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**COMMUTANTS OF SOME HAUSDORFF MATRICES**

BILLY E. RHOADES

## COMMUTANTS OF SOME HAUSDORFF MATRICES

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Let  $B(c)$  denote the Banach algebra of bounded operators over  $c$ , the space of convergent sequences. Let  $\Gamma$  and  $\Delta$  denote the subalgebras of  $B(c)$  consisting, respectively, of conservative and conservative triangular infinite matrices, and  $C$  the Cesaro matrix of order one. In this paper we investigate  $\text{Com}(C)$  in  $\Gamma$  and  $B(c)$ ,  $\text{Com}(H)$  in  $\Gamma$  and  $B(c)$  for certain Hausdorff matrices  $H$ , and some related questions.

Let  $B(c)$  denote the Banach algebra of bounded operators over  $c$ , the space of convergent sequences. Let  $\Gamma$  and  $\Delta$  denote the subalgebras of  $B(c)$  consisting, respectively, of conservative and conservative triangular infinite matrices. It is well known (see, e.g. [3, p. 77]) that the commutant of  $C$ , the Cesaro matrix of order one, in  $\Delta$  is the family  $\mathcal{H}$  of conservative Hausdorff matrices. The same proof yields the result that if  $H$  is any conservative Hausdorff triangle with distinct diagonal elements, then  $\text{Com}(H) = \mathcal{H}$  in  $\Delta$ . In this paper we investigate  $\text{Com}(C)$  in  $\Gamma$  and  $B(c)$ ,  $\text{Com}(H)$  in  $\Gamma$  and  $B(c)$  for certain Hausdorff matrices  $H$ , and some related questions.

The spaces of bounded, convergent, and absolutely convergent sequences shall be denoted by  $m$ ,  $c$ , and  $l$ .  $U$  will denote the unilateral shift, and we shall use  $A \leftrightarrow B$  to indicate that the operators  $A$  and  $B$  commute. An infinite matrix  $A$  is said to be triangular if it has only zero entries above the main diagonal, and a triangle if it is triangular and has no zeros on the main diagonal. An infinite matrix  $A$  is conservative; i.e.,  $A: c \rightarrow c$  if and only if

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \quad a_k = \lim_n a_{nk}$$

exists for each  $k$ , and  $\lim_n \sum_k a_{nk}$  exists.

The proof [2, p. 249] that  $\text{Com}(C) = \mathcal{H}$  in  $\Delta$ , uses the associativity of matrix multiplication. If  $\text{Com}(C)$  is to remain unchanged in the larger algebra  $\Gamma$ , it is necessary that  $\text{Com}(C)$  contain only triangular matrices. We are thus led to the following result, where  $e_k$  denotes the coordinate sequence with a 1 in the  $k$ th position and zeros elsewhere.

**THEOREM 1.** *Let  $A$  be a conservative triangle,  $B$  an infinite matrix with finite norm,  $B \leftrightarrow A$ . Then  $B$  is triangular if and only if*

$$(1) \quad t(A - a_{nn}I) = 0$$

and  $t \in l$  imply  $t$  lies in the span of  $(e_0, e_1, \dots, e_n)$ ,  $n = 0, 1, 2, \dots$ .

The conditions in (1) are merely a reformulation of the fact that  $B$  is triangular. For, if  $B \leftrightarrow A$ , then we obtain the system

$$(2) \quad \sum_{j=k}^{\infty} b_{nj}a_{jk} = \sum_{j=0}^n a_{nj}b_{jk}; \quad n, k = 0, 1, 2, \dots$$

Define  $t^n = \{b_{nk}\}_{k=0}^{\infty}$ ,  $n = 0, 1, 2, \dots$ ; i.e.,  $t^n$  is the  $n$ -th row of  $B$ . With  $n = 0$ , (2) can be written in the form  $t^0(A - a_{00}I) = 0$ . Thus  $b_{0k} = 0$  for  $k > 0$ . By induction, one can then show that  $b_{nk} = 0$  for  $k > n$ , and hence  $B$  is triangular.

To prove the converse, suppose (1) fails to hold for all  $n$ . Let  $N$  be the smallest such  $n$ . Then (1) has a nonzero solution outside the span of  $(e_0, e_1, \dots, e_N)$  and  $B$  is not triangular.

A matrix  $A$  is said to be of type  $M$  if it is not a right zero divisor over  $l$ : i.e.,  $tA = 0$  and  $t \in l$  imply  $t = 0$ . Therefore, an equivalent formulation of (1) is that  $(U^*)^{n+1}(A - a_{nn}I)U^{n+1}$  be of type  $M$  for each  $n = 0, 1, 2, \dots$ .

Let  $\mathcal{D}$  denote the set of conservative Hausdorff triangles with distinct diagonal entries,  $\mathcal{A}$  the algebra of all matrices with finite norm.

**COROLLARY 1.** *Let  $H \in \mathcal{D}$ . Then  $\text{Com}(H)$  in  $\Delta = \text{Com}(H)$  in  $\Gamma = \text{Com}(H)$  in  $\mathcal{A} = \mathcal{H}$  if and only if (1) is satisfied.*

The last equality follows from the fact that every Hausdorff matrix with finite norm is automatically conservative.

A matrix  $A$  is said to be factorable if  $a_{nk} = c_n d_k$  for each  $n$  and  $k$ . Examples of factorable triangular matrices are  $C$ , the Hausdorff matrices generated by  $\{a/(n+a)\}$  for  $a > 0$ , and the weighted mean methods (see [2, p. 57]).

**THEOREM 2.** *If  $A$  is a factorable triangle and  $B \leftrightarrow A$  then  $B$  is triangular.*

*Proof.* Set  $n = k = 0$  in (2) to get

$$(3) \quad \sum_{j=1}^{\infty} b_{0j}a_{j0} = 0.$$

From (2) with  $n = 0$ ,  $k = 1$ , we have

$$a_{00}b_{01} = \sum_{j=1}^{\infty} b_{0j}a_{j1} = \sum_{j=1}^{\infty} b_{0j}c_j d_1 = (d_1/d_0) \sum_{j=1}^{\infty} b_{0j}a_{j0}.$$

Since  $a_{00} \neq 0$ ,  $b_{01} = 0$  from (3). By induction one can show that  $b_{nk} = 0$  for  $k > n$ .

**COROLLARY 2.**  $\text{Com}(C)$  in  $\mathcal{A} = \text{Com}(C)$  in  $\Gamma = \text{Com}(C)$  in  $\mathcal{A} = \mathcal{H}$ .

Corollary 2 follows immediately from Theorem 2 since  $C$  is factorable.

**COROLLARY 3.** If  $A \in \mathcal{A}$ , is factorable, and has exactly one zero on the main diagonal, then  $B \leftrightarrow A$  implies  $B$  is triangular.

*Proof.* Let  $N$  be such that  $a_{NN} = 0$ . If  $N > 0$ , then the proof of Theorem 2 forces  $b_{nk} = 0$  for  $k > n$ ,  $n < N$ . For  $k > N$ ,  $n = N$  in (2) we have

$$\sum_{j=k}^{\infty} b_{nj} a_{jk} = \sum_{j=0}^N a_{Nj} b_{jk} = a_{NN} b_{Nk} = 0,$$

or

$$-b_{Nk} c_k = \sum_{j=k+1}^{\infty} b_{Nj} c_j,$$

since  $d_k \neq 0$  for  $k > N$ . The above equation leads to  $b_{Nk} c_k = 0$  which implies  $b_{Nk} = 0$ . By induction,  $b_{nk} = 0$  for  $n > N$ ,  $k > n$ .

If a factorable triangular matrix  $A$  contains at least two zeros on the main diagonal, then  $\text{Com}(A)$  in  $\mathcal{A}$  need not equal  $\text{Com}(A)$  in  $\Gamma$ . This fact is a special case of the following. A necessary condition for any conservative triangle  $A$  to satisfy  $\text{Com}(A)$  in  $\mathcal{A} = \text{Com}(A)$  in  $\Gamma$  is that  $A$  have distinct diagonal entries. For, suppose there exist integers  $i, k$ ,  $k > i \geq 0$  such that  $a_{ii} = a_{kk}$ . Then the matrix  $(U^*)^{i+1}(A - a_{ii}I)U^{i+1}$  has a zero on the main diagonal in the  $(k - i)$ th position and is therefore not of type  $M$ .

A necessary condition, therefore, for a conservative Hausdorff matrix  $H$  to satisfy  $\text{Com}(H)$  in  $\mathcal{A} = \text{Com}(H)$  in  $\Gamma$  is that  $H$  have distinct diagonal entries. The condition, however, is not sufficient. Let  $A = H + \lambda K$  where  $H$  is the Hausdorff matrix generated by  $\mu_n = (n - a)/(-a)(n + 1)$ ,  $a > 0$ ,  $K$  is the compact Hausdorff matrix generated by  $\mu_0 = 1$ ,  $\mu_n = 0$ ,  $n > 0$ , and  $\lambda$  is any real number satisfying  $-(a + 1)/a < \lambda < 0$ . We shall show that  $B = U^*(A - a_{00}I)U$  is not of type  $M$ . Thus  $\text{Com}(A)$  in  $\Gamma$  will contain nontriangular matrices.

Let  $D$  be the Hausdorff matrix generated be

$$\nu_n = \frac{\lambda(n - \varepsilon)}{-\varepsilon(n + 1)}, \quad \text{where } \varepsilon = \lambda/\delta, \delta = -\lambda - 1 - 1/a.$$

Since  $a_{00} = 1 + \lambda$ , a straightforward calculation verifies that  $D$  and  $A - a_{00}I$  agree, except for terms in the first column.  $B$  is obtained by removing the first row and first column from  $A - a_{00}I$ . Therefore  $B = U^*DU$ . By Theorem 1 of [4],  $D$  is not of type  $M$ , and a suitable sequence  $t$  is  $t_0 = 1, t_n = (-1)^n \varepsilon (\varepsilon - 1) \cdots (\varepsilon - n + 1)/n! \quad n > 0$ . Therefore  $B$  is also not of type  $M$ .

For  $\text{Com}(H)$  in  $\Delta$  to equal  $\text{Com}(H)$  in  $\Gamma$  it is not necessary that the Hausdorff matrix  $H$  be a triangle. Set  $H = \bar{H} - \mu_0 I$ , when  $\bar{H}$  is any conservative Hausdorff matrix such that  $\text{Com}(\bar{H})$  in  $\Delta = \text{Com}(\bar{H})$  in  $\Gamma$ .

We shall now examine  $\text{Com}(C)$  in  $B(c)$ .

Let  $e$  denote the sequence of all ones. If  $T \in B(c)$  then one can define continuous linear functionals  $\chi$  and  $\chi_i$  by  $\chi(T) = \lim T e - \sum_k \lim (T e_k)$  and  $\chi_i(T) = (T e)_i - \sum_k (T e_k)_i, \quad i = 1, 2, \dots$ . (See, e.g., [5, p. 241].) It is known [1, p. 8] that any  $T \in B(c)$  has the representation

$$(4) \quad T x = v \lim x + B x \quad \text{for each } x \in c,$$

where  $B$  is the matrix representation of the restriction of  $T$  to  $c_0$  and  $v$  is the bounded sequence  $v = \{\chi_i(T)\}$ .

The second adjoint of  $T$  has the matrix representation

$$(5) \quad T^{**} = \begin{pmatrix} \chi(T) & a_1 & a_2 & \cdots \\ \chi_1(T) & b_{11} & b_{12} & \cdots \\ \chi_2(T) & b_{21} & b_{22} & \cdots \\ \dots\dots\dots \end{pmatrix}$$

where the  $a_i$ 's occur in the representation of

$$\lim \circ T \in c^* \quad \text{as} \quad (\lim T)(x) = \lim (T x) = \chi(T) \lim x + \sum_k a_k x_k.$$

See, e.g., [6, p. 357].

For the matrix  $C$ , each  $\chi_i(C) = 0$ , [5, p. 241] and  $\chi(C) = 1$ , so that

$$(6) \quad C^{**} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \dots\dots\dots \end{pmatrix}.$$

Since  $C \leftrightarrow T$  if and only if  $C^{**} \leftrightarrow T^{**}$ , we may use (5) and (6) to obtain  $(C^{**} T^{**})_{00} = (T^{**} C^{**})_{00} = \chi(T)$ , and, for  $n > 0$ ,

$$(7) \quad (C^{**} T^{**})_{n0} = \frac{1}{n} \sum_{k=1}^n \chi_k(T) = \chi_n(T) = (T^{**} C^{**})_{n0}.$$

The system (7) yields  $\chi_n(T) = \chi_1(T)$ ,  $n = 1, 2, 3, \dots$ . Thus  $v = \chi_1(T)e$ . Substituting in (4) with  $\chi \in c_0$  we see that  $c$  must commute with  $B$ . Since  $B$  is a matrix and  $B \in \mathcal{A}$ , we may use Corollary 2 to obtain the following result.

**THEOREM 3.** *Let  $T \in B(c)$ . Then  $T \leftrightarrow C$  if and only if  $T$  has the form (4) with  $v = \chi_1(T)e$  and  $B \in \mathcal{H}$ .*

*Note added in proof.* The hypotheses of Theorem 1 can be modified without changing the details of the proof. For example, if  $A$  and  $B$  are any two bounded operators over  $l^p$ ,  $p > 1$ , then the conclusion of Theorem 1 holds. In particular, since  $C \in B(l^p)$  for  $p > 1$ , we get as a corollary that  $\text{Com}(C)$  in  $B(l^p)$  consists only of those Hausdorff matrices that belong to  $B(l^p)$ . Another description of  $\text{Com}(C)$  in  $B(l^2)$  appears in A. Shields and L. Wallen [Indiana Univ. Math. J., 20 (1971) 777-788].

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