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COMMUTANTS OF SOME HAUSDORFF MATRICES

BILLY E. RHOADES

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Let $B(c)$ denote the Banach algebra of bounded operators over c , the space of convergent sequences. Let Γ and Δ denote the subalgebras of $B(c)$ consisting, respectively, of conservative and conservative triangular infinite matrices, and C the Cesaro matrix of order one. In this paper we investigate $\text{Com}(C)$ in Γ and $B(c)$, $\text{Com}(H)$ in Γ and $B(c)$ for certain Hausdorff matrices H , and some related questions.

Let $B(c)$ denote the Banach algebra of bounded operators over c , the space of convergent sequences. Let Γ and Δ denote the subalgebras of $B(c)$ consisting, respectively, of conservative and conservative triangular infinite matrices. It is well known (see, e.g. [3, p. 77]) that the commutant of C , the Cesaro matrix of order one, in Δ is the family \mathcal{H} of conservative Hausdorff matrices. The same proof yields the result that if H is any conservative Hausdorff triangle with distinct diagonal elements, then $\text{Com}(H) = \mathcal{H}$ in Δ . In this paper we investigate $\text{Com}(C)$ in Γ and $B(c)$, $\text{Com}(H)$ in Γ and $B(c)$ for certain Hausdorff matrices H , and some related questions.

The spaces of bounded, convergent, and absolutely convergent sequences shall be denoted by m , c , and l . U will denote the unilateral shift, and we shall use $A \leftrightarrow B$ to indicate that the operators A and B commute. An infinite matrix A is said to be triangular if it has only zero entries above the main diagonal, and a triangle if it is triangular and has no zeros on the main diagonal. An infinite matrix A is conservative; i.e., $A: c \rightarrow c$ if and only if

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \quad a_k = \lim_n a_{nk}$$

exists for each k , and $\lim_n \sum_k a_{nk}$ exists.

The proof [2, p. 249] that $\text{Com}(C) = \mathcal{H}$ in Δ , uses the associativity of matrix multiplication. If $\text{Com}(C)$ is to remain unchanged in the larger algebra Γ , it is necessary that $\text{Com}(C)$ contain only triangular matrices. We are thus led to the following result, where e_k denotes the coordinate sequence with a 1 in the k th position and zeros elsewhere.

THEOREM 1. *Let A be a conservative triangle, B an infinite matrix with finite norm, $B \leftrightarrow A$. Then B is triangular if and only if*

$$(1) \quad t(A - a_{nn}I) = 0$$

and $t \in l$ imply t lies in the span of $(e_0, e_1, \dots, e_n), n = 0, 1, 2, \dots$.

The conditions in (1) are merely a reformulation of the fact that B is triangular. For, if $B \leftrightarrow A$, then we obtain the system

$$(2) \quad \sum_{j=k}^{\infty} b_{nj}a_{jk} = \sum_{j=0}^n a_{nj}b_{jk}; \quad n, k = 0, 1, 2, \dots$$

Define $t^n = \{b_{nk}\}_{k=0}^{\infty}, n = 0, 1, 2, \dots$; i.e., t^n is the n -th row of B . With $n = 0$, (2) can be written in the form $t^0(A - a_{00}I) = 0$. Thus $b_{0k} = 0$ for $k > 0$. By induction, one can then show that $b_{nk} = 0$ for $k > n$, and hence B is triangular.

To prove the converse, suppose (1) fails to hold for all n . Let N be the smallest such n . Then (1) has a nonzero solution outside the span of (e_0, e_1, \dots, e_N) and B is not triangular.

A matrix A is said to be of type M if it is not a right zero divisor over l : i.e., $tA = 0$ and $t \in l$ imply $t = 0$. Therefore, an equivalent formulation of (1) is that $(U^*)^{n+1}(A - a_{nn}I)U^{n+1}$ be of type M for each $n = 0, 1, 2, \dots$.

Let \mathcal{D} denote the set of conservative Hausdorff triangles with distinct diagonal entries, \mathcal{A} the algebra of all matrices with finite norm.

COROLLARY 1. *Let $H \in \mathcal{D}$. Then $\text{Com}(H)$ in $\Delta = \text{Com}(H)$ in $\Gamma = \text{Com}(H)$ in $\mathcal{A} = \mathcal{H}$ if and only if (1) is satisfied.*

The last equality follows from the fact that every Hausdorff matrix with finite norm is automatically conservative.

A matrix A is said to be factorable if $a_{nk} = c_n d_k$ for each n and k . Examples of factorable triangular matrices are C , the Hausdorff matrices generated by $\{a/(n + a)\}$ for $a > 0$, and the weighted mean methods (see [2, p. 57]).

THEOREM 2. *If A is a factorable triangle and $B \leftrightarrow A$ then B is triangular.*

Proof. Set $n = k = 0$ in (2) to get

$$(3) \quad \sum_{j=1}^{\infty} b_{0j}a_{j0} = 0.$$

From (2) with $n = 0, k = 1$, we have

$$a_{00}b_{01} = \sum_{j=1}^{\infty} b_{0j}a_{j1} = \sum_{j=1}^{\infty} b_{0j}c_j d_1 = (d_1/d_0) \sum_{j=1}^{\infty} b_{0j}a_{j0}.$$

Since $a_{00} \neq 0$, $b_{01} = 0$ from (3). By induction one can show that $b_{nk} = 0$ for $k > n$.

COROLLARY 2. $\text{Com}(C)$ in $\mathcal{A} = \text{Com}(C)$ in $\Gamma = \text{Com}(C)$ in $\mathcal{A} = \mathcal{H}$.

Corollary 2 follows immediately from Theorem 2 since C is factorable.

COROLLARY 3. If $A \in \mathcal{A}$, is factorable, and has exactly one zero on the main diagonal, then $B \leftrightarrow A$ implies B is triangular.

Proof. Let N be such that $a_{NN} = 0$. If $N > 0$, then the proof of Theorem 2 forces $b_{nk} = 0$ for $k > n$, $n < N$. For $k > N$, $n = N$ in (2) we have

$$\sum_{j=k}^{\infty} b_{nj}a_{jk} = \sum_{j=0}^N a_{Nj}b_{jk} = a_{NN}b_{Nk} = 0,$$

or

$$-b_{Nk}c_k = \sum_{j=k+1}^{\infty} b_{Nj}c_j,$$

since $d_k \neq 0$ for $k > N$. The above equation leads to $b_{Nk}c_k = 0$ which implies $b_{Nk} = 0$. By induction, $b_{nk} = 0$ for $n > N$, $k > n$.

If a factorable triangular matrix A contains at least two zeros on the main diagonal, then $\text{Com}(A)$ in \mathcal{A} need not equal $\text{Com}(A)$ in Γ . This fact is a special case of the following. A necessary condition for any conservative triangle A to satisfy $\text{Com}(A)$ in $\mathcal{A} = \text{Com}(A)$ in Γ is that A have distinct diagonal entries. For, suppose there exist integers i, k , $k > i \geq 0$ such that $a_{ii} = a_{kk}$. Then the matrix $(U^*)^{i+1}(A - a_{ii}I)U^{i+1}$ has a zero on the main diagonal in the $(k - i)$ th position and is therefore not of type M .

A necessary condition, therefore, for a conservative Hausdorff matrix H to satisfy $\text{Com}(H)$ in $\mathcal{A} = \text{Com}(H)$ in Γ is that H have distinct diagonal entries. The condition, however, is not sufficient. Let $A = H + \lambda K$ where H is the Hausdorff matrix generated by $\mu_n = (n - a)/(-a)(n + 1)$, $a > 0$, K is the compact Hausdorff matrix generated by $\mu_0 = 1$, $\mu_n = 0$, $n > 0$, and λ is any real number satisfying $-(a + 1)/a < \lambda < 0$. We shall show that $B = U^*(A - a_{00}I)U$ is not of type M . Thus $\text{Com}(A)$ in Γ will contain nontriangular matrices.

Let D be the Hausdorff matrix generated be

$$\nu_n = \frac{\lambda(n - \varepsilon)}{-\varepsilon(n + 1)}, \quad \text{where } \varepsilon = \lambda/\delta, \delta = -\lambda - 1 - 1/a.$$

Since $a_{00} = 1 + \lambda$, a straightforward calculation verifies that D and $A - a_{00}I$ agree, except for terms in the first column. B is obtained by removing the first row and first column from $A - a_{00}I$. Therefore $B = U^*DU$. By Theorem 1 of [4], D is not of type M , and a suitable sequence t is $t_0 = 1, t_n = (-1)^n \varepsilon(\varepsilon - 1) \cdots (\varepsilon - n + 1)/n! \quad n > 0$. Therefore B is also not of type M .

For $\text{Com}(H)$ in Δ to equal $\text{Com}(H)$ in Γ it is not necessary that the Hausdorff matrix H be a triangle. Set $H = \bar{H} - \mu_0 I$, when \bar{H} is any conservative Hausdorff matrix such that $\text{Com}(\bar{H})$ in $\Delta = \text{Com}(\bar{H})$ in Γ .

We shall now examine $\text{Com}(C)$ in $B(c)$.

Let e denote the sequence of all ones. If $T \in B(c)$ then one can define continuous linear functionals χ and χ_i by $\chi(T) = \lim T e - \sum_k \lim (T e_k)$ and $\chi_i(T) = (T e)_i - \sum_k (T e_k)_i, \quad i = 1, 2, \dots$. (See, e.g., [5, p. 241].) It is known [1, p. 8] that any $T \in B(c)$ has the representation

$$(4) \quad T x = v \lim x + B x \quad \text{for each } x \in c,$$

where B is the matrix representation of the restriction of T to c_0 and v is the bounded sequence $v = \{\chi_i(T)\}$.

The second adjoint of T has the matrix representation

$$(5) \quad T^{**} = \begin{pmatrix} \chi(T) & a_1 & a_2 & \cdots \\ \chi_1(T) & b_{11} & b_{12} & \cdots \\ \chi_2(T) & b_{21} & b_{22} & \cdots \\ \dots\dots\dots \end{pmatrix}$$

where the a_i 's occur in the representation of

$$\lim \circ T \in c^* \quad \text{as} \quad (\lim T)(x) = \lim (T x) = \chi(T) \lim x + \sum_k a_k x_k.$$

See, e.g., [6, p. 357].

For the matrix C , each $\chi_i(C) = 0$, [5, p. 241] and $\chi(C) = 1$, so that

$$(6) \quad C^{**} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \dots\dots\dots \end{pmatrix}.$$

Since $C \leftrightarrow T$ if and only if $C^{**} \leftrightarrow T^{**}$, we may use (5) and (6) to obtain $(C^{**} T^{**})_{00} = (T^{**} C^{**})_{00} = \chi(T)$, and, for $n > 0$,

$$(7) \quad (C^{**} T^{**})_{n0} = \frac{1}{n} \sum_{k=1}^n \chi_k(T) = \chi_n(T) = (T^{**} C^{**})_{n0}.$$

The system (7) yields $\chi_n(T) = \chi_1(T)$, $n = 1, 2, 3, \dots$. Thus $v = \chi_1(T)e$. Substituting in (4) with $\chi \in c_0$ we see that c must commute with B . Since B is a matrix and $B \in \mathcal{A}$, we may use Corollary 2 to obtain the following result.

THEOREM 3. *Let $T \in B(c)$. Then $T \leftrightarrow C$ if and only if T has the form (4) with $v = \chi_1(T)e$ and $B \in \mathcal{H}$.*

Note added in proof. The hypotheses of Theorem 1 can be modified without changing the details of the proof. For example, if A and B are any two bounded operators over l^p , $p > 1$, then the conclusion of Theorem 1 holds. In particular, since $C \in B(l^p)$ for $p > 1$, we get as a corollary that $\text{Com}(C)$ in $B(l^p)$ consists only of those Hausdorff matrices that belong to $B(l^p)$. Another description of $\text{Com}(C)$ in $B(l^p)$ appears in A. Shields and L. Wallen [Indiana Univ. Math. J., 20 (1971) 777-788].

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