CANONICAL FORMS FOR LOCAL DERIVATIONS

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Consider a field \( k \), the formal power series field \( k((x)) \) in one variable over \( k \), and a derivation \( D \) of \( k((x)) \) that maps \( k \) into itself. We wish to replace \( x \) by another generator \( y \) of \( k((x)) \) so that \( D y \) has a particularly simple expression as a function of \( y \). This is accomplished subject to certain restrictions on the differential field \( k \), some deductions are drawn, and there are extensions to the analogous problem for power series rings in several variables.

We first consider a derivation \( D \) on a noetherian local ring \( R \). If \( M \) is the maximal ideal of \( R \), then for any \( N = 1, 2, \cdots \) we have \( D M^N \subset M^{N-1} \). Thus \( D \) is automatically continuous in the natural topological ring structure of \( R \), where a basis for the neighborhoods of zero are the various powers of \( M \).

**Theorem 1.** Let \( R \) be a complete noetherian local ring containing \( Q \), \( M \) the maximal ideal of \( R \) and \( D \) a derivation of \( R \) such that \( D M \not\subset M \). Then \( M \) has a set of generators \( y_1, \cdots, y_n \) such that \( D y_1 = \cdots = D y_n = 1 \).

There is an element \( x \in M \) such that \( D x \notin M \). For any other element \( y \in M \), either \( D y \) or \( D(x + y) \) is not in \( M \). Since \( x \) and \( y \) generate the same ideal in \( R \) as do \( x \) and \( x + y \), it follows that \( M \) is generated by those of its elements \( x \) for which \( D x \notin M \), that is, for which \( D x \) is a unit in \( R \). Now if \( x \in M \) and \( D x \notin M \), we have \( D(x/Dx) - 1 = xD(1/Dx) \in M \), so that \( M \) is generated by those of its elements \( x \) satisfying \( D x - 1 \in M \). Since \( R \) is noetherian, a finite number of such \( x \)'s, say \( x_1, \cdots, x_n \), will generate \( M \). If we have elements \( y_1, \cdots, y_n \in M \) such that \( x_i - y_i \in M^2 \) for each \( i = 1, \cdots, n \) then \( y_1, \cdots, y_n \) also generate \( M \), and we shall be done with the proof if we can find such \( y_1, \cdots, y_n \) such that \( D y_1 = \cdots = D y_n = 1 \). To do this, we shall show by a successive approximation process that \( x_1, \cdots, x_n \) may be replaced by elements which differ from these by elements in successively higher powers of \( M \) in such a way that the new \( D x_1 - 1, \cdots, D x_n - 1 \) also belong to high powers of \( M \), and we shall then let each \( y_i, i = 1, \cdots, n \), be the limit of the sequence of \( x_i \)'s thus obtained. Specifically, we are reduced to showing that if \( x_1, \cdots, x_n \) generate \( M \) and \( N \geq 1 \) is an integer such that for each \( i = 1, \cdots, n \) we have \( D x_i - 1 \in M^N \), then there exist \( z_1, \cdots, z_n \in M^{N+1} \) such that each \( D(x_i + z_i) - 1 \in M^{N+1} \). Since \( D M^{N+1} \subset M^N \), it suffices
to show that the \( R \)-module homomorphism induced by \( D \)

\[
\sum_{i_1, \ldots, i_n \geq 0 \atop i_1 + \cdots + i_n = N+1} Rx^{i_1}_1 \cdots x^{i_n}_n \longrightarrow M^N/M^{N+1},
\]

according to which \( x^{i_1}_1 \cdots x^{i_n}_n \) is mapped into

\[
i_n x^{i_1-1}_1 x^{i_2}_2 \cdots x^{i_n}_n + \cdots + i_n x^{i_1}_1 \cdots x^{i_{n-1}}_{n-1} x^{i_n-1}_n + M^{N+1},
\]
is surjective. To do this it suffices to show that if \( X_1, \ldots, X_n \) are indeterminates then the \( \mathbb{Q} \)-linear map \( \delta \) from the vector space of forms of degree \( N + 1 \) in \( \mathbb{Q}[X_1, \ldots, X_n] \) into forms of degree \( N \) that is given by

\[
\delta(X^{i_1}_1 \cdots X^{i_n}_n) = X^{i_1}_1 \cdots X^{i_n}_n \left( \frac{i_1}{X_1} + \cdots + \frac{i_n}{X_n} \right)
\]
is surjective. For this, we order the set of monomials \( X^{i_1}_1 \cdots X^{i_n}_n \) of degree \( N \) in \( \mathbb{Q}[X_1, \ldots, X_n] \) lexicographically, setting \( X^{i_1}_1 \cdots X^{i_n}_n < X^{j_1}_1 \cdots X^{j_n}_n \) if, for the smallest \( p = 1, \ldots, n \) such that \( i_p \neq j_p \), we have \( i_p < j_p \). That \( \delta \) is surjective follows immediately from the remark that if \( i_1, \ldots, i_n \geq 0, i_1 + \cdots + i_n = N \), and \( q = 1, \ldots, n \) is the largest integer such that \( i_q \neq 0 \), then \( X^{i_1}_1 \cdots X^{i_n}_n \) differs from \( \delta(X^{i_1}_1 \cdots X^{i_n}_n X_q/(i_q + 1)) \) by a linear combination of monomials that are less than \( X^{i_1}_1 \cdots X^{i_n}_n \).

**Corollary.** Under the same conditions as above, there exist generators \( y, z_1, \ldots, z_{n-1} \) of \( M \) such that \( Dy = 1 \) and \( Dz_1 = \cdots = Dz_{n-1} = 0 \).

To prove this, just set \( y = y_1, z_1 = y_2 - y_1, \ldots, z_{n-1} = y_n - y_1 \).

The case of greatest interest for Theorem 1 is that of a formal power series ring \( k[[x_1, \ldots, x_n]] \) in indeterminates \( x_1, \ldots, x_n \) over a field \( k \) of characteristic zero and a derivation \( D \) on this ring that sends \( k \) into itself but does not send the maximal ideal \( M \) of the ring into itself. Since any set of generators of the maximal ideal \( M \) of a noetherian local ring \( R \) contains a minimal set of generators, in number \( \dim_{k|M}M/M^2 \), we see that in the present case new generators \( y, z_1, \ldots, z_{n-1} \) may be chosen for \( M \) such that our differential ring is the formal power series ring \( k[[y, z_1, \ldots, z_{n-1}]] \), with the derivation extended from \( k \) by means of \( Dy = 1, Dz_1 = \cdots = Dz_{n-1} = 0 \). We ask what the constants of this ring are, that is, what are the elements to which application of \( D \) gives 0? Any element of \( k[[y, z_1, \ldots, z_{n-1}]] \) can be uniquely written \( \sum_{i=0}^{\infty} f_i(z) y^i \), where each \( f_i(z) \in k[[z_1, \ldots, z_{n-1}]] \). Note that \( D \) maps \( k[[z_1, \ldots, z_{n-1}]] \) into itself and is obtained simply by applying \( D \) to the coefficients of the power series, these coefficients being elements of \( k \). We have
Therefore we get $\sum_{i=0}^{n} f_i(z)y^i$ constant if and only if for each $i \geq 0$ we have $Df_i(z) + (i + 1)f_{i+1}(z) = 0$. In other words, the constants of $k[[y, z_1, \cdots, z_{n-1}]]$ are the elements of the form

$$\sum_{i=0}^{n} (-1)^i \frac{D^i f(z)}{i!} y^i,$$

for arbitrary $f(z) \in k[[z_1, \cdots, z_{n-1}]]$. The generators $y, z_1, \cdots, z_{n-1}$ of the maximal ideal of $k[[y, z_1, \cdots, z_{n-1}]]$ are of course not unique, but may be altered by adding to each of $y, z_1, \cdots, z_{n-1}$ a constant in $M$, that is an element of the above form, with $f(z)$ having no term of degree zero, provided the new elements we obtain have their linear terms linearly independent over $k$.

To prove the next theorem, we shall have to restrict ourselves to differential fields with certain special properties. For this purpose, let us consider briefly an arbitrary ordinary differential field $k$ and a system of $n$ first order linear differential equations in $n$ unknowns over $k$, that is, a system of equations $Dy_i = \sum_{j=1}^{n} a_{ij}y_j + b_i, i = 1, \cdots, n$, where each $a_{ij}, b_i \in k$. By a solution of this system we of course mean an $n$-tuple $(y_1, \cdots, y_n)$ of elements of some differential extension field of $k$ satisfying these equations. The totality of solutions in $k$ (that is, solutions with component functions in $k$), if any, is got as usual by adding to a particular solution an arbitrary solution of the corresponding system of homogeneous differential equations $Dy_i = \sum_{j=1}^{n} a_{ij}y_j, i = 1, \cdots, n$, and, as usual, the solutions of the homogeneous system form a vector space over the subfield of constants of $k$. That this vector space is of dimension at most $n$ is an immediate consequence of the following result.

**LEMMA.** Let $a_{ij}, i, j = 1, \cdots, n$, be elements of the differential field $k$. Then any solutions in $k$ of the system of $n$ first order homogeneous differential equations in $n$ unknowns $Dy_i = \sum_{j=1}^{n} a_{ij}y_j, i = 1, \cdots, n$, that are linearly dependent over $k$ are linearly dependent over the subfield of constants of $k$.

For suppose that the solutions $z_1 = (y_{i1}, \cdots, y_{in}), \cdots, z_m = (y_{m1}, \cdots, y_{mn})$, with each $y_{ij} \in k$, are linearly dependent over $k$. We must show that $z_1, \cdots, z_m$ are linearly dependent over the subfield of constants of $k$. We may suppose that no proper subset of $z_1, \cdots, z_m$ is linearly dependent over $k$. Choose $c_1, \cdots, c_m \in k$ such that $c_1z_1 + \cdots + c_mz_m =$
0 and suppose, as we may, that \( \alpha = 1 \). For \( i = 1, \cdots, n \) we have
\[
\sum_{p=1}^{m} c_p y_{pi} = 0,
\]
so that
\[
\sum_{p=1}^{m} c_p D y_{pi} = \sum_{p=1}^{m} c_p \sum_{j=1}^{n} a_{ij} y_{pj} = \sum_{j=1}^{n} a_{ij} \sum_{p=1}^{m} c_p y_{pj} = 0.
\]
Therefore
\[
0 = D \sum_{p=1}^{m} c_p y_{pi} = \sum_{p=1}^{m} (Dc_p) y_{pi} + \sum_{p=1}^{m} c_p D y_{pi} = \sum_{p=1}^{m} (Dc_p) y_{pi}.
\]

Since \( c_1 = 1 \), we have \( \sum_{p=1}^{m} (Dc_p) y_{pi} = 0 \) for \( i = 1, \cdots, n \), or \( \sum_{p=1}^{m} (Dc_p) z_p = 0 \). Since \( z_1, \cdots, z_m \) are linearly independent over \( k \), we have \( Dc_p = 0 \) for \( p = 2, \cdots, m \). Hence \( c_1, \cdots, c_m \) are constant. Recalling that \( \sum_{p=1}^{m} c_p z_p = 0 \) completes the proof.

We say that the system of equations \( D y_i = \sum_{j=1}^{n} a_{ij} y_j + b_i \), \( i = 1, \cdots, n \), has a full set of solutions in \( k \) if it has a particular solution in \( k \) and if the corresponding system of homogeneous equations \( D y_i = \sum_{j=1}^{n} a_{ij} y_j \), \( i = 1, \cdots, n \), has \( n \) solutions in \( k \) that are linearly independent (over \( k \) or over its subfield of constants). It is worth mentioning the following result, which we shall not use, a result that is an easy consequence of standard facts: The system of equations \( D y_i = \sum_{j=1}^{n} a_{ij} y_j + b_i \), \( i = 1, \cdots, n \), with coefficients \( a_{ij}, b_i \) in the differential field \( k \), has a full set of solutions in some finitely generated differential extension field of \( k \) whose subfield of constants is algebraic over that of \( k \).

**Theorem 2.** Let \( R \) be a complete noetherian local ring containing \( \mathbb{Q} \), \( M \) the maximal ideal of \( R \), and \( D \) a derivation on \( R \) such that \( DM \subset M \). Let the differential field \( k = R/M \) be such that any system of \( n \) first order linear differential equations in \( n \) unknowns with coefficients in \( k \) has a full set of solutions in \( k \), where \( n = \dim_k M/M^2 \). Then \( M \) has a set of generators \( y_1, \cdots, y_n \) such that \( D y_i = \cdots = D y_n = 0 \).

The derivation on \( k \) is of course that induced by \( D \) via the natural surjection \( R \to k \). We denote this derivation on \( k \), somewhat incorrectly, by the same letter \( D \). For any \( N = 1, 2, \cdots \) we have \( DM^N \subset M^N \).
The map \( D \) on \( M \) induces a map \( \Delta \) on the \( k \)-vector space \( M/M^2 \), with \( \Delta(m + M^2) = Dm + M^2 \) for any \( m \in M \). The operator \( \Delta \) is not \( k \)-linear, but it is additive, and it satisfies the relation \( \Delta(ax) = (Da)x + a(\Delta x) \) for all \( a \in k, x \in M/M^2 \). Fix a \( k \)-basis \( \alpha_1, \ldots, \alpha_n \) of \( M/M^2 \). Then there exist \( a_{ij} \in k, i, j = 1, \ldots, n \), such that for each \( i = 1, \ldots, n \) we have \( \Delta\alpha_i = \sum_{j=1}^n a_{ij}\alpha_j \). For any \( u_1, \ldots, u_n \in k \) we have

\[
\Delta\left(\sum_{i=1}^n u_i\alpha_i\right) = \sum_{i=1}^n \left( (Du_i)\alpha_i + u_i \sum_{j=1}^n a_{ij}\alpha_j \right) = \sum_{i=1}^n \left( Du_i + \sum_{j=1}^n a_{ij}u_j \right)\alpha_i.
\]

By our assumptions on \( k \), there exist \( n \)-tuples of elements of \( k \), say \((u_{i1}, \ldots, u_{in})\), \((u_{i1}, \ldots, u_{i_n})\), linearly independent over \( k \), such that for each \( i, p = 1, \ldots, n \) we have \( Du_{pi} + \sum_{j=1}^n a_{ij}u_{pj} = 0 \). If we let \( \xi_p = \sum_{i=1}^n u_{pi}\alpha_i \), \( p = 1, \ldots, n \), then we get \( \Delta\xi_1 = \cdots = \Delta\xi_n = 0 \) and \( \xi_1, \ldots, \xi_n \) is a \( k \)-basis of \( M/M^2 \). Now choose \( x_1, \ldots, x_n \in M \) such that \( \xi_i = x_i + M^2, i = 1, \ldots, n \). Then \( x_1, \ldots, x_n \) is a set of generators for \( M \) and \( Dx_1, \ldots, Dx_n \in M^2 \). We have to show that \( x_1, \ldots, x_n \) can be modified so that we still have \( \xi_i = x_i + M^2, i = 1, \ldots, n \), and in addition have the stronger relations \( Dx_1 = \cdots = Dx_n = 0 \). To do this it suffices to prove, in virtue of the usual successive approximation argument and the completeness of \( R \), that if \( N = 2, 3, \ldots \) and \( x_1, \ldots, x_n \) is a set of generators of \( M \) such that \( Dx_1, \ldots, Dx_n \in M^N \), then there exist \( z_1, \ldots, z_n \in M^N \) such that \( D(x_i + z_i), \ldots, D(x_n + z_n) \in M^{N+1} \). To prove this, for each \( i = 1, \ldots, n \) write

\[
Dx_i = \sum_{i_1, \ldots, i_n=0}^{\text{?}} a_{i_1i_2\ldots i_n}x_1^{i_1} \cdots x_n^{i_n},
\]

for certain \( a_{i_1i_2\ldots i_n} \in R \), and try setting

\[
z_i = \sum_{i_1, \ldots, i_n=0}^{\text{?}} b_{i_1i_2\ldots i_n}x_1^{i_1} \cdots x_n^{i_n},
\]

with the \( b_{i_1i_2\ldots i_n} \)'s elements of \( R \) to be determined. Since for each \( i = 1, \ldots, n \)

\[
Dz_i = \sum_{i_1, \ldots, i_n=0}^{\text{?}} (Db_{i_1i_2\ldots i_n})x_1^{i_1} \cdots x_n^{i_n} \pmod{M^{N+1}},
\]

we will have \( D(x_1 + z_1), \ldots, D(x_n + z_n) \) all in \( M^{N+1} \) if, for each \( i_1, \ldots, i_n \), we have \( a_{i_1i_2\ldots i_n} + Db_{i_1i_2\ldots i_n} \in M \). Passing from \( R \) to \( R/M = k \), we are reduced to the problem of finding elements of \( k \) with prescribed derivatives. But that this is always possible is a consequence of the assumption made on \( k \) (special case of first order linear differential equations where each \( a_{ij} = 0 \)), except in the case \( n = 0 \), where the
The case of greatest interest for Theorem 2 is, as for Theorem 1, that of a formal power series ring $k[[x_1, \ldots, x_n]]$ over a field $k$ of characteristic zero in indeterminates $x_1, \ldots, x_n$ and a derivation $D$ on this ring that sends $k$ into itself and each $x_i$ into a power series with no term of degree zero. If the differential field $k$ satisfies the appropriate condition on the solvability of systems of linear differential equations, then new variables $y_1, \ldots, y_n$ may be found such that $k[[x_1, \ldots, x_n]] = k[[y_1, \ldots, y_n]]$ and $Dy_1 = \cdots = Dy_n = 0$. It is easy to compute the subring of constants of this ring. We in fact do this for a slightly more general case, where the variables $y_1, \ldots, y_n$ may satisfy certain analytic relations.

**Proposition.** Let $k$ be a differential field and let $R$ be a differential extension ring of $k$ which is a complete noetherian local ring containing nonunits $y_1, \ldots, y_n$ such that $R = k + Ry_1 + \cdots + Ry_n$ and $Dy_1 = \cdots = Dy_n = 0$. Then the constants of $R$ are just the subring $C[[y_1, \ldots, y_n]]$, where $C$ is the subfield of constants of $k$.

Clearly each formal power series in $y_1, \ldots, y_n$ with coefficients in $C$ is a constant. Suppose conversely that $x = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} y_1^{i_1} \cdots y_n^{i_n}$, with each $a_{i_1, \ldots, i_n}$ in $k$, is such that $Dx = 0$. We would like to know that this power series representation for $x$ has the property that for each $N = 1, 2, 3, \ldots$, if we consider the various $i_1, \ldots, i_n \geq 0$ such that $i_1 + \cdots + i_n = N$ and $a_{i_1, \ldots, i_n} \neq 0$, then the various elements $y_1^{i_1} \cdots y_n^{i_n} \in M^N$ are actually linearly independent over $k$ modulo $M^{N+1}$. This property of the representation of our given $x$ is not necessarily true to begin with, but working successively with $N = 1, 2, \ldots$ we can modify the $a_{i_1, \ldots, i_n}$'s so as to make this property valid. This being so, we can prove by induction on $N = 0, 1, 2, \ldots$ that all the coefficients $a_{i_1, \ldots, i_n}$ of the power series representation of $x$ are in $C$, as follows. This fact is clear for $N = 0$, and if for a certain $N > 0$ we know that each nonzero $a_{i_1, \ldots, i_n}$ is a constant whenever $i_1 + \cdots + i_n < N$, then the congruence

$$Dx \equiv \sum_{i_1 + \cdots + i_n = N} (Da_{i_1, \ldots, i_n})y_1^{i_1} \cdots y_n^{i_n} \pmod{M^{N+1}}$$

shows that $Da_{i_1, \ldots, i_n} = 0$ if $i_1 + \cdots + i_n = N$. Thus each $a_{i_1, \ldots, i_n}$ is in $C$.

A rather stringent condition is imposed on $k$ in the statement of Theorem 2. That some such condition is necessary can be seen from the example of the formal power series ring $R = k[[x]]$, where the differential field $k$ contains an element $a$ that is not of the form $-Db/b$ for any $b \in k$ and the derivation $D$ on $k$ is extended to $R$ by
setting $Dx = ax$. Any other generator of the ideal $Rx$ is of the form $y = b_1x + b_2x^2 + \cdots$, with $b_1, b_2, \cdots \in k$, $b_1 \neq 0$, and if $Dy = 0$ we get $Db_1 + b_2a = 0$, which is impossible. Or we may obtain a similar counterexample, verified by a similar argument, by supposing $k$ to contain an element $a$ which is not the derivative of any element of $k$ and then extending the derivation $D$ on $k$ to one on $k[[x]]$ by setting $Dx = ax^2$.

The remainder of this paper will consider extensions of the derivation on a differential field $k$ of characteristic zero to the formal power series field $k((x))$, the field of quotients of the formal power series ring in one variable $k[[x]]$. We begin with a number of well-known remarks. First of all, $k((x))$ has a natural topological field structure, depending only on its field structure and inducing the usual topology on $k[[x]]$, since the maximal ideal $k[[x]]$ of $k[[x]]$ can be characterized as the set of all $u \in k((x))$ such that, given any $v \in k((x))$, there exists an integer $n > 0$ such that $1 + u^n v$ has an $m$th root in $k((x))$ for an infinite number of positive integers $m$, and $k[[x]]$ is the set of all elements of $k((x))$ not having a reciprocal in $k[[x]]$. Any nonzero $u \in k((x))$ has an order, denoted $\text{ord } u$, which is that integer $m$ such that we can write $u = \sum_{n \geq m} a_n x^n$, with each $a_n \in k$ and $a_m \neq 0$, and $\text{ord } u$ does not depend on $x$. The element $x$ which, together with $k$, “generates” $k((x))$ is certainly not unique, since it can be replaced by any other element of order one. The field $k$ is of course determined to within isomorphism as the field $k[[x]]/k[[x]]$, but $k$ is not necessarily determined as a subfield of $k((x))$; for example if $k = k_0(a)$, where $k_0$ is a subfield of $k$ and $a$ is transcendental over $k_0$, then $k_0(a + x)$ could replace $k$.

We shall be interested in derivations of $k((x))$ that map $k$ into itself and are continuous. Such a derivation is given by

$$D(\sum_{n \geq m} a_n x^n) = \sum_{n \geq m} ((Da_n)x^n + na_n x^{n-1}Dx),$$

for any $\{a_n\}_{n \geq m} \subset k$. The derivation $D$ is completely determined by its action on $k$ and the knowledge of $Dx$, which can be an arbitrary element of $k((x))$. If we note that for any such $D$ the set $\{\text{ord } Du - \text{ord } u: u \in k((x)), u \neq 0, D_u \neq 0\}$ is bounded from below, we see that there exist derivations $D$ of $k((x))$ that map $k$ into itself and are not continuous, got for example by taking a transcendence basis $\{u_a\}_{a \in A}$ for $k((x))$ over $k$ and defining each $Du_a$ to be some specific element of $k((x))$, subject to the condition that the set $\{\text{ord } Du_a - \text{ord } u_a\}_{a \in A}$ is not bounded from below. (We remark that we use here the well-known fact that $k((x))$ has infinite transcendence degree over $k$. This can be shown by a cardinality argument if $k$ is at most countable and
then easily extended to any $k$, but it may be worth mentioning that
an easy differential-algebraic proof of this fact can be based on the
well-known and elementary result that if $k$ is a differential field of
characteristic zero and $K$ a differential extension field of $k$ having
the same subfield of constants, then elements of $K$ whose derivatives
are in $k$ are algebraically dependent over $k$ if and only if a linear
combination of them with constant coefficients not all zero is in $k$.
For using the continuous derivation $D$ on $k((x))$ that is given by
$Dk = 0$, $Dx = 1$, we see that the power series for $(\log (1 + ax))_{a \in \mathbb{k}}$
are algebraically independent over the subfield $k(x)$ of $k((x))$ since no
nontrivial linear combination with coefficients in $k$ of their derivatives
$\{a/(1 + ax)\}$ is the derivative of an element of $k(x)$. Or we may use
the well-known result that if $k$ and $K$ are as above then elements
of $K$ whose logarithmic derivatives are in $k$ are algebraically dependent
over $k$ if and only if a nontrivial power product of these elements is
in $k$ to show that the power series for $e^x, e^{x^2}, e^{x^3}, \cdots$ are algebraically
independent over $k(x)$.

The following two theorems concern the classification of continuous
derivations of $k((x))$ that map $k$ into itself. The analogous problem
for derivations of the field of quotients $k((x_1, \cdots, x_n))$ of the formal
power series ring $k[[x_1, \cdots, x_n]]$, where $Dx_1, \cdots, Dx_n$ are quite arbitrary,
seems considerably more difficult. Note the slight overlap (the case
$r = 0$) of the next result with Theorem 1.

**Theorem 3.** Let $k$ be a field of characteristic zero, $D$ a continuous
derivation of the formal power series field $k((x))$ that maps $k$ into
itself and does not send the maximal ideal of $k[[x]]$ into itself. Then
there exists a unique nonnegative integer $r$ and an element $y \in k((x))$
of order one (so that $k((x)) = k((y))$) such that $Dy = ay^{-r}$, for some
nonzero $a \in k$. The element $a$ is unique to within multiplication
by $(r + 1)^{th}$ powers of nonzero elements of $k$, and for given a the
element $y$ is unique to within multiplication by an $(r + 1)^{th}$ root of
unity in $k$.

We must have $\text{ord } Dx \leq 0$, for otherwise $D(k[[x]])x \subset k[[x]]x$.
Hence we can write $Dx = ax^{-r}(1 + \sum_{n=1}^{\infty} a_n x^n)$, with $r$ a nonnegative
integer and $a, a_1, a_2, \cdots \in k, a \neq 0$. The $k[[x]]$-module generated by
$D(k[[x]])$ is $k[[x]]x^{-r}$, proving that $r$ is unique. Any element $y \in k((x))$ of
order one is of the form $y = bx(1 + \sum_{n=1}^{\infty} b_n x^n)$, with $b, b_1, b_2, \cdots \in k,
b \neq 0$. The leading term of the power series for $Dy$ is $bx^{-r}$, so that
$Dy - b^{r+1}ay^{-r} \in k[[x]]x^{-r} = k[[y]]y^{-r}$. Thus the transition from $x$ to
$y$ multiplies $a$ by $b^{r+1}$. It is now immediate that the existence of a
special $y \in k((x))$ with the property prescribed in the statement of the
theorem and also the uniqueness statements about $a$ and $y$ will all
be known if it can be shown that there exist unique \( b_1, b_2, \cdots \in k \) such that if \( y = x(1 + \sum_{n=1}^{\infty} b_n x^n) \) then \( Dy = ay^{-r} \). For this particular \( y \) we have

\[
Dy = D\left(x + \sum_{n=1}^{\infty} b_n x^{n+1}\right) \\
= \sum_{n=1}^{\infty} (Db_n)x^{n+1} + \left(1 + \sum_{n=1}^{\infty} (n + 1) b_n x^n\right)ax^{-r}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right),
\]

which we want to equal

\[
ay^{-r} = ax^{-r}\left[1 - \left(\sum_{n=1}^{\infty} b_n x^n\right) + \left(\sum_{n=1}^{\infty} b_n x^n\right)^2 - \cdots\right].
\]

The condition we need is therefore

\[
a^{-1} \sum_{n=1}^{\infty} (Db_n)x^{n+r+1} + \left(1 + \sum_{n=1}^{\infty} (n + 1) b_n x^n\right)\left(1 + \sum_{n=1}^{\infty} a_n x^n\right) \\
= \left[1 - \left(\sum_{n=1}^{\infty} b_n x^n\right) + \left(\sum_{n=1}^{\infty} b_n x^n\right)^2 - \cdots\right].
\]

Both sides of this last equation have constant term 1. For any integer \( m > 0 \), the coefficient of \( x^m \) on the left is \( a^{-1} Db_{m-r-1} + (m + 1)b_m + mb_{m-1}a_1 + \cdots + 2b_1a_{m-1} + a_m \) (understanding \( b_n \) to be 0 if \( n < 1 \)), while the coefficient of \( x^m \) on the right hand side is \( -rb_m + (a \text{ specific polynomial in } b_1, \cdots, b_{m-1} \text{ with integer coefficients}) \). Therefore by letting \( m = 1, 2, 3, \cdots \) we successively find \( b_1, b_2, \cdots \in k \) such that \( Dy = ay^{-r} \), and we see that these \( b_1, b_2, \cdots \) are unique.

**Corollary.** If \( y \) is as above, the constants of \( k((x)) = k((y)) \) are precisely the elements of the form

\[
c - \frac{a^{-1} Dc}{r+1} y^{r+1} + \frac{(a^{-1} D)^2 c}{(r+1)^2 2!} y^{2(r+1)} - \frac{(a^{-1} D)^3 c}{(r+1)^3 3!} y^{3(r+1)} + \cdots, c \in k.
\]

For any subset \( \{c_n\}_{n \in \mathbb{Z}} \) of \( k \) such that \( c_n = 0 \) if \( n \) is sufficiently small, we have

\[
D(\sum c_n y^n) = \sum ((Dc_n)y^n + nc_n ay^{-r-1}) = \sum (Dc_{n-r-1} + nc_n) y^{n-r-1}.
\]

Therefore for \( \sum c_n y^n \) to be constant it is necessary and sufficient that \( Dc_{n-r-1} + nc_n = 0 \) for all \( n \), that is that \( c_n = -Dc_{n-r-1}/(na) \) if \( n \neq 0 \) and \( c_{-r-1} \) be a constant of \( k \). Therefore we must have \( c_n = 0 \) if \( n < 0 \) or \( n \equiv 0 \text{(mod } (r+1)) \), and the corollary follows directly, with \( c = c_0 \).

If we have a derivation of \( k((x)) \) that sends both the field \( k \) and the maximal ideal of \( k[[x]] \) into themselves, then this derivation is
automatically continuous and is the extension to $k((x))$ of a derivation of $k[[x]]$. Theorem 2 is directly applicable if the differential equations $Dy = ay$ and $Dy = a$ have solutions in $k$ for any $a \in k$. For a quite general differential field $k$, where this condition is not necessarily satisfied, nothing much can be said. However, complete information is also available in the special but important case in which the derivation on $k$ is trivial.

**Theorem 4.** Let $k$ be a field of characteristic zero, $D$ a nonzero derivation of the formal power series ring in one variable $k[[x]]$ that is trivial on $k$ and maps the maximal ideal of $k[[x]]$ into itself. For any $y \in k[[x]]$ of order one we can write $Dy = y'/\sum_{n=0}^{\infty} a_n y^n$, with $r \geq 1$ an integer, $a_0, a_1, \cdots \in k$, and $a_0 \neq 0$. Here $r$ and $a_{r-1}$ are uniquely determined by $D$, independent of the choice of $y$, and $a_0$ is determined to within multiplication by the $(r - 1)^{th}$ power of a nonzero element of $k$. If $r > 1$ then $y$ can be chosen such that $Dy = y/(a + cy^{r-1})$, with $a, c \in k$. If $r = 1$ then $y$ can be chosen such that $Dy = y/a$ with $a \in k$, and here $y$ is unique to within multiplication by a nonzero element of $k$.

Let us write $Dx = x'\sum_{n=0}^{\infty} \alpha_n x^n$, with $r$ an integer and $\alpha_0, \alpha_1, \cdots \in k$, $\alpha_0 \neq 0$. Then any $y \in k[[x]]$ of order one is of the form $y = bx(1 + \sum_{n=1}^{\infty} b_n x^n)$ with $b, b_1, b_2, \cdots \in k, b \neq 0$. Since $y \equiv bx \mod k[[x]][x^2]$ we have $Dy = x'\sum_{n=0}^{\infty} \alpha_n y^n$, with $a_0, a_1, \cdots \in k, a_0 = \alpha_0 b^{-1}$. In particular, $r$ is unique. Since $D$ maps $k[[x]]x$ into itself, we have $r \geq 1$. In the special case where $y = bx$, we verify immediately that $x_n = \alpha_n b^{-n-1}$, $n = 0, 1, 2, \cdots$. To complete the proof of the theorem it suffices to show that if we restrict ourselves to the case $b = 1$, that is $y = x(1 + \sum_{n=1}^{\infty} b_n x^n)$, then $a_{r-1} = a_{r-1}$ and, furthermore, that there exist certain $b, b_1, b_2, \cdots \in k$, unique if $r = 1$, such that we have $a_n = 0$ for all $n \neq 0, r - 1$. Working out $Dy$ in two ways we get

$$Dy = \frac{y'}{\sum_{n=0}^{\infty} a_n y^n} = \left(1 + \sum_{n=1}^{\infty} (n + 1) b_n x^n\right)\frac{x^r}{\sum_{n=0}^{\infty} \alpha_n x^n},$$

so that the $\alpha_n$'s and $a_n$'s are related by the identity

$$\sum_{n=0}^{\infty} \alpha_n x^n = \frac{\left(1 + \sum_{n=1}^{\infty} (n + 1) b_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\left(1 + \sum_{N=1}^{\infty} b_N x^N\right)^n\right)}{\left(1 + \sum_{n=1}^{\infty} b_n x^n\right)^r}.$$
Comparison of terms of degree zero gives $a_0 = a_0$, which we already know. To show that $a_{r-1} = a_{r-1}$ it suffices to show that for each integer $m = 0, 1, 2, \cdots$ the coefficient of $x^{r-1}$ in the power series expansion of

$$
(1 + \sum_{n=1}^{\infty} (n + 1)b_n x^n)x^m \left(1 + \sum_{N=1}^{\infty} b_N x^N \right)^{m-r}
$$

is 1 if $m = r - 1$, otherwise 0. But this last power series is $x^r y^{m-r} y'$, where $'$ refers to formal differentiation with respect to $x$. If $m \neq r - 1$ this power series is $x^r (y^{m-r+1})'(m - r + 1)$, and the coefficient of $x^{r-1}$ must be zero since the derivative with respect to $x$ of a formal power series in $x$ has no term in $x^{r-1}$. On the other hand if $m = r - 1$ the coefficient of $x^{r-1}$ in $x^m y' / y$ is clearly 1. So it remains only to show that by a suitable choice of $b_1, b_2, \ldots$, unique if $r = 1$, we can get all $a_n$'s except $a_0$ and $a_{r-1}$ to be zero. Now for any integer $m > 0$ the coefficient of $x^m$ in the right hand side of the last displayed equation is

$$
a_m + (a \text{ specific polynomial in } a_0, \ldots, a_{m-1}, b_1, \ldots, b_{m-1} \text{ with integer coefficients}) + (m + 1 - r)b_{m-1}
$$

and this should equal $\alpha_m$. Letting $m = 1, 2, \ldots, r - 2$ we successively get the values of $b_1, \ldots, b_{r-2}$ for which $a_1 = \cdots = a_{m-2} = 0$, and they are unique. We already know that $a_{r-1} = a_{r-1}$. We can now choose $b_{r-1}$ to be an arbitrary element of $k$, except in the case $r = 1$ where there is no $b_{r-1}$ to worry about, and we then successively get unique $b_r, b_{r+1}, \ldots$ in $k$ such that $a_r = a_{r+1} = \cdots = 0$. This completes the proof. Note that if $r > 1$, then for fixed $a_0, a_{r-1} \in k$ with $a_0 \neq 0$ there are many possibilities for our $y$ of order one such that $Dy = y'/(a_0 + a_{r-1}y^{r-1})$, all given by replacing $y$ by $\gamma y (1 + \sum_{n=r-1}^{\infty} b_n y^n)$, with $\gamma$ any $(r - 1)^{th}$ root of unity in $k$, $b_{r-1}$ an arbitrary element of $k$, and $b_r, b_{r+1}, \cdots$ polynomial functions of $b_{r-1}$.

In the last theorem, and also in Theorem 3 if it happens that $Dk = 0$, we can write $D = f(x)d/dx$, with $f(x) \in k((x))$. In the duality between the one dimensional vector spaces over $k((x))$ of continuous $k$-derivations and $k$-differentials of $k((x))$, the basis for the space of differentials that is dual to the basis $D$ for the space of derivations is $dx/f(x)$. We have therefore also derived canonical forms for the nonzero $k$-differentials of $k((x))$, and these are of the type $y^r dy/a$ for $r \geq 0$, $ady/y$, and $((a/y^r) + (c/y))dy$ for $r > 1$, with $a, c \in k, a \neq 0$. Note that the invariance of $a_{r-1}$ is simply the invariance of the residue of $dx/f(x)$. Note also that in these cases the constant subfield of $k((x))$ for the derivation $D$ is the same as that for the derivation $d/dx$, which is just $k$.

One can verify that if the $k$ of Theorem 3 or 4 is the field of
complex numbers (with trivial derivation) and the derivation $D$ on $k((x))$ is such that $Dx$ is a convergent power series in $x$, then the $y$ of order one for which $Dy$ is in canonical form can be chosen to be a convergent power series in $x$. The analogous comment applies to the application of Theorem 1 to the formal power series ring $k[[x_1, \cdots, x_n]]$: if $Dx_1, \cdots, Dx_n$ are convergent power series, we can get $y_1, \cdots, y_n$ to be convergent power series in $x_1, \cdots, x_n$.

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