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The expectation $E^P(\Phi)$ approximates the solution $u(z) = E^W(\Phi)$ of the Dirichlet problem for a plane domain D with boundary conditions ϕ on the boundary γ of D , where W is Wiener measure, P is the measure generated by a random walk which approximates Brownian motion beginning at z , and Φ is the functional on paths which equals the value of ϕ at the point where the path first meets γ . This paper develops a specific rate of convergence. If γ is C^2 , and P^n is generated by random walks beginning at z , with independent increments in the coordinate directions at intervals $1/n$, with mean zero, variance $1/\sqrt{n}$, and absolute third moment bounded by M , then $|E^{P^n}(\Phi) - E^W(\Phi)| \leq (CMV/\rho(z, \gamma))n^{-1/16}(\log n)^{9/8}$, where V is the total variation of ϕ on γ , $\rho(z, \gamma)$ is the distance from z to γ , and C is a constant depending only on γ .

Assume D is a Jordan region. If $z_t = x_t + iy_t$ is Brownian motion in R^2 beginning at z_0 , (cf. e.g., [5, p. 262]), and $\tau = \inf\{t: z_t \in \gamma\}$ is the first time z hits the boundary γ of D , then Φ is the functional given by $\Phi(z_\cdot) = \phi(z_\tau)$. Let $E^W(\Phi(z_\cdot)) = \int \Phi(z_\cdot) dW$ be the expectation of Φ with respect to Wiener measure W on $C([0, \infty), \mathcal{C})$. (See [8, pp. 218-19] for a definition of Brownian motion on the interval $[0, 1]$ and the corresponding Wiener measure.)

Let $g_1^1, g_2^1, g_2^2, \dots, g_k^1, g_k^2, \dots$ be a sequence of independent random variables with mean zero, variance 1, and absolute third moment bounded uniformly by $M < \infty$, and let

$$\xi_k^\alpha = g_k^\alpha/\sqrt{n}, \zeta_0 = z_0, \zeta_k = z_0 + \sum_{i=1}^k (\xi_i^1 + \sqrt{-1}\xi_i^2), t_k = k/n.$$

Let $\xi(t)$ be the continuous random broken line which has vertices (t_k, ζ_k) and is linear between vertices. Let P^n be the measure on $C([0, \infty), \mathcal{C})$ generated by this line, i.e., $P^n(S) = P(\xi(t) \in S)$.

Now by the Central Limit Theorem $P^n(\xi^\alpha(t) \leq \lambda) \rightarrow W(z_i^\alpha \leq \lambda)$, $\alpha = 1, 2$, where $\xi^\alpha(t), z_i^\alpha$ are the real and imaginary parts of $\xi(t), z_t$ respectively, (cf. e.g., [1, pp. 186-7]). More exactly one has the Barry-Esseen Theorem [3, p. 521]: For nt an integer

$$(1.1) \quad \sup_\lambda |P(\xi^\alpha(t) \leq \lambda) - N(\lambda/\sqrt{t})| \leq \frac{33}{4}M/\sqrt{nt}$$

where $N(x)$ is the normal distribution. A useful generalization of the

Central Limit Theorem is that convergence also takes place for the expectation of any functional on $C[0, 1]$ which is continuous with respect to uniform convergence on $[0, 1]$ and satisfies mild growth conditions, e.g. $\Phi(x) = \int_0^1 x(t)^2 dt$, $\sup_{0 \leq t \leq 1} x_t$, etc. Rates of convergence have been calculated for some specific one-dimensional functionals Φ , (e.g., [10], [11]). For an arbitrary functional Φ satisfying a uniform Hölder condition one can get rates of convergence using Levy distance in $C[0, t]$ ([9], see also §2 of this paper). Explicit rates of convergence are of interest for various practical problems and computer applications.

Although $\Phi(z) = \phi(z_.)$ is not continuous with respect to uniform convergence, it is continuous a.s. with respect to Wiener measure, so convergence takes place. In this paper we obtain a rate of convergence.

THEOREM. *There exists a universal constant $C^* = C^*(\gamma)$ such that*

$$(1.2) \quad |E^{P^n}(\Phi) - E^W(\Phi)| \leq \frac{C^* V(\phi) M}{\rho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8}$$

where $V(\phi)$ is the total variation of ϕ on γ , M is the bound on absolute third moments defined above, z_0 is the initial point of the paths $z_.$, and $\rho(z_0, \gamma) = \inf_s |z_0 - \gamma(s)|$ is the distance from z_0 to γ .

2. **Levy distance.** We define measures P_t^n, W_t on $C([0, t], \mathcal{C})$ by

$$P_t^n(S) = P^n(\pi^{-1}S), \quad W_t(S) = W(\pi^{-1}S),$$

where $\pi: C([0, \infty), \mathcal{C}) \rightarrow C([0, t], \mathcal{C})$ is the projection $\pi(f) = f|_{[0, t]}$. The Levy distance L between the measures P_t^n and W_t is given by

$$(2.1) \quad L(P_t^n, W_t) = \max(\epsilon_1, \epsilon_2),$$

where

$$\begin{aligned} \epsilon_1 &= \inf \{ \epsilon: P_t^n(S) \leq W_t(S^{\epsilon, t}) + \epsilon \text{ for all closed sets } S \}, \\ \epsilon_2 &= \inf \{ \epsilon: W_t(S) \leq P_t^n(S^{\epsilon, t}) + \epsilon \text{ for all closed sets } S \}, \end{aligned}$$

and

$$S^{\epsilon, t} = \{ y: \exists x \in S \ni \sup_{0 \leq s \leq t} |y(s) - x(s)| < \epsilon \}$$

is an ϵ -neighborhood of S with respect to the sup-norm on $[0, t]$.

The following proposition is a direct generalization of a result of Prokhorov ([9]) to two dimensions as is its proof.

PROPOSITION 1. *There exists an absolute constant C such that*

$$(2.2) \quad L(P_1^n, W_1) \leq CM^{1/4} n^{-1/8} (\log n)^{15/8}.$$

COROLLARY.

If $t = k/n$, k an integer, then $L(P_t^n, W_t) \leq C\sqrt{t}k^{-1/8}(\log k)^{15/8}$ for some constant C .

3. Boundedness of harmonic density. Fix a point γ_0 on γ , and a direction along γ , parametrize γ by arclength in the chosen direction from γ_0 . Let l denote the length of γ , and take the argument s of $\gamma = \gamma(s)$ modulo l .

Since γ is C^2 , there exists $R > 0$ such that any circle of radius R will meet γ in at most two points. It follows that for any two points $\gamma(a)$ and $\gamma(a + \delta)$ on γ where $0 < \delta < R$ that $\gamma([a, a + \delta])$ will lie in the intersection of the closed disks bounded by the two circles of radius R through $\gamma(a)$ and $\gamma(a + \delta)$. The case we have to eliminate is where γ is tangent to one of the circles at $\gamma(a)$ and $\gamma(a + \delta)$, but does not cross the circle, i.e., there are neighborhoods in γ of $\gamma(a)$ and $\gamma(a + \delta)$ which do not meet the closed disk bounded by the circle except at $\gamma(a)$ or $\gamma(a + \delta)$. But in this case we observe that a small rotation of the circle about one of the points $\gamma(a)$ or $\gamma(a + \delta)$ will result in three points of intersection, contradicting our assumption about γ . Furthermore, it follows from the Jordan curve theorem that the center of one of the two circles will be in D , the other center will be outside D .

We are now ready to prove the following result.

PROPOSITION 2.

$$W(z_\tau \in \gamma([a, a + \delta])) \leq B\delta/\rho(z_0, \gamma)$$

where B is an absolute constant depending only on γ .

Proof. We may assume $\delta < R$ and also δ sufficiently small that

$$2(R - (R^2 - \delta^2/4)^{1/2}) < \rho(z_0, \gamma)/2,$$

since by addition if the proposition holds for small δ , it holds for δ in general.

Let C be the circle of radius R through $\gamma(a)$ and $\gamma(a + \delta)$, with center not in D . Then

$$P_{z_0}(z_\tau \in \gamma([a, a + \delta])) \leq P_{z_0}(z_{\tau(C)} \in \delta^*)$$

where $\tau(C) = \inf \{t: z_t \in C\}$, and $\delta^* = D \cap C$. Now invert the plane with respect to the circle C , sending z_0 into $I(z_0)$. Now $I(z_t)$ is Brownian motion with a time change. (P. Lévy [7, p. 254], see also [5, pp. 279–80] for another proof of this.) However, where $I(z_t)$ first hits C is independent of any time change; “*Les propriétés intrinsèques de*

la courbe C sont invariantes par une representation conforme."

Now

$$z_{\tau(C)} \in C \implies I(z_{\tau(C)}) = z_{\tau(C)}, I(\delta^*) = \delta^* ,$$

so $P_{z_0}(z_{\tau(C)} \in \delta^*) = P_{z_0}((I(z))_{\tau'(C)} \in \delta^*)$ where $\tau'(C) = \inf \{s: (I(z))_s \in C\}$.

But the harmonic density on a circle is given by the Poisson kernel (cf. e.g., [4, p. 361 ff.]); it is bounded,

$$P_{z_0}((I(z))_{\tau'(C)} \in \delta^*) \leq \frac{2}{2\pi} |\delta^*| / \rho(I(z_0), C)$$

where $|\delta^*|$ is the length of δ^* . Now

$$R - \rho(I(z_0), C) = R^2 / (\rho(z_0, C) + R)$$

so

$$1/\rho(I(z_0), C) = (\rho(z_0, C) + R)/R\rho(z_0, C) \leq \frac{\Delta/R + 1}{\rho(z_0, C)} ,$$

where Δ is the diameter of D .

Now look at $\rho(z_0, C)$:

$$\rho(z_0, C) \geq \rho(z_0, \gamma) - 2(R - (R^2 - s^2/4)^{1/2})$$

where $s = |\gamma(a + \delta) - \gamma(a)| \leq \delta$. But δ was sufficiently small that

$$2(R - (R^2 - \delta^2/4)^{1/2}) < \rho(z_0, \gamma)/2 ,$$

and since $s \leq \delta$,

$$2(R - (R^2 - s^2/4)^{1/2}) \leq 2(R - (R^2 - \delta^2/4)^{1/2}) .$$

Hence

$$\rho(z_0, C) > \rho(z_0, \gamma)/2, \text{ also } |\delta^*| \leq \frac{\pi}{2}s \leq \frac{\pi}{2}\delta$$

and it follows that

$$W(z_\tau \in \gamma([a, a + \delta])) \leq \frac{2}{2\pi} \frac{\pi}{2} \delta \cdot 2(\Delta/R + 1) / \rho(z_0, \gamma) = B\delta / \rho(z_0, \gamma) .$$

4. Some inequalities. We shall need the following.

LEMMA.

$$(4.1) \quad \begin{aligned} W(\tau > t) &\leq \frac{4}{\pi} \exp(-\pi^2 t / 8\Delta^2) \\ P^n(\tau > t) &\leq \frac{4}{\pi} \exp(-\pi^2 t / 8\Delta^2) + AM(nt)^{-1/8} (\log nt)^{1/2} , \end{aligned}$$

where Δ is the diameter of D and A is an absolute constant.

Proof.

$$\begin{aligned} W(\tau > t) &\leq \Pr(\max_{0 \leq s \leq t} |z_s - z_0| < \Delta) \\ &\leq \Pr(\max_{0 \leq s \leq 1} |\operatorname{Re}(z_s - z_0)| < \Delta/\sqrt{t}) = T(\Delta/\sqrt{t}) \\ &\leq \frac{4}{\pi} \exp(-\pi^2 t/8\Delta^2), \end{aligned}$$

where $T(\lambda) = \Pr(\max_{0 \leq s \leq 1} |x_s| < \lambda)$. The last inequality comes from the fact that the infinite series expansion for $T(\lambda)$ [11] is alternating, with decreasing terms.

$$\begin{aligned} P^n(\tau > t) &\leq \Pr \max_{k \leq nt} (|\zeta_k - z_0| < \Delta) \\ &\leq \Pr(\max_{k \leq nt} |\operatorname{Re}(\zeta_k - z_0)/\sqrt{t}| < \Delta/\sqrt{t}). \end{aligned}$$

Now the theorem of Rosencrantz [10] applies [11] and we have

$$\begin{aligned} &\Pr(\max_{k \leq nt} |\operatorname{Re}(\zeta_k - z_0)/\sqrt{t}| < \Delta/\sqrt{t}) \\ &\leq A \cdot M(\log nt)^{1/2} (nt)^{-1/8} + T(\Delta/\sqrt{t}) \end{aligned}$$

where A is an absolute constant. But we saw above that

$$T(\Delta/\sqrt{t}) \leq \frac{4}{\pi} \exp(-\pi^2 t/8\Delta^2),$$

so

$$P^n(\tau > t) \leq \frac{4}{\pi} \exp(-\pi^2 t/8\Delta^2) + AM(nt)^{-1/8} (\log nt)^{1/2}.$$

Now we need more notation. Let $K_\lambda = \gamma([0, \lambda])$, let $(z_\tau \in K_\lambda)^{\varepsilon, \tau} \subset C([0, \infty), \mathcal{C})$ be defined by $y \in (z_\tau \in K_\lambda)^{\varepsilon, \tau}$ iff $\exists z$ such that $z_\tau \in K_\lambda$ and (for $\tau = \tau(z)$) $\sup_{0 \leq s \leq \tau} |y_s - z_s| < \varepsilon$. Let $\delta = \sqrt{\varepsilon}$, and let $K_\lambda^\delta = \gamma([0, \lambda + \delta]) \cup \gamma(l - \delta, l)$, where l is the length of γ .

PROPOSITION 3.

$$W((z_\tau \in K_\lambda)^{\varepsilon, \tau} \cap (z_\tau \notin K_\lambda^\delta)) \leq G\sqrt{\varepsilon}$$

where G is a constant depending only on γ .

Proof. Let $\tau(\partial\varepsilon) = \inf\{t: \rho(z_t, K_\lambda) < \varepsilon\}$ where $\rho(z_t, K_\lambda)$ is the distance from z_t to K_λ . Then

$$\begin{aligned} &W((z_\tau \in K_\lambda)^\varepsilon, \tau \cap (z_\tau \notin K_\lambda^\delta)) \\ &\leq W(\tau(\partial\varepsilon) < \tau, z_\tau \notin K_\lambda^\delta) + W(\tau(\partial\varepsilon) > \tau, z_\tau \notin K_\lambda^\delta, \tau(\partial\varepsilon) < \tau(s\varepsilon)) \\ &= E^W(\chi_{\tau(\partial\varepsilon) < \tau} P_{z_\tau(\partial\varepsilon)}(z_\tau \notin K_\lambda^\delta) \\ &\quad + \chi_{[\tau(\partial\varepsilon) > \tau, z_\tau \notin K_\lambda^\delta]} P_{z_\tau}(\tau(\partial\varepsilon) < \tau(s\varepsilon))) \end{aligned}$$

by the strong Markov property [1, p. 268], where $\tau(s\varepsilon) = \inf \{t: \rho(z_t, D) > \varepsilon\}$. We estimate $P_{z_\tau(\partial\varepsilon)}(z_\tau \notin K_\lambda^\delta)$:

Let $\gamma(a)$ be a point in K_λ of distance ε from $z_\tau(\partial\varepsilon)$, let T be the tangent to γ at $\gamma(a)$. Let S_i ($i = 1, 2$) be lines perpendicular to T through the points $\gamma(a - \delta)$ and $\gamma(a + \delta)$. The distance d_i from $z_\tau(\partial\varepsilon)$ to each of the lines S_i will be less than $\delta + \varepsilon$ (less than δ unless $\gamma(a)$ is an endpoint of K_λ ; let $d = \min(d_1, d_2)$). Let T' be parallel to T , at a distance $\varepsilon \cdot \sup |\gamma''|$ on the opposite side of T from $z_\tau(\partial\varepsilon)$. I now claim $\gamma([a - \delta, a + \delta]) \cap T' = \emptyset$ if $2\delta < 1/\sup |\gamma''|$. Choose coordinates such that $\gamma(a) = 0, \gamma'(a) > 0$. Then by Taylor's Theorem, for each h there exists θ such that

$$\begin{aligned} \text{Im } \gamma(a + h\delta) &= \text{Im } \gamma(a) + \text{Im } \gamma'(a) \cdot h\delta + \text{Im } \gamma''(a + \theta h\delta) \cdot h^2\delta^2/2 \\ &= \text{Im } \gamma''(a + \theta h\delta) \cdot h^2\delta^2/2. \end{aligned}$$

Hence for $|h| \leq 1, |\text{Im } \gamma(a + h\delta)| \leq \sup |\delta''| \cdot \delta^2/2 < \varepsilon \cdot \sup |\gamma''|$ and $\gamma([a - \delta, a + \delta])$ does not meet T' .

Let $\tau_{T'}$ be the first time (after $\tau(\partial\varepsilon)$) that z_t hits the line T', τ_S the first time (after $\tau(\partial\varepsilon)$) that z_t hits $S_1 \cup S_2$, and $c = \rho(z_\tau(\partial\varepsilon), T') \leq \varepsilon \cdot (\sup |\gamma''| + 1)$. Note that $\tau_{T'}$ and τ_S are independent, since the components of Brownian motion in the direction of S_i and T' are independent. We can write

$$\begin{aligned} P_{z_\tau(\partial\varepsilon)}(z_\tau \notin K_\lambda^\delta) &< P_{z_\tau(\partial\varepsilon)}(\tau_{T'} > \tau_S) + O(\delta) \\ &= \int_0^\infty P_{z_\tau(\partial\varepsilon)}(\tau_S < t) d_t P_{z_\tau(\partial\varepsilon)}(\tau_{T'} \leq t) + O(\delta). \end{aligned}$$

Now

$$\begin{aligned} P_{z_\tau(\partial\varepsilon)}(\tau_{T'} \leq t) &= P(\sup_{0 \leq s \leq t} x_s \geq c) = P(\sup_{0 \leq s \leq 1} x_s \geq c/\sqrt{t}) \\ &= \sqrt{2/\pi} \int_{c/\sqrt{t}}^\infty e^{-u^2/2} du \end{aligned}$$

(cf. e.g., [1, p. 287] and [8, p. 227]).

Hence

$$\begin{aligned} P_{z_\tau(\partial\varepsilon)}(\tau_{T'} > \tau_S) &= \int_0^\infty P_{z_\tau(\partial\varepsilon)}(\tau_S < t) \sqrt{2/\pi} \frac{1}{2} ct^{-3/2} e^{-c^2/2t} dt \\ &\leq \int_0^\infty P(\sup_{0 \leq s \leq t} |x_s| > d) (c/\sqrt{2\pi}) t^{-3/2} e^{-c^2/2t} dt \\ &\leq 2 \int_0^\infty P(\sup_{0 \leq s \leq 1} x_s > d/\sqrt{t}) (c/\sqrt{2\pi}) t^{-3/2} e^{-c^2/2t} dt \end{aligned}$$

which by a straightforward computation, is bounded by $2(d^2/c^2 + 1)^{-1/2}$.

But I claim $d \sim \delta$: choose coordinate such that $\gamma(a) = 0, \gamma'(a) = 1$. Using Taylor's Theorem we get $\delta \geq d \geq \delta - (\sup |\gamma''|/2)\delta^2 - \varepsilon$, so $d \sim \delta$. And $\delta = \sqrt{\varepsilon}, c \leq \varepsilon(\sup |\gamma''| + 1)$, so

$$2(d^2/c^2 + 1)^{-1/2} \leq G_1\varepsilon/\delta = G_1\sqrt{\varepsilon}$$

for some constant G_1 .

Now the same argument can be applied to estimate $P_{z_\tau}(\partial\varepsilon < \tau(s\varepsilon))$ (i.e., the probability that a Brownian path will move a distance $d \sim \delta = \sqrt{\varepsilon}$ in the direction tangent to the curve before it moves a distance $c = O(\varepsilon)$ in the direction normal to the curve). Hence $P_{z_\tau}(\tau(\partial\varepsilon) < \tau(s\varepsilon)) \leq G_2\sqrt{\varepsilon}$ for some constant G_2 and our proposition follows.

5. Proof of the theorem. We are now ready to prove our theorem.

$$\begin{aligned} & |E^{P^n}(\Phi) - E^W(\Phi)| \\ (5.1) \quad &= |E^{P^n}(\Phi\chi_{\tau \leq t}) - E^W(\Phi\chi_{\tau \leq t}) + E^{P^n}(\Phi\chi_{\tau > t}) - E^W(\Phi\chi_{\tau > t})| \\ &\leq E^{P^n}(\Phi\chi_{\tau \leq t}) - E^W(\Phi\chi_{\tau \leq t}) + \sup_r |\phi| (P^n(\tau > t) + W(\tau > t)). \end{aligned}$$

Looking at the first term,

$$\begin{aligned} & |E^{P^n}(\Phi\chi_{\tau \leq t}) - E^W(\Phi\chi_{\tau \leq t})| \\ &= \left| \int_0^t \phi(\gamma(\lambda))(P^n(z_\tau \in \gamma(d\lambda), \tau \leq t) - W(z_\tau \in \gamma(d\lambda), \tau \leq t)) \right| \\ &\leq |\phi(\gamma(0))| (P(\tau > t) + W(\tau > t)) + \int_0^t |P^n(z_\tau \in K_\lambda, \tau \leq t) - W(z_\tau \in K_\lambda, \tau \leq t)| \cdot |d\phi(\lambda)|. \end{aligned}$$

We estimate the integrand:

The event $(z_\tau \in K_\lambda, \tau \leq t)$ is determined by the behavior of the path up to time t , so

$$\begin{aligned} P^n(z_\tau \in K_\lambda, \tau \leq t) - W(z_\tau \in K_\lambda, \tau \leq t) &= P_t^n(z_\tau \in K_\lambda, \tau \leq t) \\ &\quad - W_t(z_\tau \in K_\lambda, \tau \leq t). \end{aligned}$$

We can use the corollary of Proposition 1 to get

$$\begin{aligned} P_t^n(z_\tau \in K_\lambda, \tau \leq t) &\leq W_t((z_\tau \in K_\lambda, \tau \leq t)^{\varepsilon, t}) + \varepsilon \\ &\leq W_t(z_\tau \in K_\lambda, \tau \leq t) + W_t((z_\tau \in K_\lambda, \tau \leq t)^{\varepsilon, t}) \\ &\quad - (z_\tau \in K_\lambda^2, \tau \leq t) + W_t(z_\tau \in K_\lambda^2 - K_\lambda, \tau \leq t) + \varepsilon, \end{aligned}$$

where $\varepsilon = \varepsilon(n, t) = CM^{1/4}n^{-1/8}t^{3/8}(\log nt)^{15/8}$.

Now $y \in (z_\tau \in K_\lambda, \tau \leq t)^{\varepsilon, t}$ means $\exists z$ such that $\tau \leq t, z_\tau \in K_\lambda,$

$\sup_{0 \leq s \leq t} |y_s - z_s| < \varepsilon$. As this condition does not depend on y_s for $s > t$,

$$\begin{aligned} &W_t((z_\tau \in K_\lambda, \tau \leq t)^{\varepsilon, t} - (z_\tau \in K_\lambda^s, \tau \leq t)) \\ &\leq W((z_\tau \in K_\lambda, \tau \leq t)^{\varepsilon, \tau} - (z_\tau \in K_\lambda^s, \tau \leq t)) \\ &\leq W((z_\tau \in K_\lambda)^{\varepsilon, \tau} - (z_\tau \in K_\lambda^s)) + W(\tau > t). \end{aligned}$$

Applying Propositions 2 and 3, we then have

$$\begin{aligned} &P^n(z_\tau \in K_\lambda, \tau \leq t) - W(z_\tau \in K_\lambda, \tau \leq t) \\ &\leq (G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t). \end{aligned}$$

We apply the above argument to the complement $\gamma - K_\lambda$ of K_λ in γ .

$$\begin{aligned} &P^n(z_\tau \in \gamma - K_\lambda, \tau \leq t) - W(z_\tau \in \gamma - K_\lambda, \tau \leq t) \\ &\leq (G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t). \end{aligned}$$

It follows that

$$\begin{aligned} &|P^n(z_\tau \in K_\lambda, \tau \leq t) - W(z_\tau \in K_\lambda, \tau \leq t)| \leq (G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} \\ &\quad + W(\tau > t) + P^n(\tau > t). \end{aligned}$$

We can now estimate the integral

$$\begin{aligned} (5.3) \quad &\int_0^t |P^n(z_\tau \in K_\lambda, \tau \leq t) - W(z_\tau \in K_\lambda, \tau \leq t)| \cdot |d\phi(\lambda)| \\ &\leq ((G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t) + P^n(\tau > t))V(\phi) \end{aligned}$$

where $V(\phi)$ is the total variation of ϕ on γ .

Combining the results of (5.1), (5.2), and (5.3), (we have)

$$\begin{aligned} &|E^{P^n}(\Phi) - E^W(\Phi)| \leq \phi(\gamma(0))(P^n(\tau > t) + W(\tau > t)) \\ &\quad + V(\phi)(G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t) + P^n(\tau > t) \\ &\quad + \sup_\tau |\phi| (P^n(\tau > t) + W(\tau > t)) \\ &\leq V(\phi)(G + 2\beta/\rho(z_0, \gamma))\sqrt{\varepsilon} + (V(\phi) + 2 \sup_\tau |\phi|) \cdot (P^n(\tau > t) \\ &\quad + W(\tau > t)). \end{aligned}$$

This estimate is minimized by choosing t so as to balance the factors $\sqrt{\varepsilon}$ and $(P^n(\tau > t) + W(\tau > t))$. So setting

$$t = \min \left\{ s : s \geq \frac{1}{2}(\Delta/\pi)^2 \log n, sn \text{ an integer} \right\},$$

we get

$$\begin{aligned} &P^n(\tau > t) + W(\tau > t) \\ &\leq \frac{8}{\pi} n^{-1/16} + A \cdot M n^{-1/8} \left(\frac{1}{2}(\Delta/\pi)^2 \log n \right)^{1/8} (\log nt)^{1/2} \\ &\leq A_1 M n^{-1/16} (\log n)^{5/8}, \end{aligned}$$

and

$$\begin{aligned} \sqrt{\varepsilon} &= \sqrt{C} M^{1/8} n^{-1/16} t^{3/16} (\log nt)^{15/16} \\ &\leq \sqrt{C} M^{1/8} n^{-1/16} A_2 (\log n)^{9/8} \end{aligned}$$

where A_1, A_2 are absolute constants. Hence

$$\begin{aligned} |E^{P^n}(\Phi) - E^W(\Phi)| &\leq V(\phi) M \frac{G\Delta + 2B}{\rho(z_0, \gamma)} \sqrt{C} A_2 n^{-1/16} (\log n)^{9/8} \\ &\quad + (V(\phi) + 2 \sup_{\tau} |\phi|) A_1 M n^{-1/16} (\log n)^{5/8} \\ &\leq (3V(\phi) + 2 \inf_{\tau} |\phi|) M \left(\frac{G\Delta + 2B}{\rho(z_0, \gamma)} \sqrt{C} A_2 + A_1 \right) n^{-1/16} (\log n)^{9/8}. \end{aligned}$$

But integration is linear, so we may assume $\phi(p) = 0$ for some p in γ , as we are taking the difference of expectations.

Letting $C^* = 3((G\Delta + 2B)\sqrt{C} A_2 + \Delta A_1)$ we have

$$|E^{P^n}(\Phi) - E^W(\Phi)| \leq \frac{C^* V(\phi) M}{\rho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8}.$$

COROLLARY. *If O is any subset of γ consisting of a finite number k of intervals, then*

$$|P_{z_0}^n(z_\tau \in O) - W_{z_0}(z_\tau \in O)| \leq \frac{2kC^* M}{\rho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8}.$$

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