

Pacific Journal of Mathematics

ON ABSOLUTE DE LA VALLÉE POUSSIN SUMMABILITY

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Gronwall proved that $(C, r) \subseteq (V - P)$ for $r \geq 0$, where (C, r) and $(V - P)$ denote Cesàro and de la Vallée Poussin summability. It is proved in this paper that $|C, r| \subseteq |V - P|$ for $r \geq 0$.

1. Introduction. Let

$$V_n = \sum_{k=1}^n \frac{(n!)^2}{(n-k)!(n+k)!} a_k \quad (n \geq 0).$$

If $\lim_{n \rightarrow \infty} V_n = s$, we say that the series is summable $(V - P)$ to s .
 If

$$\sum_{n=1}^{\infty} |V_n - V_{n-1}| < \infty.$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|V - P|$.

Hyslop [2] proved that the $(V - P)$ method is equivalent to the $(A, 2)$ method defined by

$$\lim_{x \rightarrow 0^+} \sum_{n=0}^{\infty} a_n e^{-n^2 x} = s$$

for all series $\sum_{n=0}^{\infty} a_n$ which satisfy the condition $a_n = O(n^c)$, where c is any constant, and that the inclusion $(A, 2) \subseteq (V - P)$ is false without restriction.

Kuttner [3] has shown that $(V - P) \subseteq (A, 2)$ without restriction.

Gronwall [1] proved that $(C, r) \subseteq (V - P)$ for $r \geq 0$, where (C, r) denotes the Cesàro summability of order r .

In this paper, we shall prove

THEOREM A. $|C, r| \subseteq |V - P|$ for $r \geq 0$.

2. Proof of Theorem A. Since it is well-known that $|C, r|$ implies $|C, r'|$ for $-1 < r \leq r'$, it is enough to consider the case r an integer. Now, writing

$$V_n = v_0 + v_1 + \dots + v_n,$$

we find that

$$(1) \quad \begin{cases} v_0 = a_0, \\ v_n = \sum_{k=1}^n \frac{((n-1)!)^2}{(n-k)!(n+k)!} k^2 a_k \quad (n \geq 1). \end{cases}$$

Now write $\tau_k = \tau_k^r$ for the (C, r) mean of the sequence $\{ka_k\}$; thus the assumption that $\sum_{n=0}^{\infty} a_n$ is summable $|C, r|$ is equivalent to

$$(2) \quad \sum_{n=0}^{\infty} \frac{|\tau_n|}{n} < \infty .$$

If we take $((n - 1)!/(n - k!)(n + k)!$ as meaning 0 whenever $k > n$, we deduce from (1) by n partial summations that, for $n \geq 1$,

$$v_n = \sum_{k=1}^n \Delta_k^r \left\{ \frac{((n - 1)!^2 k}{(n - k)!(n + k)!} \right\} \binom{k + r}{k} \tau_k .$$

Now it is well-known that in order that the series-to-series transformation

$$b_n = \sum_{k=0}^{\infty} \alpha_{nk} a_k$$

should be that $\sum_{n=0}^{\infty} |b_n|$ converges whenever $\sum_{n=0}^{\infty} |a_n|$ does so, it is necessary and sufficient that

$$\sum_{n=0}^{\infty} |\alpha_{nk}|$$

should be bounded. Thus it is enough to show that, for $k \geq 1$,

$$(3) \quad \sum_{n=k}^{\infty} \left| \Delta_k^r \left\{ \frac{((n - k)!^2 k}{(n - k)!(n + k)!} \right\} \right| = O(k^{-r-1}) .$$

It is easily seen by induction on r that

$$\Delta_k^r \left\{ \frac{((n - 1)!^2 k}{(n - k)!(n + k)!} \right\} = \frac{A^r(n, k)((n - 1)!^2}{(n - k)!(n + k + r)!} ,$$

where $A^r(n, k)$ is defined inductively by

$$(4) \quad \begin{cases} A^0(n, k) = k , \\ A^{r+1}(n, k) = (n + k + r + 1)A^r(n, k) - (n - k)A^r(n, k + 1) . \end{cases}$$

Write $P_j(k)$ for a polynomial in k of degree not exceeding j , possibly different at each occurrence (thus $P_0(k)$ denotes a constant). We deduce from (4) by induction that

$$A^{2s}(n, k) = \sum_{j=0}^s P_{2j+1}(k) n^{s-j} ,$$

$$A^{2s+1}(n, k) = \sum_{j=0}^{s+1} P_{2j}(k) n^{s+1-j} .$$

Hence, uniformly in the ranges stated

$$A^r(n, k) = \begin{cases} O(n^{(r+1)/2}) & (1 \leq k \leq n^{1/2}), \\ O(K^{r+1}) & (n^{1/2} < k \leq n). \end{cases}$$

Next, for large n uniformly in $k \leq n^{2/3}$ we have, by Stirling's formula

$$\frac{(n!)^2}{(n-k)!(n+k)!} = O(H(n, k)),$$

where

$$H(n, k) = \left(1 - \frac{k}{n}\right)^{-n+k-1/2} \left(1 + \frac{k}{n}\right)^{-n-k-1/2}.$$

We have

$$\log H(n, k) = -\frac{k^2}{n} + O\left(\frac{k^3}{n^2}\right).$$

Now since we supposing that $k \leq n^{2/3}$ we have

$$(5) \quad \exp\left\{O\left(\frac{k^3}{n^2}\right)\right\} = O(1)$$

so that

$$\frac{(n!)^2}{(n-k)!(n+k)!} = O\left\{\exp\left(-\frac{k^2}{n}\right)\right\}.$$

This will not apply if $k > n^{2/3}$. Since we cannot then assert (5). However, for fixed n , $(n!)^2/(n-k)!(n+k)!$ is a decreasing function of k so that, for $k > n^{2/3}$,

$$\frac{(n!)^2}{(n-k)!(n+k)!} = O\{\exp(-n^{1/3})\}.$$

Also, it is trivial that

$$\frac{((n-1)!)^2}{(n-k)!(n+k+2)!} = \frac{(n!)^2}{(n-k)!(n+k)!} O(n^{-r-2}).$$

Combining these results, we find that

$$A_k^r \left\{ \frac{((n-1)!)^2 k}{(n-k)!(n+k)!} \right\} = \begin{cases} O(n^{-(r+3)/2}) & (1 \leq k \leq n^{1/2}), \\ O\left(\frac{k^{r+1}}{n^{r+2}} \exp\left(-\frac{k^2}{n}\right)\right) & (n^{1/2} < k \leq n^{2/3}), \\ O(n^{-1} \exp(-n^{-(1/3)})) & (n^{2/3} < k \leq n). \end{cases}$$

Thus the sum (3) is

$$O\left\{\sum_{k \leq n < k^{3/2}} \frac{1}{n} \exp(-n^{-(1/3)})\right\} + O\left\{\sum_{k^{3/2} \leq n < k^2} \frac{k^{r+1}}{n^{r+2}} \exp\left(-\frac{k^2}{n}\right)\right\} + O\left\{\sum_{n \geq k^2} \frac{1}{n^{(r+3)/2}}\right\}$$

$$= O(I_1) + O(I_2) + O(I_3),$$

say. It is clear that

$$I_1 = O(k^{-r-1}),$$

$$I_3 = O(k^{-r-1})$$

so we need consider only I_2 . Now for fixed k

$$\frac{k^{r+1}}{y^{r+2}} \exp\left(-\frac{k^2}{y}\right)$$

is increasing for $y < y_0$ and decreasing for $y > y_0$, where $y_0 = y_0(k) = k^2/(r + 2)$. Hence

$$(6) \quad I_2 \leq k^{r+1} \int_{k^{3/2-1}}^{k^2+1} \frac{1}{y^{r+2}} \exp\left(-\frac{k^2}{y}\right) dy + \frac{k^{r+1}}{y_0^{r+2}} \exp\left(-\frac{k^2}{y_0}\right).$$

The second term on the right of (6) is a constant multiple of k^{-r-3} . The first does not exceed

$$k^{r+1} \int_0^\infty \frac{1}{y^{r+2}} \exp\left(-\frac{k^2}{y}\right) dy.$$

Putting $y = k^2/w$, this becomes

$$k^{-r-1} \int_0^\infty w^r e^{-w} dw = \Gamma(r + 1)k^{-r-1},$$

hence the result.

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