ON ABSOLUTE DE LA VALLÉE POUSSIN SUMMABILITY

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Gronwall proved that \((C, r) \subseteq (V - P)\) for \(r \geq 0\), where \((C, r)\) and \((V - P)\) denote Cesàro and de la Vallée Poussin summability. It is proved in this paper that \(|C, r| \subseteq |V - P|\) for \(r \geq 0\).

1. Introduction. Let

\[
V_n = \sum_{k=1}^{n} \frac{(n!)^2}{(n-k)! (n+k)!} a_k \quad (n \geq 0).
\]

If \(\lim_{n \to \infty} V_n = s\), we say that the series is summable \((V - P)\) to \(s\).

If

\[
\sum_{n=1}^{\infty} |V_n - V_{n-1}| < \infty.
\]

The series \(\sum_{n=0}^{\infty} a_n\) is said to be summable \(|V - P|\).

Hyslop [2] proved that the \((V - P)\) method is equivalent to the \((A, 2)\) method defined by

\[
\lim \sum_{x=0}^{\infty} a_x e^{-nt} = s
\]

for all series \(\sum_{n=0}^{\infty} a_n\) which satisfy the condition \(a_n = 0(n^c)\), where \(c\) is any constant, and that the inclusion \((A, 2) \subseteq (V - P)\) is false without restriction.

Kuttner [3] has shown that \((V - P) \subseteq (A, 2)\) without restriction.

Gronwall [1] proved that \((C, r) \subseteq (V - P)\) for \(r \geq 0\), where \((C, r)\) denotes the Césaro summability of order \(r\).

In this paper, we shall prove

**THEOREM A.** \(|C, r| \subseteq |V - P|\) for \(r \geq 0\).

2. Proof of Theorem A. Since it is well-known that \(|C, r|\) implies \(|C, r'|\) for \(-1 < r \leq r'\), it is enough to consider the case \(r\) an integer. Now, writing

\[
V_n = v_0 + v_1 + \cdots + v_n,
\]

we find that

\[
\begin{align*}
v_0 &= a_0, \\
v_n &= \sum_{k=1}^{n} \frac{((n-1)!)^2}{(n-k)! (n+k)!} k^2 a_k \quad (n \geq 1).
\end{align*}
\]
Now write $\tau_k = \tau^r_k$ for the $(C, r)$ mean of the sequence $\{ka_k\}$; thus the assumption that $\sum_{n=0}^{\infty} a_n$ is summable $|C, r|$ is equivalent to

$$\sum_{n=0}^{\infty} \frac{|\tau^r_n|}{n} < \infty.$$ 

If we take $((n-1)!/(n-k)!(n+k)!$ as meaning 0 whenever $k > n$, we deduce from (1) by $n$ partial summations that, for $n \geq 1$,

$$v_n = \sum_{k=1}^{n} \Delta^r_k \left\{ \frac{(n-k)!(k^r)}{(n-k)!(n+k)!} \right\}(k+r)\tau^r_k.$$ 

Now it is well-known that in order that the series-to-series transformation

$$b_n = \sum_{k=0}^{\infty} a_{nk}a_k$$

should be that $\sum_{n=0}^{\infty} |b_n|$ converges whenever $\sum_{n=0}^{\infty} |a_n|$ does so, it is necessary and sufficient that

$$\sum_{n=0}^{\infty} |a_{nk}|$$

should be bounded. Thus it is enough to show that, for $k \geq 1$,

$$\sum_{n=k}^{\infty} \left| \Delta^r_k \left\{ \frac{(n-k)!(k^r)}{(n-k)!(n+k)!} \right\} \right| = O(k^{-r-1}).$$

It is easily seen by induction on $r$ that

$$\Delta^r_k \left\{ \frac{(n-k)!(k^r)}{(n-k)!(n+k)!} \right\} = \frac{A^r(n, k)((n-1)!)^r}{(n-k)!(n+k+r)!} ,$$

where $A^r(n, k)$ is defined inductively by

$$A^0(n, k) = 1 ,$$

$$A^{r+1}(n, k) = (n + k + r + 1)A^r(n, k) - (n - k)A^r(n, k + 1).$$

Write $P_j(k)$ for a polynomial in $k$ of degree not exceeding $j$, possibly different at each occurrence (thus $P_j(k)$ denotes a constant). We deduce from (4) by induction that

$$A^{2s}(n, k) = \sum_{j=0}^{s} P_{2s+1}(k)n^{s-j},$$

$$A^{2s+1}(n, k) = \sum_{j=0}^{s-1} P_{2s}(k)n^{s+1-j}.$$ 

Hence, uniformly in the ranges stated
\[ A'(n, k) = \begin{cases} O(n^{(r+1)/2}) & (1 \leq k \leq n^{1/2}) , \\ O(K^{r+1}) & (n^{1/2} < k \leq n) . \end{cases} \]

Next, for large \( n \) uniformly in \( k \leq n^{2/3} \) we have, by Stirling's formula
\[
\frac{(n!)^2}{(n - k)!(n + k)!} = O(H(n, k)) ,
\]
where
\[
H(n, k) = \left( 1 - \frac{k}{n} \right)^{n + k - 1/2} \left( 1 + \frac{k}{n} \right)^{-n - k + 1/2} .
\]

We have
\[
\log H(n, k) = -\frac{k^2}{n} + O\left( \frac{k^3}{n^2} \right) .
\]

Now since we supposing that \( k \leq n^{2/3} \) we have
\[
\exp \left\{ O\left( \frac{k^2}{n^2} \right) \right\} = O(1)
\]
so that
\[
\frac{(n!)^2}{(n - k)!(n + k)!} = O\left\{ \exp \left( \frac{-k^2}{n} \right) \right\} .
\]

This will not apply if \( k > n^{2/3} \). Since we cannot then assert (5).
However, for fixed \( n \), \( (n!)^2/(n - k)!(n + k)! \) is a decreasing function of \( k \) so that, for \( k > n^{2/3} \),
\[
\frac{(n!)^2}{(n - k)!(n + k)!} = O\left\{ \exp \left( -n^{4/3} \right) \right\} .
\]

Also, it is trivial that
\[
\frac{((n - 1)!)^2}{(n - k)!(n + k + 2)!} = \frac{(n!)^2}{(n - k)!(n + k)!} O(n^{-r}) .
\]

Combining these results, we find that
\[
A\left\{ \frac{((n - 1)!)^2}{(n - k)!(n + k)!} \right\} = \begin{cases} O(n^{-(r+2)/2}) & (1 \leq k \leq n^{1/2}) , \\ O\left( \frac{k^{r+1}}{n^{r+2}} \exp \left( -\frac{k^2}{n} \right) \right) & (n^{1/2} < k \leq n^{2/3}) , \\ O(n^{-1} \exp \left( -n^{-1/3} \right)) & (n^{2/3} < k \leq n) . \end{cases}
\]

Thus the sum (3) is
\[
O\left\{ \sum_{k \leq n < k^{3/2}} \frac{1}{n} \exp \left( -n^{-(1/3)} \right) \right\} + O\left\{ \sum_{k^{3/2} \leq n < k^2} \frac{k^{r+1}}{n^{r+2}} \exp \left( -\frac{k^2}{n} \right) \right\} + O\left\{ \sum_{n \geq k^2} \frac{1}{n^{(r+3)/2}} \right\} = O(I_1) + O(I_2) + O(I_3),
\]
say. It is clear that 
\[
I_1 = O(k^{-r-1}),
\]
\[
I_3 = O(k^{-r-1})
\]
so we need consider only \(I_2\). Now for fixed \(k\)
\[
\frac{k^{r+1}}{y^{r+2}} \exp \left( -\frac{k^2}{y} \right)
\]
is increasing for \(y < y_0\) and decreasing for \(y > y_0\), where \(y_0 = y_0(k) = k^2/(r + 2)\). Hence
\[
I_2 \leq k^{r+1} \int_{k^{3/2} - 1}^{k^{3/2}} \frac{1}{y^{r+2}} \exp \left( -\frac{k^2}{y} \right) dy + \frac{k^{r+1}}{y_0^{r+2}} \exp \left( -\frac{k^2}{y_0} \right).
\]
The second term on the right of (6) is a constant multiple of \(k^{-r-3}\). The first does not exceed
\[
k^{r+1} \int_{0}^{\infty} \frac{1}{y^{r+2}} \exp \left( -\frac{k^2}{y} \right) dy.
\]
Putting \(y = k^2/w\), this becomes
\[
k^{-r-1} \int_{0}^{\infty} w^r e^{-w} dw = \Gamma(r + 1)k^{-r-1},
\]
hence the result.

REFERENCES


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