

# Pacific Journal of Mathematics

**A SEPARABLY CLOSED RING WITH NONZERO TORSION PIC**

ANDY R. MAGID

## A SEPARABLY CLOSED RING WITH NONZERO TORSION PIC

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**We give an example of a ring and a rank one projective module over that ring such that the square of the module is free but the module does not become free over any separable extension of the ring.**

Every ideal class in the ring of integers in a number field can be split by an unramified extension. Over a commutative ring which is an algebra over the rationals every torsion element of Pic of the ring is split by a separable extension [3]. These examples suggest the question: is the torsion part of Pic of a separably closed ring trivial? We will exhibit a ring which shows the answer is negative. The ring arises as a slight modification of an example of Swan [5].

For any commutative ring  $k$ ,  $k^\times$  denotes the group of units of  $k$ ,  $Cl(k)$  denotes the divisor class group if  $k$  is a domain, and  $Qu(k)$  denotes the group of quadratic extensions of  $k$ . We use  $Z$  for the integers and  $Q$  for the rationals.

**DEFINITION.** For any commutative ring  $k$ , let  $k(S^1) = k[X_0, X_1]/(X_0^2 + X_1^2 - 1)$ . Let  $t_i$  be the image of  $X_i$  in  $k(S^1)$ .  $k(S^1)$  is graded mod 2; let  $k(P^1)$  be the even graded piece and  $L(k)$  the odd.

**LEMMA 1.**  $L(k)$  is a projective  $k(P^1)$ -module of rank 1 whose tensor square is free.

*Proof.* It suffices to check the first assertion for  $k = Z$ . By the argument in [5, p. 271]  $L(Z)$  is projective of rank 1. The multiplication in  $k(S^1)$  defines a homomorphism

$$L(k) \otimes_{k(P^1)} L(k) \rightarrow k(P^1)$$

whose image contains  $t_0^2 + t_1^2 = 1$  and is thus an isomorphism.

We will show that  $L(Z)$  cannot be split by a separable extension of  $Z(P^1)$ . We begin by collecting some facts about the rings involved.

- LEMMA 2.** (a)  $Q(i)(P^1) = Q(i)[v, v^{-1}]$  where  $v = (t_0 + it_1)^2$   
 (b)  $Q(P^1)^x = Q^x$   
 (c)  $Z/2Z(P^1)$  is a polynomial ring (in one variable) over  $Z/2Z$   
 (d)  $L(Z/2Z)$  is freely generated by  $t_0 + t_1$

*Proof.* Let  $K = Q(i)$ . Then  $K(S^1) = K[u, u^{-1}]$  where  $u = t_0 + it_1$ .

$K[v, v^{-1}]$  is contained in  $K(P^1)$  and  $K[u, u^{-1}]$  is separable of rank two over both rings. Thus  $K[v, v^{-1}] = K(P^1)$ ; this gives (a). Let  $g$  be the automorphism of  $K(P^1)$  defined by  $g(i) = -i$  (so  $g(v) = v^{-1}$ ). Then the ring of  $g$ -invariants is  $Q(P^1)$ . Since  $K(P^1)^x = K^x\{v\}$ , the only invariant units are in  $Q^x$ , whence (b). For (c), we observe that  $Z/2Z[X_0, a]$  where  $a^2 = 1$  is isomorphic to  $Z/2Z(S^1)$  when  $X_0$  is sent to  $t_0$  and  $a$  to  $t_0 + t_1$ . This isomorphism is graded and under it,  $Z/2Z(P^1)$  corresponds to  $Z/2Z[X_0, a]$ . For (d), let  $f_i = t_i(t_0 + t_1)$ . Then  $f_i(t_0 + t_1) = t_i$ , so  $t_0 + t_1$  generates  $L(Z/2Z)$ .

We remark that (c) implies that  $Z/2Z(P^1)^x = \{1\}$  and (b) implies that  $Z(P^1)^x = \{\pm 1\}$ .

**PROPOSITION 3.**  $Z(P^1)$  is a normal domain with  $\text{Pic}(Z(P^1)) = \{1, L(Z)\}$ .

*Proof.* We begin by establishing analogous assertions for  $Q(P^1)$ . Let  $K, v, g$  be as in the proof of Lemma 2 and let  $G = \{1, g\}$ . Then  $K(P^1)$  is a Galois extension of  $Q(P^1)$  with group  $G$ . Let  $U = X(P^1)^x$ . Since by Lemma 2 (a)  $K(P^1)$  is a UFD,  $Q(P^1)$  is normal and there is an isomorphism  $\text{Cl}(Q(P^1)) \rightarrow H^1(G, U)$  [4, p. 55]. To compute the latter group, we use the exact sequence of  $G$ -modules

$$1 \longrightarrow V \longrightarrow U \longrightarrow K^x \longrightarrow 1$$

where  $V$  is the subgroup generated by  $v$ . Since by Lemma 2 (b)  $U^G = Q^x = (K^x)^G$  and since  $H^1(G, K^x) = 1$  by Hilbert's Theorem 90, the cohomology sequence of the above sequence shows that  $H^1(G, U)$  and  $H^1(G, V)$  are isomorphic. Since  $V$  is  $G$ -isomorphic with  $Z$  with  $g$  acting by multiplication by  $-1$ , we see that  $H^1(G, V)$  (and hence  $\text{Cl}(Q(P^1))$ ) is of order two. Now let  $S$  be the multiplicative set in  $Z(P^1)$  generated by the integer primes. The integer primes remain prime in  $Z(P^1)$  (for the odd primes this is trivial and for two it follows from Lemma 2 (c)). Since  $S^{-1}Z(P^1) = Q(P^1)$  it follows from [4, p. 21] that  $Z(P^1)$  is a normal domain with  $\text{Cl}(Z(P^1)) = \text{Cl}(Q(P^1))$ . By [5, Thm. 4, p. 271]  $L(Z)$  cannot be generated by a single element. It follows that  $\{1, L(Z)\} = \text{Pic}(Z(P^1)) = \text{Cl}(Z(P^1))$ .

Next, we show that every connected Galois extension of  $Z(P^1)$  is abelian.

**LEMMA 4.** Let  $K$  be an algebraically closed field of characteristic zero. Then every connected Galois extension of  $K[X, X^{-1}]$  is cyclic.

*Proof.* We may assume  $K$  is the complex numbers. Let  $T$  be a connected Galois extension of  $K[X, X^{-1}]$  and let  $E$  be the quotient field of  $T$ . Let  $M$  and  $P$  be the Riemann surfaces of  $E$  and  $K(X)$

respectively ( $P$  is just the Riemann sphere, of course). The inclusion of  $K(X)$  in  $E$  displays  $M$  as a local branched covering of  $P$  ramified only above  $0$  and  $\infty$ . The branching order formula [3, Cor. 3, p. 225] shows that  $M$  has genus zero and only one branch point over each of  $0$  and  $\infty$ . It follows that the covering map is the  $n^{\text{th}}$  power map, where  $n = [T: K[X, X^{-1}]]$  and hence that  $T = K[X, X^{-1}](\sqrt[n]{X})$ . Then the Galois group of  $T$  is cyclic of order  $n$ .

**PROPOSITION 5.** *Every connected Galois extension of  $Z(P^1)$  is cyclic.*

*Proof.* Let  $K$  be the algebraic closure of  $Q$  and  $R$  the ring of all algebraic integers. Let  $T$  be a connected Galois extension of  $Z(P^1)$  with group  $G$ . Then  $T \otimes_Z R$  is a Galois extension of  $R(P^1)$ , and hence is a product of copies of a connected Galois extension  $T_0$  of  $R(P^1)$ . The Galois group  $H$  of  $T_0$  is a subgroup of  $G$ .  $T_0 \otimes_R K$  is a connected Galois extension of  $K(S^1) = K[v, v^{-1}]$  and hence by Lemma 4  $H$  is cyclic. Choose a homomorphism (necessarily an injection) of  $T$  into  $T_0$ . Then  $T^H$  is a separable  $Z(P^1)$ -subalgebra of  $T_0^H = R(P^1)$ . Thus  $T^H$  is contained in  $S(P^1)$  where  $S$  is the ring of integers in some finite extension of  $Q$ . Since  $S(P^1)$  is  $Z(P^1)$ -projective and  $T^H$  is  $Z(P^1)$ -separable,  $T^H$  is a  $T^H$ -summand of  $S(P^1)$ . Let  $f$  denote the evaluation  $Z(S^1) \rightarrow Z$  (and also its restriction to  $Z(P^1)$ ) at the point  $(1, 0)$ . Then  $T^H \otimes_f Z$  is a separable  $Z$ -subalgebra of  $S(P^1) \otimes_f Z = S$ . It follows that  $T^H \otimes_f Z = Z$  and hence  $Z(P^1) = T^H$ .

**THEOREM 6.**  *$L(Z)$  cannot be split by a separable extension of  $Z(P^1)$ .*

*Proof.* It will suffice to prove the theorem for connected Galois extensions; let  $T$  be such an extension with group  $G$ . If  $T$  splits  $L(Z)$  then by Lemma 3  $\text{Pic}(Z(P^1)) = H^1(G, T^X)$ . It follows that  $G$  has even order. Since by Proposition 5  $G$  is cyclic, it will suffice to show that  $Z(P^1)$  has no connected quadratic extensions. Since for any normal domain  $k$  with quotient field  $L$  the map  $Qu(k) \rightarrow Qu(L)$  is injective,  $Qu(Z(P^1))$  is contained in  $Qu(Q(P^1))$ . To compute this latter group, we use the exact sequence of [1, p. 129] valid for any ring  $k$  containing  $1/2$ :

$$1 \longrightarrow k^X / (K^X)^2 \longrightarrow Qu(k) \longrightarrow 2\text{-Pic}(k) \longrightarrow 1$$

where the first map sends  $a$  to  $k[X]/(X^2 - a)$ , the second sends  $T$  to  $T/k$  and the fourth group in the sequence is the two-torsion part of  $\text{Pic}$ . Let  $T$  be a quadratic extension of  $Z(P^1)$  and let  $I = T/Z(P^1)$ . Let  $T_0 = T \otimes_Z Q$  and let  $I_0 = T_0/Q(P^1)$ . If  $I = 1$  and thus also  $I_0 = 1$ , the above exact sequence shows that  $T_0 = Q(\sqrt{a})(P^1)$  for some  $a$

in  $Q$ , since by Lemma 2 (b)  $Q(P^1)^x = Q^x$ . Let  $f: Z(P^1) \rightarrow Z$  and  $h: Q(P^1) \rightarrow Q$  be induced by evaluation at  $(1, 0)$ . Then  $T \otimes_f Z \otimes_z Q = T_0 \otimes_k Q$ , since the first is  $Q \times Q$  and the second  $Q(\sqrt{a})$ ,  $a$  is in  $Q$  and  $T_0$  and therefore  $T$  must be the trivial extension. To treat the case  $I = L(Z)$  we use the following exact sequence, which is part of a sequence due to Small [6]: for any ring  $k$ ,

$$Qu(k) \longrightarrow \text{Pic}_{(2)}(k) \longrightarrow U'(k)$$

where the middle group is those two-torsion elements of  $\text{Pic}(k)$  which become free over  $k/2k$  and the end group is  $(k/4k)^x$  modulo the subgroup generated by the squares and the image of  $k^x$ ; the first map sends  $T$  to  $T/k$  and the second sends  $I$  to the class of  $g(m, m)$  where  $g: I \otimes_k I \rightarrow I$  is an isomorphism and  $m$  is in  $I$  with the image of  $m$  becoming a basis for  $I/2I$ . By Lemma 2 and the remarks following we see that  $U'(Z(P^1)) = Z/4Z(P^1)^x/\{\pm 1\}$ , and by Lemma 2d and Proposition 3 we see that  $\text{Pic}_{(2)}(Z(P^1)) = \{1, L(Z)\}$ . To compute the image of  $L(Z)$  in  $U'(Z(P^1))$  we choose for  $g$  the multiplication map (Lemma 1) and for  $m$  the element  $t_0 + t_1$  (Lemma 2 (d)). Then  $g(m, m) = 1 + 2t_0t_1$  has non-trivial image in  $Z/4Z(P^1)$ . Thus the case  $I = L(Z)$  does not occur.

We conclude with some remarks: one can define  $k(P^n)$  and  $k(S^n)$  in a similar manner for  $n$  larger than 1. The arguments given here can be extended to cover these rings, except for Lemma 4. Presumably the analogue of Theorem 6 remains valid, however.

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