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**BOUNDARY RESPECTING MAPS OF 3-MAINFOLDS**

BENNY DAN EVANS

## BOUNDARY RESPECTING MAPS OF 3-MANIFOLDS

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**This paper is about maps of compact 3-manifolds which map the boundary of the domain (possibly nonhomeomorphically) into the boundary of the range. F. Waldhausen has shown that such a map between compact, orientable, irreducible 3-manifolds with nonempty, incompressible boundary is homotopic to a homeomorphism if and only if the map induces an isomorphism at the fundamental group level. The main theorem of this paper states that the above theorem remains valid if the assumption of incompressible boundary is dropped.**

**A study of disk sums of bounded 3-manifolds will be required in order to prove the above-mentioned theorem. This investigation involves theorems about disk sums of bounded 3-manifolds analogous to the classical Kneser theorem for closed 3-manifolds.**

The reader may wish to consult [11] for a proof of Waldhausen's theorem mentioned above and [4] and [7] for variations of Waldhausen's theorem related to the theorems proved in this paper.

All spaces and maps in this paper are assumed to belong to the precise linear category, and each subspace that we shall discuss is taken to be piecewise linearly embedded. If  $A$  is a subcomplex of the simplicial complex  $X$ , we use the notation  $U(A, X)$  to denote a regular neighborhood of  $A$  in a second derived subdivision of  $X$ .

If  $X$  is a manifold, we use the notation  $\text{bd } X$  and  $\text{int } X$  to denote the boundary of  $X$  and the interior of  $X$  respectively.

A 3-manifold  $M$  is said to be *irreducible* if each 2-sphere in  $M$  is the boundary of some 3-cell in  $M$ .

A compact 2-manifold  $F$  embedded in a manifold  $M$  is *properly embedded* in  $M$  if  $F \cap \text{bd } M = \text{bd } F$ . A compact 2-manifold  $F$  properly embedded in a 3-manifold  $M$  is *incompressible* in  $M$  if for each disk  $D$  in  $M$  such that  $D \cap F = \text{bd } D$ , there exists a disk  $D'$  in  $F$  such that  $\text{bd } D = \text{bd } D'$ .

Let  $F$  denote a 2-manifold properly embedded in a 3-manifold  $M$ , and let  $J$  denote a loop in  $M$  that meets  $F$  transversely. We define the symbol  $[J, F]$  to be 0 if  $J$  meets  $F$  an even number of times, and  $[J, F] = 1$  if  $J$  meets  $F$  an odd number of times. Observe that if  $J^*$  is a loop in  $M$  that is homotopic to  $J$ , then  $[J, F] = [J^*, F]$ .

Let  $M$  and  $N$  denote compact, orientable  $n$ -manifolds, and let  $f: (M, \text{bd } M) \rightarrow (N, \text{bd } N)$  be a map. Having chosen generators  $\alpha$  and  $\beta$  for  $H_n(M, \text{bd } M)$  respectively, we say that  $f$  has *degree*  $k$  if and only if  $f_*(\alpha) = k\beta$ . In general we shall not be concerned with the sign of the degree of a map. We note for future reference that if  $f: (M, \text{bd } M) \rightarrow (N, \text{bd } N)$  has degree  $\pm 1$ , then  $f_*: \Pi_1(M) \rightarrow \Pi_1(N)$  is an epimorphism [2].

A compact 3-manifold  $M$  is a *cube-with-handles* if and only if  $M$  is homeomorphic with  $U(\Gamma, R^3)$  where  $\Gamma$  is some connected graph embedded in  $R^3$ .

If  $\alpha$  and  $\beta$  are loops in a manifold  $M$  based at a common point  $*$ , we use the notation  $\alpha \sim \beta$  to indicate that  $\alpha$  is homotopic to  $\beta$  relative to the base point  $*$ . Where there is no danger of confusion, we shall also use this notation to denote homotopic maps.

II. Disk sum decompositions. Kneser's theorem [10] for connected sums of closed 3-manifolds with fundamental group a free product is not in general true for disk sums of bounded 3-manifolds [5]. However as we shall shortly show, each bounded 3-manifold has a natural decomposition into disk sums determined by its fundamental group. This is made more precise in Theorem 2.4. I wish to thank William Jaco for collaborating with me on the proofs of the theorems of this section.

A group  $G$  is said to be *indecomposable* if  $G$  cannot be written as a nontrivial free product. Let  $G$  be a finitely generated group. If  $A_1, \dots, A_k$  are indecomposable groups which are not infinite cyclic, and if  $F$  is a free group such that  $A_1 * \dots * A_k * F$  is isomorphic with  $G$ , then  $A_1 * \dots * A_k * F$  is said to be a *free decomposition* for  $G$ . Each finitely generated group has a free decomposition which is unique up to isomorphism and order of factors. (see page 245 of [6].)

LEMMA 2.1. *Let  $M$  be a compact 3-manifold with nonempty boundary and suppose  $\pi_1(M) \approx G * Z$  for some group  $G$ . Then there exists a properly embedded disk  $D$  in  $M$  which does not separate  $M$ . Furthermore  $\pi_1(M - D) \approx G$ .*

*Proof.* Let  $K$  denote a  $C - W$  complex such that  $\pi_1(K) \approx G$  and  $\pi_i(K) = 0$  for  $i > 1$ . Let  $A$  denote a simple closed curve. Let  $K \vee A$  denote the space obtained by attaching a point of  $A$  to a point of  $K$ . Then by Van Kampen's theorem,  $\pi_1(K \vee A) \approx G * Z$ . Observe that the universal cover of  $K \vee A$  is contractible. Hence  $\pi_i(K \vee A) = 0$  for each  $i > 1$ .

Since  $\pi_1(K \vee A) \approx \pi_1(M)$  and  $\pi_i(K \vee A) = 0$  for  $i > 1$ , we can construct a map  $f: M \rightarrow K \vee A$  such that  $f_*: \pi_1(M) \rightarrow \pi_1(K \vee A)$  is an isomorphism. Let  $x$  denote a point on  $A - K$ . We may assume that the map  $f$  is transverse with respect to  $x$ . Then  $f^{-1}(x) = F_1 \cup \dots \cup F_r$  consists of a collection of mutually exclusive, 2-sided, properly embedded 2-manifolds in  $M$ . We wish to change  $f$  by a homotopy in such a way that each component of  $f^{-1}(x)$  becomes either a disk or a 2-sphere.

Let  $i: F_j \rightarrow M$  denote the inclusion map. Then  $f_* \circ i_*: \pi_1(F_j) \rightarrow \pi_1(K \vee A)$  is the trivial map. Thus  $i_*: \pi_1(F_j) \rightarrow \pi_1(M)$  is also the zero homomorphism. If some  $F_i$  is not simply connected, then there is a loop  $l$  on  $F_i$  such that  $l$  is not contractible in  $F_i$ , but  $l$  does contract in  $M$ . By the usual surgery on a map that represents a contraction of  $l$ , we are able to find a loop  $l^*$  on  $F_j$  (possibly  $i \neq j$ ) such that  $l^*$  does not contract in  $F_j$ , but  $l^*$  has a contraction in  $M$  that meets  $\bigcup_{i=1}^r F_i$  only in  $l^*$ . Then since each  $F_i$  is 2-sided, we may apply the loop theorem [9] to  $\text{cl}(M - \bigcup_{i=1}^r U(F_i, M))$  to obtain a disk  $D$  in  $M$  such that  $D \cap \bigcup_{i=1}^r F_i = D \cap F_j = \text{bd } D$ , and  $\text{bd } D$  is not contractible in  $F_j$ .

We consider  $U(D, M)$  as a product  $D' \times [0, 1]$  with  $D \subset D' \times 1/2$ . Observe that  $F_j \cap U(D, M)$  is an annulus  $R$ .  $R$  separates  $D' \times [0, 1]$  into two components, one of which we identify with  $D \times [0, 1]$ . Define  $g: M \rightarrow K \vee A$  as follows. Set  $g \equiv f$  on  $\text{cl}(M - D' \times [0, 1] \cup \text{bd } D \times [0, 1/4] \cup \text{bd } D \times [3/4, 1])$ . Set  $g(D \times 1/4 \cup D \times 3/4) = x$ . Since  $\pi_2(K \vee A) = 0$ , we can extend  $g$  over  $D \times [0, 1/4]$ ,  $D \times [3/4, 1]$ , and  $\text{cl}(D' \times [0, 1] - (D \times [0, 1/4] \cup D \times [3/4, 1]))$  in such a way that  $g^{-1}(x) \cap D' \times [0, 1] = \text{bd } D \times [0, 1/4] \cup D \times 1/4 \cup D \times 3/4 \cup \text{bd } D \times [3/4, 1]$ . Observe that  $g$  differs from  $f$  only on the interior of the 3-cell  $D' \times [0, 1]$ . Hence since  $\pi_i(K \vee A) = 0$  for  $i > 1$ , it follows that  $g$  is homotopic to  $f$ .

By deformations of the above type, we can construct a map  $h: M \rightarrow K \vee A$  such that  $h_*: \pi_1(M) \rightarrow \pi_1(K \vee A)$  is an isomorphism and each component of  $h^{-1}(x)$  is a simply connected 2-manifold.

Let  $l$  denote a simple closed curve in  $M$  such that  $h(l)$  is homotopic to the simple closed curve  $A$ . Then  $[h(l), x] = [A, x] = 1$ . Hence  $1 = [l, h^{-1}(x)] \equiv \sum_{i=1}^r [l, F_i] \pmod{2}$ . It follows that for some  $i$ ,  $[l, F_i] = 1$ . (In particular,  $h^{-1}(x) \neq \emptyset$ .) If  $F_i$  is a disk, we set  $D = F_i$ . If  $F_i$  is a 2-sphere, let  $P$  be an arc joining  $F_i$  to  $\text{bd } M$ . We then construct  $D$  from  $F_i$  by first removing the interior of the disk  $U(P, M) \cap F_i$  from  $F_i$  and then adjoining the obvious annulus in  $\text{bd } U(P, M)$ .

Then by Van Kampen's theorem,  $\pi_1(M) \approx \pi_1(\text{cl}(M - U(D, M)))_* Z$ . It follows from uniqueness of a free decomposition that  $G \approx \pi_1(M - D)$ .

Lemma 2.1 provides an easy proof of the following well known theorem.

**COROLLARY 2.2:** *Let  $M$  be a compact, orientable, irreducible 3-manifold. Then  $\pi_1(M)$  is free if and only if  $M$  is a cube-with-handles.*

The proof of Corollary 2.2 is a straightfoward application of Lemma 2.1 and is omitted.

**LEMMA 2.3:** *Let  $M$  be a compact 3-manifold with nonempty boundary such that  $\pi_1(M) \approx A_1 * A_2$  with  $A_1 \neq 1 \neq A_2$ . If each properly embedded disk in  $M$  separates  $M$ , then there exists a disk  $D$  in  $M$  that separates  $M$  into two nonsimply connected components.*

*Proof.* Construct  $C - W$  complexes  $K_1, K_2$  such that  $\pi_1(K_i) \approx A_i, \pi_j(K_i) = 0, i = 1, 2, j > 1$ . Let  $L$  be obtained from  $K_1$  and  $K_2$  by adjoining one end point of an arc  $A$  to  $K_1$  and attaching the other end point to  $K_2$ . Let  $x$  denote the midpoint of  $A$ . By Van Kampen's theorem,  $\pi_1(L) \approx \pi_1(K_1) * \pi_1(K_2)$ . Since  $L$  is aspherical, there exists a map  $f: M \rightarrow L$  such that  $f_*: \pi_1(M) \rightarrow \pi_1(L)$  is an isomorphism.

Using the same techniques as in Lemma 2.1, we can change  $f$  by a series of homotopies so that  $f^{-1}(x) = E_1, \dots, E_r$  consists of properly embedded disks and 2-spheres. Since each properly embedded disk (and hence each 2-sphere) in  $M$  separates  $M$ , the 2-manifold  $E_1$  separates  $M$  into two components whose closures we denote by  $M_1$  and  $M_2$ . Suppose  $M_1$  is simply connected. Let  $M_2^* = \text{cl}(M_2 - U(E_1, M)), M_1^* = M_1 \cup U(E_1, M)$ . Define  $g: M \rightarrow L$  as follows. Let  $g|_{M_2^*} = f|_{M_2^*}$ . We may assume that  $g$  is a level preserving map on  $U(E_1, M)$  so that  $g(M_1^* \cap M_2^*)$  is a single point  $p$  on  $A$ . Thus we may let  $g(M_1^*) = p$ .

Observe first of all that  $E_1$  does not occur as a component of  $g^{-1}(x)$ . Secondly since  $i_*: \pi_1(M_2^*) \rightarrow \pi_1(M)$  is an isomorphism, it follows from the following commutative diagram that  $g_*: \pi_1(M) \rightarrow \pi_1(L)$  is an isomorphism.

$$\begin{array}{ccc} \pi_1(M_2^*) & \xrightarrow{i_*} & \pi_1(M) \\ \downarrow i_* & & \downarrow g_* \\ \pi_1(M) & \xrightarrow{f_*} & \pi_1(L) . \end{array}$$

Thus if each 2-manifold  $E_i$  ( $1 \leq i \leq r$ ) separates  $M$  into two components, one of which is simply connected, then altering  $f$  as above we can construct a map  $h: M \rightarrow L$  such that  $h_*: \pi_1(M) \rightarrow \pi_1(L)$  is an isomorphism and  $h^{-1}(x) = \emptyset$ . But then  $h(M) \subset K_1$ .

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{h_*} & \pi_1(K_1) \\ & \searrow h_* & \downarrow i_* \\ & & \pi_1(L) \end{array}$$

Then from the above diagram we see that  $i_*: \pi_1(K_i) \rightarrow \pi_1(L)$  is an epimorphism. It follows that  $\pi_1(K_2) = 1$  contrary to the hypothesis of the lemma. Hence some component  $E_i$  separates  $M$  into two components neither of which is simply connected. If  $E_i$  is a 2-sphere, we change it to a properly embedded disk by the same technique used at the end of Lemma 2.1.

**THEOREM 2.4.** *Let  $M$  be a compact 3-manifold with nonempty boundary. Suppose  $\pi_1(M) \approx A_1^* \cdots * A_k^* F_r$  is a free decomposition for  $\pi_1(M)$  where  $r$  is the rank of  $F_r$ . Then there exist disks  $D_1, \dots, D_{k+r-1}$  separating  $M$  into  $k$  components  $M_1, \dots, M_k$ . And there exists an isomorphism  $\psi: \pi_1(M) \rightarrow A_1^* \cdots * A_k^* F_r$  such that  $\psi i_* \pi_1(M_i) = A_j$  ( $1 \leq i \leq k$ ).*

*Outline of proof.* Let  $G = \pi_1(M)$ . If  $G$  has a free factor of  $Z$ , we write  $G = G_1 * Z$ . We apply Lemma 2.1 to find a disk  $D_1$  such that  $M^1 = \text{cl}(M - U(D_1, M))$  and  $\pi_1(M^1) \approx G_1$ . If  $G_1$  has a free factor of  $Z$ , we apply Lemma 2.1 again to construct  $M^2$ . Since  $M$  is compact,  $G$  is finitely generated. Hence Grushko's theorem (page 191 of [6]) assures us that this process must eventually yield a compact 3-manifold  $M^r$  such that each properly embedded disk in  $M^r$  separates  $M^r$ . If  $\pi_1(M^r)$  is decomposable, we apply Lemma 2.3 to find a properly embedded disk that separates  $M^r$  into two nonsimply connected components  $M_1, M_2$ . We apply Lemma 2.3 to each of the components  $M_1, M_2$ . Again by Grushko's theorem this process must terminate.

We have a collection  $D_1, \dots, D_p$  of properly embedded disks in  $M$  separating into components  $M_1, \dots, M_q$  such that  $\pi_1(M_i)$  is neither decomposable nor infinite cyclic for each  $i(1 \leq i \leq q)$ . By Van Kampen's theorem,  $\pi_1(N) \approx \pi_1(M_1) * \cdots * \pi_1(M_q) * F$  where  $F$  is a free group. The theorem now follows from uniqueness of the free decomposition for  $\pi_1(M)$ .

### III. Boundary respecting maps.

**THEOREM 3.1.** *Let  $M$  and  $N$  denote compact, orientable, irreducible 3-manifolds with nonempty boundaries. Let  $f(M, \text{bd } M) \rightarrow (N, \text{bd } N)$  be a degree 1 map such that  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is a monomorphism. Then there exists a homotopy  $h_i: (M, \text{bd } M) \rightarrow (N, \text{bd } N)$  such that  $h_0 = f$  and  $h_1$  is a homeomorphism.*

*Proof.* Since  $f$  is a degree 1 map, it follows that  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an epimorphism and hence is an isomorphism. By Lemma 2 of [7] it suffices to show that  $f$  is homotopic to map  $g: (M, \text{bd } M) \rightarrow (N, \text{bd } N)$  such that  $g|_{\text{bd } M}: \text{bd } M \rightarrow \text{bd } N$  is a homeomorphism. We

proceed to show this.

It follows from the sphere theorem [3] that  $M$  and  $N$  are aspherical manifolds. Thus,  $f: M \rightarrow N$  is a homotopy equivalence. Since  $f$  is a degree 1 map, we have for each  $q$  that the following diagram commutes where  $\lambda$  is the usual Lefschetz duality isomorphism.

$$\begin{array}{ccc} H_q(M, \text{bd } M) & \xrightarrow{\lambda} & H^{n-q}(M) \\ \downarrow f_* & & \uparrow f^* \\ H_q(N, \text{bd } N) & \xrightarrow{\lambda} & H^{n-q}(N) . \end{array}$$

Hence  $f_*: H_q(M, \text{bd } M) \rightarrow H_q(N, \text{bd } N)$  is an isomorphism for each  $q$ . Thus we may apply the five lemma to the following diagram to conclude that  $f_*: H_q(\text{bd } M) \rightarrow H_q(\text{bd } N)$  is an isomorphism for each  $q$ .

$$\begin{array}{ccccccccc} H_{q+1}(M) & \rightarrow & H_{q+1}(M, \text{bd } M) & \rightarrow & H_q(\text{bd } M) & \rightarrow & H_q(M) & \rightarrow & H_q(M, \text{bd } M) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_{q+1}(N) & \rightarrow & H_{q+1}(N, \text{bd } N) & \rightarrow & H_q(\text{bd } N) & \rightarrow & H_q(N) & \rightarrow & H_q(N, \text{bd } N) . \end{array}$$

In particular since each boundary component of  $M$  and  $N$  is orientable and  $f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$  is an isomorphism, it follows that each boundary component of  $N$  has a unique boundary component of  $M$  in its preimage.

Let  $X_1, \dots, X_r, Y_1, \dots, Y_r$  denote the boundary components of  $M$  and  $N$  respectively; assume notation has been chosen so that  $f(X_i) \subset Y_i$  ( $1 \leq i \leq r$ ). Put  $f_i = f|_{X_i}$ . It follows from the above remarks that  $f_i$  is a degree 1 map for each  $i$  ( $1 \leq i \leq r$ ). Thus  $f_{i*}: \pi_1(X_i) \rightarrow \pi_1(Y_i)$  is an epimorphism. Furthermore

$$f_*: H_1(X_1) \oplus \dots \oplus H_1(X_r) \rightarrow H_1(Y_1) \oplus \dots \oplus H_1(Y_r)$$

is an isomorphism with  $f_*(H_1(X_i)) \subset H_1(Y_i)$  for each  $i$  ( $1 \leq i \leq r$ ). It follows that  $f_{i*}: H_1(X_i) \rightarrow H_1(Y_i)$  is an isomorphism for each  $i$  ( $1 \leq i \leq r$ ). Since the rank of the first homology of a closed orientable 2-manifold determines the genus of the 2-manifold, it follows that  $X_i$  is homeomorphic to  $Y_i$  for each  $i$  ( $1 \leq i \leq r$ ). Then since fundamental group of a closed 2-manifold is necessarily Hopfian [1], it follows that  $f_{i*}: \pi_1(X_i) \rightarrow \pi_1(Y_i)$  is an isomorphism for each  $i$  ( $1 \leq i \leq r$ ). Neilsen's theorem [8] now assures us that  $f_i: X_i \rightarrow Y_i$  is in fact homotopic to a homeomorphism.

Thus we are able to construct a map  $g: (M, \text{bd } M) \rightarrow (N, \text{bd } N)$  such that  $g$  is homotopic to  $f$ , and  $g|_{\text{bd } M}: \text{bd } M \rightarrow \text{bd } N$  is a homeomorphism. This completes the proof of Theorem 3.1.

**THEOREM 3.2.** *Let  $M$  and  $N$  denote compact, orientable, irredu-*

cible 3-manifolds with nonempty boundary. Let  $f: (M, \text{bd } M) \rightarrow (N, \text{bd } N)$  be a map such that  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an isomorphism. Then  $f$  is homotopic to a homeomorphism. Furthermore, if  $M$  is not the product of a compact 2-manifold with the unit interval, the homotopy above can be chosen so that it maps  $\text{bd } M$  into  $\text{bd } N$ .

*Proof.* Suppose  $N$  is a cube-with-handles. Then since the fundamental group of  $M$  is free of the same rank as  $\pi_1(N)$ , it follows that  $M$  is also a cube-with-handles, and there exists a homeomorphism  $g: N \rightarrow M$ . The map  $gf$  then induces an automorphism  $\alpha$  on the fundamental group of  $M$ . Then applying a theorem of Zieschang [12], there exists a homeomorphism  $h: M \rightarrow M$  such that  $h_* = \alpha$ . Then the maps  $f$  and  $g^{-1}h$  induce identical isomorphisms from the fundamental group of  $M$  onto the fundamental group of  $N$ . Then since  $N$  is an aspherical manifold, it follows that  $f$  is homotopic to the homeomorphism  $g^{-1}h$ . This completes the proof of Theorem 3.2 when  $N$  is a cube-with-handles.

We continue the proof assuming  $N$  is not a cube-with-handles. Let  $\pi_1(N) \approx B_1 * \dots * B_n * F$  denote a free decomposition for  $\pi_1(N)$ . Let  $D_1, \dots, D_d$  denote the collection of properly embedded disks in  $N$  whose existence is guaranteed by Theorem 2.4. Let

$$\text{cl} \left( N - \bigcup_{j=1}^d U(D_j, N) \right) = N_1, \dots, N_n,$$

and let  $\psi: \pi_1(N) \rightarrow B_1 * \dots * B_n * F$  be an isomorphism such that

$$\psi i_* (\pi_1(N_j)) = B_j \quad (1 \leq j \leq n).$$

Since  $N$  is not a cube-with-handles, we have the  $n \geq 1$  so that no component  $N_j$  ( $1 \leq j \leq n$ ) is simply connected.

Let  $f^{-1}(\bigcup_{j=1}^d D_j) = \bigcup_{i=1}^e E_i$  where for each  $i$ ,  $E_i$  is a properly embedded 2-manifold in  $M$ . As a consequence of the sphere theorem [3], both  $M$  and  $N$  are aspherical manifolds. Hence applying the techniques of §2, we may assume that each component  $E_i$  is simply connected. Furthermore, if  $E_i$  is a 2-sphere, then  $E_i$  bounds a 3-cell  $R$  in  $M$ . It is easy then to change  $f$  on a regular neighborhood of  $R$  so that  $E_i$  no longer occur as a component of  $f^{-1}(\bigcup_{j=1}^d D_j)$ . Thus we may assume that each component  $E_i$  ( $1 \leq i \leq e$ ) is a disk properly embedded in  $M$ .

We identify distinguished regular neighborhoods  $U(E_i, M)$  and  $U(D_j, N)$  with  $E_i \times [0, 1]$  and  $D_j \times [0, 1]$  respectively. We may assume that these neighborhoods have been chosen so that

$$f^{-1} \left( \bigcup_{j=1}^d U(D_j, N) \right) = \bigcup_{i=1}^e U(E_i, M),$$

and that if  $f(E_i) \subset D_j$ , then  $f|_{E_i \times [0,1]}: E_i \times [0, 1] \rightarrow D_j \times [0, 1]$  is a level



preserving map.

Let  $\text{cl}(M - \bigcup_{i=1}^m U(E_i, M)) = \bigcup_{i=1}^m M_i$ . Let  $x$  (resp.  $y$ ) denote the base point of  $M$  (resp.  $N$ ), and let  $x_i$  (resp.  $y_j$ ) denote the base point of  $M_i$  ( $1 \leq i \leq m$ ) (resp.  $N_j$  ( $1 \leq j \leq n$ )). We choose the base points so that  $f(x) = y$ , and  $f(\{x_i\}) = \{y_j\}$ . Let  $\alpha_i$  (resp.  $\beta_j$ ) denote arcs in  $M$  (resp.  $N$ ) joining  $x_i$  to  $x$  (resp.  $y_j$  to  $y$ ), and let the inclusion maps at the fundamental group level be defined along these arcs.

Finally, we assume that  $f(\text{int } M) \subset \text{int}(N)$ , and we put  $f_i = f|_{M_i}$  ( $1 \leq i \leq m$ ).

By Van Kampen's theorem, there is an isomorphism  $\varphi: \pi_1(M) \rightarrow A_1 * \dots * A_m * F'$  such that  $\varphi i_* \pi_1(M_i) = A_i$  ( $1 \leq i \leq m$ ). Note that some of the groups  $A_i$  may be trivial.

**LEMMA A.** *Let  $N_j$  be a component of  $\text{cl}(N - \bigcup_{j=1}^n U(D_j, N))$ . Then there exists a unique component  $M_i$  of  $\text{cl}(M - \bigcup_{i=1}^m U(E_i, M))$  such that  $f_{i*}: \pi_1(M_i) \rightarrow \pi_1(N_j)$  is an isomorphism. If  $M_p$  is any other component of  $\text{cl } M - \bigcup_{i=1}^m U(E_i, M)$  such that  $f(M_p) \subset N_j$ , then  $M_p$  is a 3-cell.*

*Proof.* Observe first of all that as a consequence of Van Kampen's theorem, the inclusion induced homomorphisms  $i_*: \pi_1(M_i) \rightarrow \pi_1(M)$  and  $i_*: \pi_1(N_j) \rightarrow \pi_1(N)$  are monomorphisms. For each  $i$  such that  $f(M_i) \subset N_j$  we have a commutative diagram.

$$\begin{array}{ccc} \pi_1(M_i) & \xrightarrow{f_{i*}} & \pi_1(N_j) \\ \downarrow i_* & & \downarrow i_* \\ \pi_1(M) & \xrightarrow{f_*} & \pi_1(N) . \end{array}$$

We conclude from this diagram that  $f_{i*}: \pi_1(M_i) \rightarrow \pi_1(N_j)$  is a monomorphism.

Let  $\lambda: A_1 * \dots * A_m * F' \rightarrow B_1 * \dots * B_n * F$  be the isomorphism defined by the composition  $\psi f_* \varphi^{-1}$ . Consider the factor  $B_j = \psi i_* (\pi_1(N_j))$  of  $B_1 * \dots * B_n * F$ . The group  $B_1 * \dots * B_n * F$  can be written as a free product  $\lambda(A_1) * \dots * \lambda(A_m) * \lambda(F')$ . By the Kurosh subgroup theorem (Corollary 4.9.1 of [6]),  $B_j$  is a free product of conjugates of subgroups of  $\lambda(A_1), \dots, \lambda(A_m)$  and a free group. But  $B_j$  is neither decomposable nor free. Thus for some element  $t$ , and for some  $i$ ,  $t^{-1} B_j t < \lambda(A_i)$ . Since  $B_j \neq 1$ , it follows that  $A_i \neq 1$ .

Let  $z$  denote an element of  $A_i$ . Choose a loop  $\bar{z}$  in  $M_i$  such that  $\varphi i_* (\bar{z}) = z$ . Let  $f(M_i) \subset N_q$ .

$$\begin{aligned} \lambda(z) &= \psi f_* (\alpha_i^{-1} \bar{z} \alpha_i) \\ \lambda(z) &= \psi (f(\alpha_i^{-1}) f(\bar{z}) f(\alpha_i)) \end{aligned}$$

$$\begin{aligned} \lambda(z) &= \psi(f(\alpha_i^{-1})\beta_q i_* f(\bar{z})\beta_q^{-1} f(\alpha_i)) \\ \lambda(z) &= ghg^{-1} \end{aligned}$$

where  $g = \psi(f(\alpha_i^{-1})\beta_q)$  and  $h = \psi i_* f(\bar{z}) \in B_q$ . Thus  $g^{-1}\lambda(A_i)g < B_q$ . Hence  $g^{-1}t^{-1}B_jtg < g^{-1}\lambda(A_i)g < B_q$ . But no nontrivial factor of a free product can be conjugate to a subgroup of any other factor. It follows that  $j = q, tg \in B_j$ , and  $g^{-1}\lambda(A_i)g = B_j$ . Recalling that  $g = \psi(f(\alpha_i)B_j^{-1})$ , it is now straightforward to show that  $f_{i_*}: \pi_1(M_i) \rightarrow \pi_1(N_j)$  is an epimorphism and hence an isomorphism.

Finally if  $f(M_p) \subset N_j$ , and if  $A_p$  is not the trivial group, then the above argument applied to  $A_p$  yields  $C^{-1}\lambda(A_p)C = B_j = g^{-1}\lambda(A_i)g$  (for some  $C$ ). Then  $gC^{-1}\lambda(A_p)Cg^{-1} = \lambda(A_i)$ . It follows that  $i = p$ . Hence if  $f(M_p) \subset N_j$  and if  $p \neq i$ , then  $\pi_1(M_p)$  is the trivial group. Furthermore, since each component of  $\text{cl}(M - \bigcup_{i=1}^e U(E_i, M))$  is irreducible, it follows that  $M_p$  is a 3-cell. This completes the proof of Lemma A.

It is our aim to show that  $f$  is a degree 1 map except possibly in the case that  $M$  is homeomorphic with the product of a closed 2-manifold with the unit interval. (In which case Waldhausen's theorems [11] will be applicable to complete the proof of Theorem 3.2.) Theorem 3.1 will then apply to prove Theorem 3.2. A study of the map  $f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$  seems to be the only accessible route to this end. As will become clear toward the end of the proof, the crucial obstruction to gaining the information we need about

$$f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$$

is the possibility that distinct boundary components of some component  $M_i$  of  $\text{cl}(M - \bigcup_{i=1}^e U(E_i, M))$  may be mapped under  $f$  into a single boundary component of some component  $N_j$  of  $\text{cl}(N - \bigcup_{j=1}^d U(D_j, N))$ . The succeeding three lemmas (particularly Lemma D) are concerned with eliminating this possibility. They constitute the most important (and easily the most difficult) steps in the proof of Theorem 3.2.

A component  $M_i$  of  $\text{cl}(M - \bigcup_{i=1}^e U(E_i, M))$  is *f-singular* if there exist components  $X_1$  and  $X_2$  of  $\text{bd } M_i$  such that  $f(X_1) \cup f(X_2)$  is contained in a single boundary component of a component  $N_j$  of  $\text{cl}(N - \bigcup_{j=1}^d U(D_j, N))$ . Consider an arbitrary component  $M_i$  of  $\text{cl}(M - \bigcup_{i=1}^e U(E_i, M))$ , and let  $f(M_i) \subset N_j$ . If  $M_i$  is *f-singular* then  $M_i$  has disconnected boundary and so cannot be a 3-cell. Thus according to Lemma A,  $f_{i_*}: \pi_1(M_i) \rightarrow \pi_1(N_j)$  is an isomorphism. Furthermore, since  $\pi_1(N_j)$  is neither infinite cyclic nor a nontrivial free product, it follows that  $M_i$  and  $N_j$  each have incompressible boundary. Hence we may apply the theorems of Waldhausen [11] to conclude that  $f_i: M_i \rightarrow N_j$  is homotopic to a homeomorphism. Further, if  $M_i$  is not the

product of a closed 2-manifold with the unit interval, then the homotopy can be taken to respect the boundary of  $M_i$  and  $N_j$ . It follows that if  $M_i$  is an  $f$ -singular component of  $\text{cl}(M - \bigcup_{i=1}^e (E_i, M))$  such that  $f(M_i) \subset N_j$ , then  $M_i$  and  $N_j$  are each homeomorphic with the product of a closed 2-manifold with the unit interval.

We require one further definition before beginning a more serious analysis of  $f$ -singular components of  $\text{cl}(M - U_{i=1}^e (E_i, M))$ .

We shall think of loops and arcs as maps with  $[0, 1]$  as domain. A loop  $\theta$  in  $M$  is  $f$ -closed if  $\theta$  can be divided into arcs  $\theta = \sigma_1 \tau_1 \cdots \sigma_k \tau_k$  with the following properties:

- (i) Each  $\sigma_i$  is an arc in  $\bigcup_{i=1}^e U(E_i, M)$ .
- (ii) Each  $\tau_i$  is an arc in  $\text{cl}(M - \bigcup_{i=1}^e U(E_i, M))$ .
- (iii) For each  $i$ ,  $f \circ \tau_i$  is a loop in  $N$ .

LEMMA B. *Let  $M_1 = G \times [0, 1]$  be an  $f$ -singular component of  $\text{cl}(M - U_{i=1}^e U(E_i, M))$  with  $f(M_1) \subset N_1 = H \times [0, 1]$  where  $G$  and  $H$  are homeomorphic 2-manifolds. If  $f(G \times \{0, 1\}) \subset H \times 0$ , then  $f^{-1}(H \times 1) = \phi$ .*

*Proof.* First of all we note that since  $f(\text{int } M) \subset \text{int } N$ , we have that  $f^{-1}(H \times 1) \subset \bigcup_{i=1}^m \text{bd } M_i$ . Suppose  $w$  is a point in  $\bigcup_{i=1}^m \text{bd } M_i$  such that  $f(w) \in H \times 1$ . Let  $f^{-1}(N_1) = M_1, M_{i_1}, \dots, M_{i_l}$  where  $M_{i_1}, \dots, M_{i_l}$  are 3-cells. Define a map  $g: M \rightarrow N$  as follows. Let  $g$  and  $f$  coincide outside  $\text{int}(M_1 \cup M_{i_1} \cdots \cup M_{i_l})$ . Define  $g$  on  $M_1$  so that  $g|_{M_1}$  is homotopic relative to the boundary of  $M_1$  to  $f|_{M_1}$ , and  $g(M_1) \subset H \times 0$ . On each 3-cell  $M_{i_j}$  ( $1 \leq j \leq l$ ) let  $g$  be a contraction of  $f|_{\text{bd } M_{i_j}}$  in  $\text{bd } N_1$ . Observe that  $g^{-1}(H \times 1/2) = \phi$ . Since  $N$  is aspherical, it is easy to see that  $g$  is homotopic to  $f$  relative to  $\bigcup_{i=1}^m \text{bd } M_i$ . Thus  $g_*: \pi_1(M) \rightarrow \pi_1(N)$  is an isomorphism.

Let  $\gamma$  be an arc in  $M$  joining  $w$  to a point in  $G \times 0$ . Then  $g \circ \gamma(0) \in H \times 1$ ,  $g \circ \gamma(1) \in H \times 0$ , and  $g \circ \gamma([0, 1]) \cap H \times 1/2 = \phi$ . Let  $\gamma^*$  be an arc in  $H \times [0, 1]$  joining  $g \circ \gamma(1)$  with  $g \circ \gamma(0)$ , and let  $\mu$  be a loop in  $M$  such that  $g \circ \mu \sim (g \circ \gamma) \gamma^*$ . Then since  $g^{-1}(H \times 1/2) = \phi$ , we have that  $0 = [g(\mu), H \times 1/2] = [(g \circ \gamma) \gamma^*, H \times 1/2] = 1$ . This contradiction completes the proof of Lemma B.

LEMMA C. *If  $\theta$  is an  $f$ -closed loop in  $M$ , then  $[\theta, E_i] = 0$  for each  $i$  ( $1 \leq i \leq e$ ).*

*Proof.* Let  $\theta = \sigma_1 \tau_1 \cdots \sigma_k \tau_k$  where the arcs  $\sigma_i$  and  $\tau_i$  satisfy the conditions (i), (ii), (iii) in the definition of an  $f$ -closed loop. Observe that since each  $f \circ \tau_i$  is a loop in  $N$ , there are at most two components  $N_1, N_2$  of  $\text{cl}(N - \bigcup_{j=1}^d U(D_j, N))$  and a single disk  $D_1$  such that  $f \circ \theta \subset$

$N_1 \cup N_2 \cup D_1 \times [0, 1]$ . (Possibly  $N_1 = N_2$ .) Let  $M_1$  and  $M_2$  be the unique components of  $\text{cl}(M - \bigcup_{i=1}^e U(E_i, M))$  such that  $f_{i*}: \pi_1(M_i) \rightarrow \pi_1(N_i)$  ( $i = 1, 2$ ) is an isomorphism.

Observe that as a consequence of Lemma B and the theorems of Waldhausen [11], it is the case that for each  $r$  such that  $f(\tau_r(0)) \in N_1$  (resp.  $f(\tau_r(0)) \in N_2$ ), there is a point  $v_r$  in  $M_1$  (resp.  $M_2$ ) such that  $f(v_r) = f(\tau_r(0))$  ( $1 \leq r \leq k$ ). (If  $M_1$  is  $f$ -singular, then by Lemma B  $f(\text{bd } M_1)$  and  $f(\tau_r(0))$  must lie in the same component of  $\text{bd } N_1$ . If  $M_1$  is not  $f$ -singular, then Waldhausen's theorems apply.)

Let  $\theta_r$  be an arc in  $M$  joining  $\tau_r(0)$  to  $v_r$  ( $1 \leq r \leq k$ ). Then  $f \circ \theta_r$  is a loop in  $N$ . Since  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an isomorphism, we may take  $f \circ \theta_r$  to be a contradictible loop. (If necessary replace  $\theta_r$  by  $\theta_r \theta_r^*$  where  $\theta_r^*$  is a loop in  $M$  based at  $v_r$  such that  $(f \circ \theta_r)^{-1} \sim f \circ \theta_r^*$ .) For each  $s$  ( $1 \leq s \leq k$ ) such that  $f \circ \tau_s \subset N_1$  (resp.  $N_2$ ) let  $\tau_s^*$  be a loop in  $M_1$  (resp.  $M_2$ ) based at  $v_s$  such that  $f \circ \tau_s^* \sim (f \circ \tau_s)^{-1}$ .

Consider the loop  $\theta^* = \sigma_1(\theta_1 \tau_1^* \theta_1^{-1} \tau_1) \cdots \sigma_k(\theta_k \tau_k^* \theta_k^{-1} \tau_k)$ . Observe first of all that since  $[\theta_r \tau_r^* \theta_r^{-1}, E_i] = 0$  for each  $r$  and  $i$  ( $1 \leq r \leq k, 1 \leq i \leq e$ ), it follows that  $[\theta^*, E_i] = [\theta, E_i]$  for each  $i$  ( $1 \leq i \leq e$ ). Also,  $f \circ (\theta_r \tau_r^* \theta_r^{-1} \tau_r)$  is a contractible loop in  $N$  for each  $r$  ( $1 \leq r \leq k$ ). Hence  $f \circ \theta^* \sim (f \circ \sigma_1)(f \circ \sigma_2) \cdots (f \circ \sigma_k)$ . Since  $(f \circ \sigma_k)(f \circ \sigma_2) \cdots (f \circ \sigma_k)$  is contained in the 3-cell  $U(D_1, N)$ , it follows that  $f \circ \theta^* \sim 1$ . Thus  $\theta^* \sim 1$ . Hence  $0 = [\theta^*, E_i] = [\theta, E_i]$  for each  $i$  ( $1 \leq i \leq e$ ). This completes the proof of Lemma C.

Lemma D. *If  $M$  is not the product of a closed 2-manifold with the unit interval, then no component of  $\text{cl}(M - \bigcup_{i=1}^e U(E_i, M))$  is  $f$ -singular.*

*Proof.* Suppose  $M_1$  is an  $f$ -singular component of

$$\text{cl}\left(M - \bigcup_{i=1}^e U(E_i, M)\right),$$

and suppose notation has been chosen so that  $f(M_1) \subset N_1$ . Let  $M_1 = G \times [0, 1]$ ,  $N_1 = H \times [0, 1]$ , and  $f(G \times \{0, 1\}) \subset H \times 0$  where  $G$  and  $H$  are homeomorphic 2-manifolds.

If  $\bigcup_{j=1}^d U(D_j, N) \cap H \times 0 = \phi$ , then  $\bigcup_{i=1}^d U(E_i, M) \cap M_1 = \phi$ . Since  $M$  is connected and  $f^{-1}(D_j) \neq \phi$  for any  $j$  ( $1 \leq j \leq d$ ), it would follow that  $e = d = 0$ , and  $M = M_1$  completing the proof of the lemma.

Thus, we assume that there is at least one disk  $D_1$  such that  $D_1 \times 0 \subset H \times 0$ . This together with the assumption that  $M_1$  is  $f$ -singular shall eventually lead us to a contradiction.

If  $f(E_i) \subset D_j$ , then  $f|_{\text{bd } E_i}$  induces a homeomorphism  $f_*^i: \pi_1(\text{bd } E_i) \rightarrow \pi_1(\text{bd } D_j)$ . We say that  $f$  is *nondegenerate* at  $E_i$  if  $f_*^i$  is not the trivial homeomorphism. Observe that if  $f$  were degenerate at each

component of  $f^{-1}(D_1 \times 0) \cap G \times 0$ , then we could construct a map  $g: G \times 0 \rightarrow H \times 0$  such that  $g$  is homotopic to  $f|_{G \times 0}$  and  $g^{-1}(\text{int } D_1 \times 0) = \phi$ . This is not possible since  $f|_{G \times 0}$  is homotopic to a homeomorphism from  $G \times 0$  onto  $H \times 0$ . Thus there exists a disk  $E_2 \times 0$  in  $G \times 0$  and similarly there exists a disk  $E_3 \times 0$  in  $G \times 1$  such that  $f(E_2 \times 0) = f(E_3 \times 0) = D_1 \times 0$ , and  $f$  is nondegenerate at  $E_2$  and  $E_3$ . Let  $D_1 \times 1 \subset N_2$ ,  $E_2 \times 1 \subset M_2^1$ ,  $E_3 \times 1 \subset M_3^1$  (possibly  $N_2 = N_1$  in which case it may also occur that  $M_2^1 = M_1$  or  $M_3^1 = M_1$ ).

We shall now begin construction of a somewhat complicated (in the sense that there are numerous technicalities involved in its construction)  $f$ -closed loop  $\theta$  in  $M$  which will enable us to reach a contradiction to the assumption that  $M_1$  is  $f$ -singular and  $M \neq M_1$ . The loop  $\theta$  is constructed from two basic arcs  $\theta_2$  with  $\theta_2(0) \in E_2 \times 0$  and  $\theta_3$  with  $\theta_3(0) \in E_3 \times 0$ .

Specifically, we wish  $\theta_2$  to be divisible into arcs  $\theta_2 = \sigma_0 \tau_1 \sigma_1 \cdots \tau_k \sigma_k$  (possibly  $k = 0$ ) with the following properties. (For notational convenience, put  $E_2^0 = E_2$ .)

- (1) Each  $\sigma_i$  is an arc in  $E_2^i \times [0, 1]$  such that the endpoints of  $\sigma_i$  lie in distinct components of  $E_2^i \times \{0, 1\}$ .
- (2) If  $i \neq j$ , then  $E_2^i \neq E_2^j$ .
- (3) Each  $\tau_i$  is an arc in a component  $M_2^i$  of  $\text{cl}(M - \bigcup_{i=1}^k U(E_i, M))$ , and  $f \circ \tau_i$  is a loop in  $N$ .
- (4) Each  $M_2^i$  ( $1 \leq i \leq k$ ) is a 3-cell.
- (5) If  $M_2^i = M_2^j$ , then either  $|i - j|$  is odd or  $i = j$ .
- (6)  $f$  is nondegenerate at each disk  $E_2^i$  and  $f(E_2^i) \subset D_1$ .
- (7)  $\theta_2(1)$  lies in a component  $M_2^{k+1}$  of  $\text{cl}(M - \bigcup_{i=1}^k U(E_i, M))$ , and  $M_2^{k+1}$  is not a 3-cell.
- (8)  $f(\theta_2(1)) \in D_1 \times 1$ .

Suppose we have constructed the arcs  $\theta_2$  with properties (1) through (8) above and  $\theta_3$  with similar properties except that  $\theta_3(0) \in E_3 \times 0$ . Then the proof of Lemma D can be completed as follows. Let  $\theta_3(1)$  lie in  $M_3^{l+1}$  where  $M_3^{l+1}$  is not a 3-cell. Since  $f \circ \theta_3(1) \cup f \circ \theta_2(1) \subset D_1 \times 1$ ,  $f \circ \theta_3(0) \cup f \circ \theta_2(0) \cup f \circ \theta_2(0) \subset D_1 \times 0$ , and  $f$  is nondegenerate at the disks  $E_2^0, E_2^k, E_3^0, E_3^l$  we can move the endpoints of  $\theta_2$  and  $\theta_3$  slightly so that  $f \circ \theta_2(0) = f \circ \theta_3(0)$  and  $f(\theta_2(1)) = f(\theta_3(1))$ .

Since  $f(\theta_2(1)) \in D_1 \times 1 \subset N_2$ ,  $f(\theta_3(1)) \in D_1 \times 1 \subset N_2$ ,  $\theta_2(1) \in M_2^{k+1}$ , and  $\theta_3(1) \in M_3^{l+1}$ , it follows that  $f(M_2^{k+1}) \cup f(M_3^{l+1}) \subset N_2$ . But neither  $M_2^{k+1}$  nor  $M_3^{l+1}$  is a 3-cell. It follows from Lemma A that  $M_2^{k+1} = M_2^{l+1}$ .

Let  $\tau_0$  be an arc in  $M_1$  joining  $\theta_3(0)$  with  $\theta_2(0)$ , and let  $\tau_*$  be an arc in  $M_2^{k+1}$  joining  $\theta_2(1)$  with  $\theta_3(1)$  (possibly  $\tau_*$  is a loop). Then  $\theta = \theta_2 \tau_* \theta_3^{-1} \tau_0$  is an  $f$ -closed loop in  $M$ . It follows from (2) that  $\theta_2$  meets  $E_2$  exactly once, and it follows from (4) and (8) that  $\theta_3$  does not meet  $E_2$ . Hence  $\theta$  is an  $f$ -closed loop in  $M$  such that  $[\theta, E_2] = 1$ . This is not consistent with the conclusion of Lemma C.

Thus, in order to prove Lemma D, we need only show that  $\theta_2$  can be constructed with the properties (1) through (8) above. (The construction of  $\theta_3$  will be similar to that of  $\theta_2$ .)

Let  $\mathcal{A}$  denote the collection of all arcs  $\mu$  in  $M$  such that  $\mu$  can be divided into arcs  $\mu = \sigma_0\tau_1\sigma_1 \cdots \tau_s\sigma_s$  satisfying properties (1) through (6) above. If  $\mu \in \mathcal{A}$  and  $\mu = \sigma_0\tau_1\sigma_1 \cdots \tau_s\sigma_s$ , we shall say that  $\mu$  has length  $s$ .

The set  $\mathcal{A}$  is not empty since the arc  $\sigma_0$  (of length zero) in  $E_2 \times [0, 1]$  that joins  $E_2 \times 0$  with  $E_2 \times 1$  is a member of  $\mathcal{A}$ . Note also that as a consequence of (2), each element of  $\mathcal{A}$  has length not greater than  $e$ . Thus there is an element of maximal length in  $\mathcal{A}$ . We choose  $\theta_2 = \sigma_0\tau_1\sigma_1 \cdots \tau_k\sigma_k$  to be an element of maximal length in  $\mathcal{A}$ . In order to complete the proof of Lemma D, we must show that  $\theta_2$  satisfies properties (7) and (8).

Suppose  $\theta_2$  does not satisfy (7); that is, suppose  $M_2^{k+1}$  is a 3-cell. We have two cases to consider; each of which shall lead to a contradiction.

*Case 1.* There exists an  $i$  such that  $M_2^{k+1} = M_2^i$  and  $|k + 1 - i|$  is even.

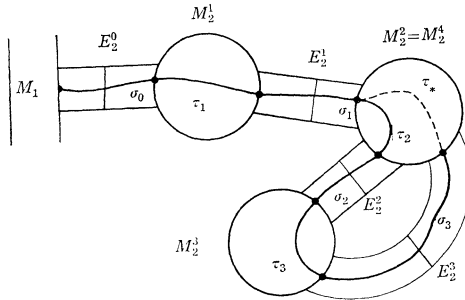


Figure 1

Observe that since  $f|_{E_i \times [0,1]}: E \times [0, 1] \rightarrow D_j \times [0, 1]$  is a level preserving map for each  $i$  and  $j$  such that  $f(E_i) \subset D_j$ , it follows from (1) and (3) above that  $f(\sigma_i(1))$  and  $f(\sigma_j(1))$  lie in the same component of  $D_1 \times \{0, 1\}$  if and only if  $|i - j|$  is even.

Since  $|k + 1 - i|$  is even,  $f(\sigma_j(1))$  and  $f(\sigma_{i-1}(1))$  lie in the same component of  $D_1 \times \{0, 1\}$ . Thus we can move the endpoint of  $\sigma_k$  so that  $f \circ \sigma_k(1) = f \circ \sigma_{i-1}(1)$ . Let  $\tau_*$  be an arc in  $M_2^i$  joining  $\sigma_k(1)$  with  $\sigma_{i-1}(1)$ . Then the loop  $\theta^* = \sigma_0\tau_1\sigma_1 \cdots \tau_k\sigma_k\tau_*\sigma_{i-1}^{-1}\tau_{i-1}^{-1} \cdots \tau_1^{-1}\sigma_0^{-1}$  is an  $f$ -closed loop. But  $[\theta^*, E_2^k] = 1$  contrary to the conclusion of Lemma C.

*Case 2.* If  $M_2^{k+1} = M_2^i$  for any  $i$ , then  $|k + 1 - i|$  is odd.

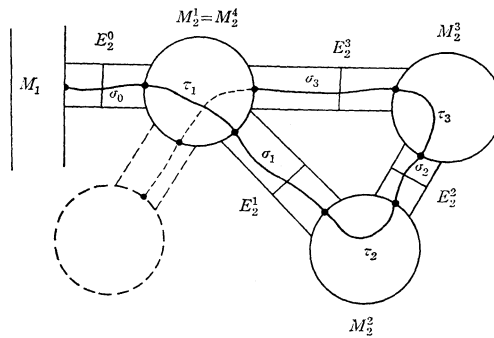


Figure 2

With no loss of generality in discussing this case, we assume that  $f(\sigma_k(1)) \in D_1 \times 0$ . There is a map  $g: (\text{bd } M_2^{k+1}) - \text{int } E_2^k \times 0 \rightarrow \text{bd } N_1$  such that  $g|_{\text{bd } E_2^k \times 0} = f|_{\text{bd } E_2^k \times 0}$  and  $g^{-1}(D_1 \times 0)$  consists of  $\text{bd } E_2^k \times 0$  and disks  $E_i \times 0$  ( $E_i \neq E_2^k$ ) on which  $f$  is nondegenerate. If  $g^{-1}(D_1 \times 0) = \text{bd } E_2^k \times 0$ , then  $g$  is a contraction of  $f|_{\text{bd } E_2^k \times 0}$  in  $\text{bd } N_2 - \text{int } D_1 \times 0$  contrary to the fact that  $f$  is nondegenerate at  $E_2^k$ . Thus there exists a disk  $E_2^{k+1} \times 0$  in  $\text{bd } M_2^{k+1}$  such that  $E_2^{k+1} \neq E_2^k$ , and  $f$  is nondegenerate at  $E_2^{k+1}$ .

Let  $\tau_{k+1}$  be an arc in  $M_2^{k+1}$  joining the endpoint of  $\sigma_k$  with a point in  $E_2^{k+1} \times 0$  such that  $f \circ \tau_{k+1}$  is a loop in  $N$ . Let  $\sigma_{k+1}$  be an arc in  $E_2^{k+1} \times [0, 1]$  joining  $\tau_{k+1}(1)$  with a point in  $E_2^{k+1} \times 1$ . We shall show that  $\theta_2^* = \sigma_0 \tau_1 \sigma_1 \cdots \tau_{k+1} \sigma_{k+1}$  satisfies the conditions (1) through (6) contradicting the maximality of  $\theta_2$ . The only nontrivial verification required is that (2) holds for  $\theta_2^*$ . If  $M_2^{k+1} \neq M_2^i$  for each  $i < k + 1$ , then (2) follows immediately. Thus suppose  $M_2^{k+1} = M_2^i, i < k + 1$ . Since  $k + 1 - i$  is odd, the disks  $E_2^{i-1} \times 1, E_2^i \times 1, E_2^k \times 0, E_2^{k+1} \times 0$  meet  $M_2^i$ . No other disks in the sequence  $E_2^0 \times 0, E_2^0 \times 1, \dots, E_2^k \times 0, E_2^k \times 1$  meet  $M_2^i$ . For suppose  $E_2^{j-1} \times \varepsilon$  and  $E_2^j \times \varepsilon$  meet  $M_2^i$  where  $\varepsilon$  is either 0 or 1 and  $i \neq j \neq k + 1$ . Then  $M_2^i = M_2^j$ . If  $\varepsilon = 1$ , then since  $f(\sigma_{i-1}(1))$  and  $f(\sigma_{j-1}(1))$  lie in the same component of  $D_1 \times \{0, 1\}$ , it follows that  $|i - j|$  is even. But this is inconsistent with property (5) for  $\theta_2$ . On the other hand, if  $\varepsilon = 0$ , then arguing just as above  $M_2^j = M_2^{k+1}$  and  $|k + 1 - j|$  is even contrary to our assumptions in the case we are considering. This proves property (2) for  $\theta_2^*$  and so contradicts the maximality of  $\theta_2$ . We conclude that  $M_2^{k+1}$  is not a 3-cell.

Property (8) remains. Suppose  $f(\theta_2(1)) \in D_1 \times 0 \subset N_1$ . Then

$$f(M_2^{k+1}) \subset N_1,$$

and it follows from Lemma A that  $M_2^{k+1} = M_1$ . We can move  $\theta_2(1)$  slightly so that  $f(\theta_2(0)) = f(\theta_2(1))$ . Let  $\tau_*$  denote an arc in  $M_1$  joining  $\theta_2(1)$  with  $\theta_2(0)$ . Then  $\tau_* \theta_2$  is an  $f$ -closed loop with  $[\tau_* \theta_2, E_2] = 1$ .

It follows that  $f \circ \theta_2(1) \in D_1 \times 1$ .

Thus  $\theta_2$  satisfies the conditions (1) through (8) above, and as already noted this is sufficient to prove Lemma D.

We are now prepared to complete the proof of Theorem 3.2 by showing that if  $M$  is not the product of a compact 2-manifold with the unit interval, then  $f$  is a degree 1 map. To this end, let  $Y$  denote a boundary component of  $N$ . Let  $\Delta$  denote a 2-simplex in  $\text{cl}(Y - \bigcup_{j=1}^d U(D_j, N))$ . Then  $\Delta$  lies in a boundary component  $Y_0$  of a component  $N_j$  of  $\text{cl}(N - \bigcup_{j=1}^d U(D_j, N))$ . By Lemma A,  $f^{-1}(N_j) = M_{i_0}, M_{i_1}, \dots, M_{i_s}$  where  $f_{i_0}: \pi_1(M_{i_0}) \rightarrow \pi_1(N_j)$  is an isomorphism and  $M_{i_1}, \dots, M_{i_s}$  are all 3-cells. By Lemma D, there is a unique component  $X_0$  of  $\text{bd } M_{i_0}$  such that  $f(X_0) \subset Y_0$ .

Observe that  $f^{-1}(\Delta)$  consists of a disjoint collection of simplices  $\Delta_0^i, \dots, \Delta_0^{i_0}, \Delta_1^i, \dots, \Delta_1^{i_1}, \dots, \Delta_s^i, \dots, \Delta_s^{i_s}$  in  $\text{bd } M$  where  $\Delta_r^i, \dots, \Delta_r^{i_r} \subset \text{bd } M_{i_r}$  ( $0 \leq r \leq s$ ). For a 2-simplex  $\sigma$  in a 2-manifold  $F$ , we use the notation  $*\sigma$  to denote the generator of the infinite cyclic summand of the simplicial chain group of  $F$  associated with  $\sigma$ . Observe that as a consequence of Waldhausen's theorems [10],  $f|_{X_0}: X_0 \rightarrow Y_0$  is homotopic to a homeomorphism. Also, since the higher homotopy of a closed 2-manifold of genus greater than zero is trivial, we have that  $f|_{\text{bd } M_{i_r}}: \text{bd } M_{i_r} \rightarrow Y_0$  is homotopic to a constant map for each  $r$  ( $1 \leq r \leq s$ ). It follows that

$$\sum_{i=1}^{t_r} f(*\Delta_i^r) = \begin{cases} \pm * \Delta, & r = 0 \\ 0, & r > 0. \end{cases}$$

Let  $X$  be the unique component of  $\text{bd } M$  which meets  $X_0$ . Then assuming notation has been properly chosen,

$$X \cap f^{-1}(\Delta) = \{\Delta_0^i, \dots, \Delta_0^{i_0}, \dots, \Delta_p^i, \dots, \Delta_p^{i_p}\}.$$

Put  $A = X \cap f^{-1}(\Delta)$ . Then

$$\sum_{\sigma \in A} f(*\sigma) = \sum_{i=1}^{t_0} f(*\Delta_0^i) + \sum_{j=1}^p \sum_{i=1}^{t_j} f(*\Delta_j^i) = \pm * \Delta.$$

Since the degree of a map between closed manifolds can be computed locally, it follows that  $f|_X: X \rightarrow Y$  is a degree 1 map. The same argument applies to show that if  $W$  is any component of  $\text{bd } M$  other than  $X$  such that  $f(W) \subset Y$ , then  $f|_W: W \rightarrow Y$  is a degree zero map.

According to the above remarks, the homeomorphism

$$f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$$

may be described as follows. Let  $X_1, \dots, X_k, Y_1, \dots, Y_q$  denote the boundary components of  $M$  and  $N$  respectively. Assume notation has



been chosen so that  $f|_{X_1}$  is a degree 1 map of  $X_i$  onto  $Y_i$  for each  $i \leq q$ . Then  $f|_{X_j}$  is a degree zero map of  $X_j$  into some boundary component of  $N$  for each  $j > q$ . Let  $\alpha_i$  (resp.  $\beta_j$ ) denote the generator of  $H_2(X_i)$  (resp.  $H_2(Y_j)$ ) ( $1 \leq i \leq h, 1 \leq j \leq q$ ). Then if  $f_*$  is the homomorphism of  $H_2(\text{bd } M)$  to  $H_2(\text{bd } N)$  induced by  $f$ , we have

$$f_*(\alpha_i) = \begin{cases} \pm \beta_i, & i \leq q \\ 0, & i > q. \end{cases}$$

Let  $\partial: H_3(M, \text{bd } M) \rightarrow H_2(\text{bd } M)$  denote the usual boundary homomorphism, and let  $\gamma$  denote the generator of  $H_3(M, \text{bd } M)$ . Then  $\partial(\gamma) = \varepsilon_1\alpha_1 + \dots + \varepsilon_h\alpha_h$  where  $\varepsilon_i = \pm 1$  ( $1 \leq i \leq h$ ). Thus with the above description of  $f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$ , we observe the following two important facts.

(i) The image of the map  $\partial: H_3(M, \text{bd } M) \rightarrow H_2(\text{bd } M)$  meets the kernel of  $f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$  only in the trivial element.

(ii)  $f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$  is an epimorphism.

Since  $M$  and  $N$  are aspherical manifolds, it follows that  $f$  is a homotopy equivalence. Thus  $f_*: H_2(M) \rightarrow H_2(N)$  is an isomorphism. Thus from the following commutative diagram we conclude that the kernel of  $f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$  is subset of the image of

$$\partial: H_3(M, \text{bd } M) \rightarrow H_2(\text{bd } M) .$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_3(M, \text{bd } M) & \xrightarrow{\partial} & H_2(\text{bd } M) & \longrightarrow & H_2(M) \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ 0 & \longrightarrow & H_3(N, \text{bd } N) & \longrightarrow & H_2(\text{bd } N) & \longrightarrow & H_2(N) . \end{array}$$

But we have already observed that these two subgroups meet only in the trivial element. It follows that  $f_*: H_2(\text{bd } M) \rightarrow H_2(\text{bd } N)$  has trivial kernel and hence is an isomorphism. Finally, we apply the five lemma to the above diagram to conclude that  $f_*: H_3(M, \text{bd } M) \rightarrow H_3(N, \text{bd } N)$  is an isomorphism. An application of Theorem 3.1 now completes the proof of Theorem 3.2.

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