.e. in the pair (f, g) .

Let $j: X^2\omega \rightarrow \omega$ be the usual recursive one-to-one onto pairing function with first and second inverses k, l respectively. Let $\omega^n = \{j(x, n) \mid x < \omega\}$ and $\alpha^n = \alpha \cap \omega^n$. $\varphi_n(\gamma)$ is a partial recursive function which enumerates with index n all partial recursive functions whose range and domain are contained in Q . Let $(n_s, \gamma_s, \delta_s)$ be a total recursive enumeration of $\{(n, \gamma, \delta) \mid \varphi_n(\gamma) = \delta\}$. We define (f, g) as follows. $f(2s+1) = (\alpha_{2s+1}, \beta_{2s+1})$ where $\alpha_{2s+1} = \delta_s - \gamma_s$, $\beta_{2s+1} = \gamma_s$, and $g(2s+1) = 2n_s + 1$. $f(2s) = (\alpha_{2s}, \beta_{2s})$ where $\alpha_{2s}, \beta_{2s}, \bigcup_{r < 2s} (\alpha_r \cup \beta_r)$ are pairwise disjoint, both α_{2s}, β_{2s} are subsets of $\omega^{k(k(s))}$ each containing $l(k(s)) + 1$ elements, and $g(2s) = 2k(s)$. Let $\xi: \omega \rightarrow Q$ be the priority sequence associated with (f, g) and let $\xi = \bigcup_{s < \omega} \xi_s$ be the priority set associated with (f, g) . Throughout the remainder of this section ξ_s, ξ will keep these meanings. Let $\eta = \omega - \xi$.

LEMMA 2. η^m is infinite for each $m < \omega$.

Proof. Suppose η^m contained n elements. Let $k = 2j(m, n)$ and let $u < \omega$ be such that for no $s > u$ and $j \leq k$ is R_j either met or injured at stage s . Choose $s > u$ such that $g(s) = k$. Since $s > u$, $\xi_s = \xi_{s-1}$ and hence (1), (2), or (3) hold at stage s . By construction of f , both (1) and (2) fail at stage s , so (3) must hold. Hence there is an $r < s$, $r > 0$, $g(r) = g(s)$ such that $\alpha_r \not\subseteq \xi_{r-1}$, $\alpha_r \subseteq \xi_r$, and $\xi_{s-1} \cap \beta_r = \emptyset$. β_r contains $n + 1$ elements of ω^m , and for $t \geq s$, $\xi_t \cap \beta_r = \emptyset$ because otherwise R_k would be injured at stage t . Hence $\beta_r \subseteq \eta$.

Let M be the class of partial recursive functions φ whose range and domain are both included in Q , which are increasing, i.e., if $\alpha \in \text{domain}(\varphi)$ then $\alpha \subseteq \varphi(\alpha)$, and which are monotone, i.e., if $\alpha, \beta \in \text{domain}(\varphi)$ and $\alpha \subseteq \beta$ then $\varphi(\alpha) \subseteq \varphi(\beta)$.

LEMMA 3. For each $\varphi \in M$ there is a $\gamma \in Q$, $\gamma \subseteq \eta$ such that for all $\alpha \in Q$, $\gamma \subseteq \alpha \subseteq \eta$ either $\alpha \notin \text{domain}(\varphi)$ or $\varphi(\alpha) = \alpha$ or $\varphi(\alpha) \cap \xi \neq \emptyset$.

Proof. Let φ have index n in the enumeration of partial recursive functions above and let $k = 2n + 1$. Choose $u < \omega$ such that for no $s > u$ and $j \leq k$ is R_j either met or injured at stage s . Choose $\gamma \in Q$ such that $\eta \cap \bigcup_{r \leq u} \beta_r \not\subseteq \gamma \subseteq \eta$. Suppose there were an $\alpha \in Q$, $\gamma \subseteq \alpha \subseteq \eta$ such that $\alpha \in \text{domain}(\varphi)$ and $\alpha \not\subseteq \varphi(\alpha)$. Then for some s , $\alpha_s = \varphi(\alpha) - \alpha$, $\beta_s = \alpha$, and $g(s) = k$. $s > u$ for otherwise $\alpha \not\subseteq \gamma$. Hence $\xi_s = \xi_{s-1}$ so that (1), (2), or (3) must hold at stage s . Since $\xi_{s-1} \subseteq \xi$ and $\alpha \subseteq \eta$, (1) fails at stage s . If there were an $r < s$, $r > 0$, $g(r) = j < k$ such that R_j is met at stage r and $\xi_{s-1} \cap \beta_r = \emptyset$ then $\beta_r \subseteq \eta$ since otherwise R_j would be injured at some stage $t > u$. Hence $\beta_r \subseteq \alpha$ and so (2) fails at stage s . Hence (3) holds at stage s and there is an $r < s$, $r > 0$, $g(r) = g(s) = k$ such that $\alpha_r \not\subseteq \xi_{r-1}$, $\alpha_r \subseteq \xi_r$ and $\xi_{s-1} \cap \beta_r = \emptyset$. $\beta_r \subseteq \eta$ for otherwise R_k would be injured at some stage $t > u$. $\emptyset \neq \alpha_r = \varphi(\beta_r) - \beta_r \subseteq \xi$ and by monotonicity we have $\varphi(\alpha) \cap \xi \neq \emptyset$.

Let \mathcal{A} = the isols and for $\alpha \in P$ let $\text{Req}(\alpha)$ = the recursive equivalence type of α . \mathcal{A}_z (the cosimple isols) = the set of those isols which can be represented as $\text{Req}(\alpha)$ where $\omega - \alpha$ is r.e. and $\mathcal{A}_z^\infty = \mathcal{A}_z - \omega$, i.e., the infinite cosimple isols. For the rest of this section we fix the following notation $S = \{y_m \mid m < \omega\}$ where $y_m = \text{Req}(\eta^m)$ for $m < \omega$.

THEOREM 1. S is an infinite subset of \mathcal{A}_z^∞ .

Proof. First we show that η is immune. η is infinite by Lemma 2. Suppose that η contained an infinite r.e. subset β which we assume is enumerated by a total recursive function $b(n)$. Define a recursive function $\varphi: Q \rightarrow Q$ by letting $\varphi(\alpha) = \alpha \cup \{b\}$ where b is the first element

in the enumeration $b(n)$ which does not occur in α . Since $\varphi \in M$ there is a $\gamma \in Q$, $\gamma \subseteq \eta$ such that for all $\alpha \in Q$, $\gamma \subseteq \alpha \subseteq \eta$ either $\alpha \notin \text{domain}(\varphi)$ or $\varphi(\alpha) = \alpha$ or $\varphi(\alpha) \cap \xi \neq \emptyset$. Since all three of these possibilities are ruled out by our construction, η is immune. ξ is r.e. since it is r.e. in (f, g) , the latter clearly being recursive. Each ξ^m is r.e. since it is the intersection of two r.e. sets and each η^m is immune since it is an infinite subset of an immune set. Thus $S \subseteq A_z^\infty$. Next we show that if $m < n < \omega$ then $y_m \neq y_n$. Otherwise there would be a one-one partial recursive function p which maps η^m onto η^n . Define a recursive function $\varphi: Q \rightarrow Q$ by letting $\varphi(\alpha) = \alpha \cup p(\alpha^m)$. Since $\varphi \in M$ there is a $\gamma \in Q$, $\gamma \subseteq \eta$ which satisfies the conclusion of Lemma 3. By our construction $\varphi(\alpha) = \alpha$ for each $\alpha \in Q$, $\gamma \subseteq \alpha \subseteq \eta$. Choose $x \in \eta^m - p^{-1}(\gamma)$ so that $p(x) \in \varphi(\gamma \cup \{x\}) = \gamma \cup \{x\}$. Since $p(x) \notin \gamma$ we have $p(x) = x$ so that $\eta^m \cap \eta^n \neq \emptyset$, a contradiction.

A set A of isols is called *strongly recursively independent* if for each $n > 0$, $R \subseteq X^{n+1}\omega$ satisfying $(\forall v_0, \dots, v_{n-1} \in \omega)(\exists! v_n \in \omega)(v_0, \dots, v_n) \in R$, and distinct elements $x_0, \dots, x_{n-1} \in A$, if there is an $x_n \in A$ such that $(x_0, \dots, x_n) \in R_A$ then R is the graph of an eventually recursive combinatorial function.

THEOREM 2. *S is strongly recursively independent.*

Proof. Let $n, R, x_0, \dots, x_{n-1}$ be as in the hypothesis of the above definition. Using Theorem 3.1 of [10] $((R \times S)_A = R_A \times S_A$, and $\omega_A = A$) there is no loss of generality in assuming that $x_m = y_m$ for $m < n$. Suppose there is a $z \in A$ such that $(y_0, \dots, y_{n-1}, z) \in R_A$. For $\alpha \in P$ let $\bar{\alpha} = (\alpha^0, \dots, \alpha^{n-1})$. Then there is a $\zeta \in z$ and a recursive R -frame F such that $(\bar{\eta}, \zeta) \in \mathcal{A}(F)$. If $\alpha = (\alpha_0, \dots, \alpha_{n-1})$, where possibly some of the α_i are themselves tuples, and $C_F(\alpha)$ is defined, define $C_F^\vee(\alpha_0, \dots, \alpha_{n-1}) = \beta_i$ where $C_F(\alpha) = (\beta_0, \dots, \beta_{n-1})$ and $\alpha_j \leq \beta_j$ for $j < n$ (here \vee occurs as a superscript to exactly one α_i and \leq denotes componentwise inclusion). Now define a partial recursive function φ whose range and domain are included in Q as follows. $\varphi(\alpha) = \alpha \cup \beta$ where $\bar{\beta} = C_F^\vee(\bar{\alpha}^\vee, \emptyset)$. Since $\varphi \in M$, Lemma 3 yields a $\gamma \in Q$, $\gamma \subseteq \eta$ such that for all $\alpha \in Q$, $\gamma \subseteq \alpha \subseteq \eta$, either $\alpha \notin \text{domain}(\varphi)$ or $\varphi(\alpha) = \alpha$ or $\varphi(\alpha) \cap \xi \neq \emptyset$. Since $(\bar{\eta}, \zeta) \in \mathcal{A}(F)$ only the second of these alternatives can occur, i.e., $\varphi(\alpha) = \alpha$. But this implies that $C_F(\alpha, \emptyset) = (\alpha, C_F^\vee(\alpha, \emptyset^\vee))$ for each $\alpha \in A = \{\alpha \in X^n Q \mid \bar{\gamma} \leq \alpha \leq \bar{\eta}\}$. If $\alpha \in A$ let $\psi(\alpha) = C_F^\vee(\alpha, \emptyset^\vee)$ so that $(\alpha, \psi(\alpha)) \in F$. Let \wedge denote componentwise intersection and observe that $(\alpha \wedge \alpha', \emptyset) \leq (\alpha, \psi(\alpha)) \wedge (\alpha', \psi(\alpha')) = (\alpha \wedge \alpha', \psi(\alpha) \cap \psi(\alpha'))$ so that $\psi(\alpha \wedge \alpha') \subseteq \psi(\alpha) \cap \psi(\alpha')$. But then our hypothesis that R is the graph of a function give (i) $\psi(\alpha \wedge \alpha') = \psi(\alpha) \cap \psi(\alpha')$ and (ii) $\alpha \sim \alpha'$ componentwise implies $\psi(\alpha) \sim \psi(\alpha')$ (here \sim denotes equal cardinality)

for every $\alpha, \alpha' \in A$. For $m < n$ let p_m be a one-to-one function mapping ω onto $\eta^m - \gamma^m$. No recursive condition is placed on the p_m 's. For $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in X^n\omega$ let $p(\alpha) = (p_0(\alpha_0), \dots, p_{n-1}(\alpha_{n-1}))$ and define a function $\varphi: X^nQ \rightarrow Q$ by $\varphi(\alpha) = \psi(\bar{\gamma} \vee p(\alpha))$ where \vee denotes component-wise union. ψ inherits properties (i) and (ii) from ψ and is therefore a combinatorial operator inducing a combinatorial function $r: X^n\omega \rightarrow \omega$ such that if γ^m has c_m elements then $(x_0 + c_0, \dots, x_{n-1} + c_{n-1}, r(x_0, \dots, x_{n-1})) \in R$ for every $(x_0, \dots, x_{n-1}) \in X^n\omega$. Thus R is the graph of an eventually combinatorial function. Let $B = \{(\alpha, \beta) \in (X^nQ) \times Q \mid \alpha \wedge \bar{\gamma} = \emptyset \text{ and } C_r(\alpha \vee \bar{\gamma}, \emptyset) = (\alpha \vee \bar{\gamma}, \beta)\}$. B is r.e. and if we let $|\cdot|$ denote componentwise cardinality, $\tilde{R} = \{x \in X^{n+1}\omega \mid (\exists \alpha)(x = |\alpha| \text{ and } \alpha \in B)\}$ is an r.e. set which is the graph of r . Thus R is the graph of an eventually recursive combinatorial function.

2. Applications.

COROLLARY 1. *Let $n > 0$ and $R \subseteq X^{n+1}\omega$ the graph of a function r . Then (i) r is eventually recursive combinatorial if and only if for each $x \in X^nA_\infty$ there is exactly one $y \in A$ such that $(x, y) \in R_A$, (ii) r is almost recursive combinatorial if and only if for each $x \in X^nA_\infty$ there is exactly one $y \in A$ such that $(x, y) \in R_A$. This solves the problem left open on the middle of page 247 of [7].*

A set A of isols is called *recursively independent* if for each $n > 0$, $R \subseteq X^n\omega$, and distinct elements $x_0, \dots, x_{n-1} \in A$, if $(x_0, \dots, x_{n-1}) \in R_A$ then there exists $\alpha \in Q$ such that $X^n(\omega - \alpha) \subseteq R$.

THEOREM 3. *If A is a strongly recursively independent set of isols, then A is recursively independent.*

Proof. For $R \subseteq X^n\omega$ let $\tilde{R} = \{(x, 0) \mid x \in R\} \cup \{(x, 1) \mid x \in X^n\omega - R\}$. \tilde{R} is the graph of the characteristic function r of R and satisfies $(\forall x)(x \in R \rightarrow (x, 0) \in \tilde{R})$ in ω and hence in A . Let x_0, \dots, x_{n-1} be distinct elements of A and suppose that $(x_0, \dots, x_{n-1}) \in R_A$. Then $(x_0, \dots, x_{n-1}, 0) \in \tilde{R}_A$ which implies that r is eventually combinatorial. Since eventually combinatorial functions are eventually monotone, r is eventually equal to 0 or eventually equal to 1. If r is eventually equal to 1 there is an $\alpha < \omega$ such that $(\forall x_0, \dots, x_{n-1})(x_0 + \alpha, \dots, x_{n-1} + \alpha) \notin R$ holds in ω and hence in A . This implies that at least one of the x_i 's in A is finite, in fact $< \alpha$. Clearly no member of a strongly recursively independent set can be finite so r must be eventually equal to 0.

COROLLARY 2. *S is recursively independent.*

A set A of isols is called *independent* if for each $n > 1$ and distinct elements $x_0, \dots, x_{n-1} \in A$, $x_0 \not\leq x_1 + \dots + x_{n-1}$.

THEOREM 4. *If A is a recursively independent set of isols then A is independent.*

Proof. Fix n and let $R = \{(x_0, \dots, x_{n-1}) \in X^n \omega \mid x_0 \leq x_1 + \dots + x_{n-1}\}$. Then $(\forall x_0, \dots, x_{n-1}, y)(x_0 + y = x_1 + \dots + x_{n-1} \rightarrow (x_0, \dots, x_{n-1}) \in R)$ holds in ω and hence in A . Thus if x_0, \dots, x_{n-1} are distinct elements of A and $x_0 \leq x_1 + \dots + x_{n-1}$ then $(x_0, \dots, x_{n-1}) \in R_A$, a contradiction.

COROLLARY 3. *S is independent.*

Let A be a set partially ordered by a relation R . We say that $\langle A, R \rangle$ can be *embedded* in Λ if there exists an order isomorphism $h: A \rightarrow \Lambda$ with respect to the usual ordering of Λ . The following theorem is of interest when compared with results in [5].

THEOREM 5. *Every countable partially ordered set can be embedded in the cosimple isols.*

Proof. By a result of [9] there is a recursive relation R which partially orders ω in such a way that every countable partial ordering can be embedded in $\langle \omega, R \rangle$. We prove our theorem by embedding $\langle \omega, R \rangle$ in Λ_s . Assume that R is reflexive and define $h(n) = \cup \{\gamma^m \mid mRn\}$. $h(n)$ is an immune set whose complement in ω is $\cup \{\omega^m \mid \sim mRn\} \cup (\xi \cap \cup \{\omega^m \mid mRn\})$ which clearly is an r.e. set. Thus $\text{Req}(h(n)) \in \Lambda_s$. If n_0Rn_1 then $h(n_0) \subseteq h(n_1)$ and $h(n_0), h(n_1) - h(n_0)$ are recursively separated by the recursive set $\cup \{\omega^m \mid mRn_0\}$. Thus $\text{Req}(h(n_0)) \leq \text{Req}(h(n_1))$ by the identity function. If $\sim mRn$ but $\text{Req}(h(m)) \leq \text{Req}(h(n))$ then for some one-one partial recursive function p , $\eta^m \subseteq \text{domain}(p)$ and p maps η^m into $\cup \{\gamma^j \mid j \neq m\}$. Define a partial recursive function φ whose range and domain are included in Q by $\varphi(\alpha) = \alpha \cup p(\alpha^m)$. Then $\varphi(\alpha) = \alpha$ for all $\alpha \in Q$, $\gamma \subseteq \alpha \subseteq \eta$ where γ is given by Lemma 3. If $x \in \eta^m - p^{-1}(\gamma)$ then $p(x) \in \varphi(\gamma \cup \{x\}) = \gamma \cup \{x\}$. Since $p(x) \notin \gamma$ we have $p(x) = x$ so $\eta^m \cap \cup \{\gamma^j \mid j \neq m\} \neq \emptyset$, a contradiction. Thus $\text{Req}(h(m)) \not\leq \text{Req}(h(n))$ and we see that $\text{Req}(h(n))$ is the required order isomorphism.

There are several ways to extend Theorem 5. If R' is a partial ordering of Λ say that $\langle A, R \rangle$ is *R' -embedded* in Λ if the mapping $h: A \rightarrow \Lambda$ is an $R - R'$ order isomorphism. For R' we are interested in $<_A$, the canonical extension to Λ of $R = \{(x, y) \in X^2 \omega \mid x < y\}$, and \leq_A ,

the canonical extension to A of $R = \{(x, y) \in X^2 \mid x \leq y\}$. Then we have

COROLLARY 4. *Every countable partially ordered set can be both $<_A$ and \leq_A embedded in A_z .*

Proof. Let R, h be as in the proof of Theorem 5. First we show that if mRn and $m \neq n$ then $\text{Req}(h(m)) <_A \text{Req}(h(n))$. Since mRn and $m \neq n$, $h(m) \subsetneq h(n)$. Let $x \in h(n) - h(m)$ and let $F = \{(\alpha, \beta) \in X^2 Q \mid x \notin \alpha \text{ and } \alpha \cup \{x\} \subseteq \beta\}$. F is a recursive $<$ -frame from which $(h(m), h(n))$ is attainable. Next we show that if $\sim mRn$ then $\sim \text{Req}(h(m)) \leq_A \text{Req}(h(n))$. For suppose that F is a recursive \leq -frame and $(h(m), h(n)) \in \mathcal{A}(F)$. Since $\sim mRn$, $\gamma^m \subseteq h(m)$ and $\gamma^m \cap h(n) = \emptyset$. Now define a partial recursive function φ whose range and domain are included in Q as follows. $\varphi(\alpha) = \alpha \cup \beta$ where $\beta = \beta_0 \cup \beta_1$ and $C_F(\alpha^m, \emptyset) = (\beta_0, \beta_1)$. Since $\varphi \in M$ Lemma 3 gives the required γ . If $\alpha \subseteq \gamma^m$ then $\varphi(\alpha \cup \gamma) = \alpha \cup \gamma$ which implies that if $C_F(\alpha^m \cup \gamma^m, \emptyset) = (\beta_0, \beta_1)$ then $\beta_1 \subseteq \gamma$. Since $\alpha \subseteq \gamma^m$ can be made as large as we please, F is not a \leq -frame. Hence $(h(m), h(n)) \notin \mathcal{A}(F)$. To complete our proof recall that for $u, v \in A$, $u < v \rightarrow u <_A v \rightarrow u \leq_A v$, $u \leq v \rightarrow u \leq_A v$, and $u = v \rightarrow u \leq_A v$, all implied by a meta-theorem.

Another way to extend Theorem 5 is to try to embed partially ordered sets into the cosimple regressive isols A_{rz} . Let τ be an infinite retraceable set enumerated in increasing order as $\{t_n \mid n < \omega\}$ where $t_n < t_{n+1}$ for $n < \omega$ and let p be a partial recursive special (cf. [3]) retracing function for τ . τ is called *T-retraceable* if for every partial recursive function h there is $u < \omega$ such that for all $n > u$ either $t_n \notin \text{domain}(h)$ or $h(t_n) < t_{n+1}$. By a result of [8] *T-retraceable* sets with r.e. complement exist and they are necessarily immune. Fix τ in the following discussion as *T-retraceable* with r.e. complement, p is a total recursive function (cf. P.5 of [3]) and $p^*(x) = (\mu n)(p^n(x) = p^{n+1}(x))$.

LEMMA 4. *If $\alpha, \beta \subseteq \tau$ are infinite recursively equivalent sets then $\alpha \cap \beta \neq \emptyset$.*

Proof. Let h be a partial recursive one-to-one function mapping α onto β . By *T-retraceability* there is a $u < \omega$ such that for all $n > u$, if $t_n \in \alpha$ then $h(t_n) \leq t_n$ (since $h(t_n) < t_{n+1}$ and $h(t_n) \in \beta \subseteq \tau$), and if $t_n \in \beta$ then $h^{-1}(t_n) \leq t_n$. Recast the second conclusion as $t_n \leq h(t_n)$ if $h(t_n) \in \beta$ and $p^*(h(t_n)) > u$. Choose $u' > u$ such that $p^*(h(t_n)) > u'$ for

$n > u'$. Thus if $n > u'$ and $t_n \in \alpha$ then $h(t_n) \leq t_n \leq h(t_n)$, i.e., $t_n \in \alpha \cap \beta$.

THEOREM 6. *Every countable partially ordered set can be embedded in the cosimple regressive isols.*

Proof. Let R be as in the proof of Theorem 5, $\sigma = \omega - \tau$, and for $m < \omega$ let $\mu^m = \{x < \omega \mid k(p^*(x)) = m\}$. For any set α let $\alpha^m = \alpha \cap \mu^m$. Our embedding is defined by $h(n) = \cup \{\tau^m \mid mRn\}$. To show that $h(n)$ is retraceable note that $h(n)$ is recursively separated from $\tau - h(n)$ by the recursive set $\cup \{\mu^m \mid mRn\}$ and then use P.5 of [2] to conclude that $h(n)$ is retraceable. $h(n)$ is an immune set whose complement in ω is $\cup \{\mu^m \mid \sim mRn\} \cup (\sigma \cap \cup \{\mu^m \mid mRn\})$ which is clearly r.e. Thus $\text{Req}(h(n)) \in A_{rz}$. If n_0Rn_1 then $h(n_0) \subseteq h(n_1)$ and $h(n_0), h(n_1) - h(n_0)$ are recursively separated by the recursive set $\cup \{\mu^m \mid mRn_0\}$. Therefore $\text{Req}(h(n_0)) \leq \text{Req}(h(n_1))$. If $\sim mRn$ but $\text{Req}(h(m)) \leq \text{Req}(h(n))$ then for some one-to-one partial recursive function q , $\tau^m \subseteq \text{domain}(q)$ and q maps τ^m into $\cup \{\tau^j \mid j \neq m\}$. Since this is ruled out by Lemma 4 $\text{Req}(h(n))$ is the required isomorphism.

Let $S_r = \{z_m \mid m < \omega\}$ where $z_m = \text{Req}(\tau^m)$ for $m < \omega$.

COROLLARY 5. *S_r is an infinite independent but not recursively independent subset of A_{rz} .*

Proof. We only show that S_r is not recursively independent. The rest is obvious from the proof of Theorem 6. If $n > m$ then $j(n, x) > j(m, x)$. Define a function q on μ^n by $q(x) = (\mu y)(\exists s \leq p^*(x))(y = p^s(x) \text{ and } p^*(y) = j(m, l(p^*(x))))$. q is readily seen to be a many one partial recursive function which maps τ^n one-to-one onto τ^m . Hence $\text{Req}(\tau^n) \leq^* \text{Req}(\tau^m)$ and therefore $\text{Req}(\tau^n) \leq_A \text{Req}(\tau^m)$ by Theorem 2.1 of [1] (which asserts that \leq^* agrees with \leq_A on A_r).

Corollary 5 explains why in Theorem 6 our method did not give \leq_A or $<_A$ embeddings. In [6] it is shown that $\text{Req}(\tau)$ where τ is T -retraceable is *universal*, i.e., $\{\text{Req}(\tau)\}$ is recursively independent. Call $x \in A$ *strongly universal* if for each $R \subseteq X^2\omega$ satisfying $(\forall v_0)(\exists! v_1)(v_0, v_1) \in R$, if there is a $y \in A$ such that $(x, y) \in R_A$ then R is the graph of an eventually recursive increasing function.

THEOREM 7. *$\text{Req}(\tau)$ is strongly universal.*

Proof. Let R be as in the hypothesis of the above definition. Assume there is an isolated $\zeta \subseteq \omega$ and recursive R -frame F such that (τ, ζ) is attainable from F . For $x < \omega$ let $s(x) = \{p^n(x) \mid n \leq$

$p^*(x)\}$ and let $\gamma = \{x < \omega \mid (s(x), \emptyset) \in F^*\}$. Clearly $s(x)$ is a recursive function and γ is an r.e. set. For $x \in \gamma$ let $h(x)$ = the maximum element of $C_F^*(s(x)', \emptyset)$. $h(x)$ is partial recursive and since $\tau \subseteq \gamma$, defined on all of τ and maps τ into τ . Hence there is a $u < \omega$ such that $h(t_n) \leq t_n$ for $n > u$. This implies that $C_F^*(s(x)', \emptyset) = s(x)$ for all $x \in \tau$ with $p^*(x) > u$, and consequently for such x we have $C_F(s(x), \emptyset) = (s(x), C_F^*(s(x), \emptyset')) \in F$. Clearly the set $\{(m, n) \mid (\exists x)(x \in \tau \text{ and } p^*(x) > u \text{ and } m = |s(x)| \text{ and } n = |C_F^*(s(x), \emptyset')|)\}$ is the graph of an increasing function contained in R and has a recursive extension $\{(m, n) \mid (\exists \alpha, \beta)(\alpha, \beta) \in F \text{ and } m = |\alpha| \text{ and } n = |\beta|)\}$ so R itself is the graph of an eventually recursive increasing function.

COROLLARY 6. $\text{Req}(\tau)$ is universal.

Proof. Just as in the proof of Theorem 3 using the fact that a bounded eventually increasing function is eventually constant. q.e.d.

Using a somewhat different priority method, we have been able to extend the results of this paper to the cosimple regressive isols. They are stated below without proof for comparison with Corollary 1. If $f: \omega \rightarrow \omega$ let $\Delta f(x) = f(x+1) - f(x)$. If $f: X^n \omega \rightarrow \omega$ and $i < n$ let Δ_i be just like Δ except that it is applied to the i th argument of f . Let Δ = the composition $\Delta_0 \cdots \Delta_{n-1}$. Then Δf is well defined and is consistent with the notation for the one variable case. f is called *recursive increasing* if f is recursive and $\Delta \hat{f}(x) \geq 0$ for all $x \in X^n \omega$, where $\hat{f}(x_0 + 1, \dots, x_{n-1} + 1) = f(x_0, \dots, x_{n-1})$ and $\hat{f}(x) = 0$ otherwise. *Eventually recursive increasing* and *almost recursive increasing* are canonically defined.

THEOREM 8. Let $n > 0$ and $R \subseteq X^{n+1} \omega$ the graph of a function r . Then r is almost recursive increasing if and only if for each $x \in X^n A_{r_z}$ there is a $y \in A$ such that $(x, y) \in R_A$.

A function $f: X^n \omega \rightarrow \omega$ is called *recursive regular* if there are unary recursive increasing functions g_i for $i < n$ such that $f(x_0, \dots, x_{n-1}) = \text{minimum}\{g_0(x_0), \dots, g_{n-1}(x_{n-1})\}$. *Eventually recursive regular* and *almost recursive regular* are canonically defined.

THEOREM 9. Let $n > 0$ and $R \subseteq X^{n+1} \omega$ the graph of a function r . Then r is almost recursive regular if and only if for each $x \in X^n A_{r_z}$ there is a $y \in A_r$ such that $(x, y) \in R_A$.

In both of these theorems the y in question, when it exists, is actually an element of A_z .

REFERENCES

1. J. Barback, *Two notes on recursive functions and regressive isols*, TAMS, **144** (1969), 77-94.
2. J. C. E. Dekker, *The minimum of two regressive isols*, Math. Zeitschr., **83** (1964), 345-366.
3. J. C. E. Dekker and J. R. Myhill, *Retraceable sets*, Canad. J. Math., **10** (1958), 357-373.
4. E. Ellentuck, *On the form of functions which preserve regressive isols*, submitted to Composito Math.
5. ———, *Uncountable suborderings of the isols*, submitted to Composito Math.
6. J. Gersting, *A rate of growth criterion for universality of regressive isols*, Pacific J. Math., **31** (1969), 669-677.
7. L. Hay, *The cosimple isols*, Ann. of Math., **83** (1966), 231-256.
8. T. G. McLaughlin, *Hereditarily retraceable isols*, Bull. Amer. Math. Soc., **73** (1967), 113-115.
9. A. Mostowski, *Über gewisse universelle Relationen*, Ann. Soc. Polon. Math., **17** (1938), 117-118.
10. A. Nerode, *Extensions to isols*, Ann. of Math., **73** (1961), 362-403.
11. G. E. Sacks, *Degrees of Unsolvability*, Ann. of Math. Studies 55 (1963), Princeton.

Received April 27, 1971. Research supported in part by NSF contract GP-11509.

RUTGERS UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California 94305

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY

University of Washington
Seattle, Washington 98105

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 42, No. 3

March, 1972

Catherine Bandle, <i>Extensions of an inequality by Pólya and Schiffer for vibrating membranes</i>	543
S. J. Bernau, <i>Topologies on structure spaces of lattice groups</i>	557
Woodrow Wilson Bledsoe and Charles Edward Wilks, <i>On Borel product measures</i>	569
Eggert Briem and Murali Rao, <i>Normpreserving extensions in subspaces of $C(X)$</i>	581
Alan Seymour Cover, <i>Generalized continuation</i>	589
Larry Jean Cummings, <i>Transformations of symmetric tensors</i>	603
Peter Michael Curran, <i>Cohomology of finitely presented groups</i>	615
James B. Derr and N. P. Mukherjee, <i>Generalized quasicenter and hyperquasicenter of a finite group</i>	621
Erik Maurice Ellentuck, <i>Universal cosimple isols</i>	629
Benny Dan Evans, <i>Boundary respecting maps of 3-manifolds</i>	639
David F. Fraser, <i>A probabilistic method for the rate of convergence to the Dirichlet problem</i>	657
Raymond Taylor Hoobler, <i>Cohomology in the finite topology and Brauer groups</i>	667
Louis Roberts Hunt, <i>Locally holomorphic sets and the Levi form</i>	681
B. T. Y. Kwee, <i>On absolute de la Vallée Poussin summability</i>	689
Gérard Lallement, <i>On nilpotency and residual finiteness in semigroups</i>	693
George Edward Lang, <i>Evaluation subgroups of factor spaces</i>	701
Andy R. Magid, <i>A separably closed ring with nonzero torsion pic</i>	711
Billy E. Rhoades, <i>Commutants of some Hausdorff matrices</i>	715
Maxwell Alexander Rosenlicht, <i>Canonical forms for local derivations</i>	721
Cedric Felix Schubert, <i>On a conjecture of L. B. Page</i>	733
Reinhard Schultz, <i>Composition constructions on diffeomorphisms of $S^p \times S^q$</i>	739
J. P. Singhal and H. M. (Hari Mohan) Srivastava, <i>A class of bilateral generating functions for certain classical polynomials</i>	755
Richard Alan Slocum, <i>Using brick partitionings to establish conditions which insure that a Peano continuum is a 2-cell, a 2-sphere or an annulus</i>	763
James F. Smith, <i>The p-classes of an H^*-algebra</i>	777
Jack Williamson, <i>Meromorphic functions with negative zeros and positive poles and a theorem of Teichmüller</i>	795
William Robin Zame, <i>Algebras of analytic functions in the plane</i>	811