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A CLASS OF BILATERAL GENERATING FUNCTIONS FOR CERTAIN CLASSICAL POLYNOMIALS

J. P. SINGHAL AND H. M. (HARI MOHAN) SRIVASTAVA

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In this paper the authors first prove a theorem on bilateral generating relations for a certain sequence of functions. It is then shown how the main result can be applied to derive a large variety of bilateral generating functions for the Bessel, Jacobi, Hermite, Laguerre and ultraspherical polynomials, as well as for their various generalizations. Some recent results given by W. A. Al-Salam [1], S. K. Chatterjea [2], M. K. Das [3], S. Saran [6] and the present authors [9] are thus observed to follow fairly easily as special cases of the theorem proved in this paper.

Let the sequence of functions $\{S_n(x) | n = 0, 1, 2, \dots\}$ be generated by

$$(1) \quad \sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x) t^n = \frac{f(x, t)}{[g(x, t)]^m} S_m(h(x, t)),$$

where m is a nonnegative integer, the $A_{m,n}$ are arbitrary constants, and f, g, h are arbitrary functions of x and t .

In the present paper we first prove the following

THEOREM. *For the $S_n(x)$ generated by (1), let*

$$(2) \quad F[x, t] = \sum_{n=0}^{\infty} a_n S_n(x) t^n,$$

where the $a_n \neq 0$ are arbitrary constants.

Then

$$(3) \quad \begin{aligned} f(x, t) F[h(x, t), yt/g(x, t)] \\ = \sum_{n=0}^{\infty} S_n(x) \sigma_n(y) t^n, \end{aligned}$$

where $\sigma_n(y)$ is a polynomial of degree n in y defined by

$$(4) \quad \sigma_n(y) = \sum_{k=0}^n a_k A_{k, n-k} y^k.$$

We also show how this theorem can be applied to derive a large number of bilateral generating functions for those classical polynomial systems that satisfy a relationship like (1). In particular, we discuss the cases of the Bessel, Jacobi, Hermite, Laguerre and ultraspherical polynomials.

2. **Proof of the theorem.** If we substitute for the coefficients $\sigma_n(y)$ from (4) on the right-hand side of (3), we shall get

$$\begin{aligned} \sum_{n=0}^{\infty} S_n(x)\sigma_n(y)t^n &= \sum_{n=0}^{\infty} S_n(x)t^n \sum_{k=0}^n a_k A_{k,n-k} y^k \\ &= \sum_{k=0}^{\infty} a_k (yt)^k \sum_{n=0}^{\infty} A_{k,n} S_{n+k}(x)t^n \\ &= f(x, t) \sum_{k=0}^{\infty} a_k S_k(h(x, t)) \{yt/g(x, t)\}^k, \end{aligned}$$

by using (1), and the theorem follows on interpreting this last expression by means of (2).

3. **Applications.** As a first instance of the applications of our theorem, we recall the following known generating function for the ultraspherical polynomials [5, p. 280]:

$$(5) \quad \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{\lambda}(x)t^n = \rho^{-m-2\lambda} P_m^{\lambda}\left(\frac{x-t}{\rho}\right),$$

where $\rho = (1 - 2xt + t^2)^{-1/2}$.

Formula (5) is of type (1) with $f = \rho^{-2\lambda}$, $g = \rho$, $h = (x - t)/\rho$, and $A_{m,n} = \binom{m+n}{n}$, and therefore, our theorem, when applied to the ultraspherical polynomials, gives us

COROLLARY 1. *If*

$$(6) \quad F[x, t] = \sum_{n=0}^{\infty} a_n P_n^{\lambda}(x)t^n,$$

then

$$(7) \quad \rho^{-2\lambda} F\left[\frac{x-t}{\rho}, \frac{yt}{\rho}\right] = \sum_{n=0}^{\infty} P_n^{\lambda}(x)b_n(y)t^n,$$

where, as well as in what follows,

$$(8) \quad b_n(y) = \sum_{k=0}^n \binom{n}{k} a_k y^k.$$

Corollary 1 was proved recently by Chatterjea [2]. Note that in his long and involved derivation of Corollary 1, Chatterjea made use of the following formula of Tricomi:

$$(9) \quad P_n^{\lambda}\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{(-1)^n(x^2-1)^{\lambda+(1/2)n}}{n!} \frac{d^n}{dx^n} \{(x^2-1)^{-\lambda}\}.$$

Evidently, in view of the known generating function (5), formula (7)

would follow from (6) and (8) in a straightforward manner, without using (9).

Next we consider the Laguerre polynomials which satisfy the relationship [5, p. 211]

$$\begin{aligned}
 (10) \quad & \sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\lambda)}(x) t^n \\
 & = (1-t)^{-1-\lambda-m} \exp\left(\frac{-xt}{1-t}\right) L_m^{(\lambda)}\left(\frac{x}{1-t}\right),
 \end{aligned}$$

which is of type (1) with $f = (1-t)^{-1-\lambda} \exp\{-xt/(1-t)\}$, $g = (1-t)$, $h = x/(1-t)$, and $A_{m,n} = \binom{m+n}{n}$. Thus we arrive at the following special case of our theorem:

COROLLARY 2. *If*

$$(11) \quad F[x, t] = \sum_{n=0}^{\infty} a_n L_n^{(\lambda)}(x) t^n,$$

then

$$\begin{aligned}
 (12) \quad & (1-t)^{-1-\lambda} \exp\left(\frac{-xt}{1-t}\right) F\left[\frac{x}{1-t}, \frac{yt}{1-t}\right] \\
 & = \sum_{n=0}^{\infty} L_n^{(\lambda)}(x) b_n(y) t^n.
 \end{aligned}$$

Corollary 2 provides us with the corrected version of a result proved earlier by Al-Salam [1, p. 134].

On the other hand, if we consider the formula (see, for instance, [4], p. 58)

$$\begin{aligned}
 (13) \quad & \sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\lambda-m-n)}(x) t^n \\
 & = (1+t)^{\lambda-m} e^{-xt} L_m^{(\lambda-m)}(x(1+t)),
 \end{aligned}$$

we shall obtain the following particular case of our theorem:

COROLLARY 3. *If*

$$(14) \quad F[x, t] = \sum_{n=0}^{\infty} a_n L_n^{(\lambda-n)}(x) t^n,$$

then

$$\begin{aligned}
 (15) \quad & (1+t)^{\lambda} e^{-xt} F[x(1+t), yt/(1+t)] \\
 & = \sum_{n=0}^{\infty} L_n^{(\lambda-n)}(x) b_n(y) t^n.
 \end{aligned}$$

For the simple Bessel polynomials defined by [5, p. 293]

$$(16) \quad y_n(x) = {}_2F_0 \left[-n, n+1; -; -\frac{1}{2}x \right],$$

we have [3, p. 409]

$$(17) \quad \sum_{n=0}^{\infty} y_{m+n}(x) \frac{t^n}{n!} \\ = (1 - 2xt)^{-(m+1)/2} \exp \left\{ \frac{1 - \sqrt{(1 - 2xt)}}{x} \right\} y_m \left(\frac{x}{\sqrt{(1 - 2xt)}} \right),$$

and on comparing (17) with (1) we are led to the following result of Das [3, p. 410]:

COROLLARY 4. *If*

$$(18) \quad F[x, t] = \sum_{n=0}^{\infty} a_n y_n(x) \frac{t^n}{n!},$$

then

$$(19) \quad (1 - 2xt)^{-1/2} \exp \left\{ \frac{1 - \sqrt{(1 - 2xt)}}{x} \right\} F \left[\frac{x}{\sqrt{(1 - 2xt)}}, \frac{yt}{\sqrt{(1 - 2xt)}} \right] \\ = \sum_{n=0}^{\infty} y_n(x) b_n(y) \frac{t^n}{n!}.$$

Similarly, if we compare (1) with the known formula [5, p. 197]

$$(20) \quad \sum_{n=0}^{\infty} H_{m+n}(x) \frac{t^n}{n!} = \exp(2xt - t^2) H_m(x - t),$$

where $H_n(x)$ denotes the Hermite polynomial of degree n in x , we shall obtain a class of bilateral generating functions for these polynomials, given by

COROLLARY 5. *If*

$$(21) \quad F[x, t] = \sum_{n=0}^{\infty} \frac{a_n}{n!} H_n(x) t^n,$$

then

$$(22) \quad \exp(2xt - t^2) F[x - t, yt] \\ = \sum_{n=0}^{\infty} H_n(x) b_n(y) \frac{t^n}{n!}.$$

For the Jacobi polynomials we first observe that the special case

$y = 1$ of the bilinear generating relation (21), p. 465 of Srivastava [8] leads us to the elegant formula

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-m-n, \beta-m-n)}(x) t^n \\
 (23) \quad & = \left\{ 1 + \frac{1}{2} (x+1)t \right\}^{\alpha-m} \left\{ 1 + \frac{1}{2} (x-1)t \right\}^{\beta-m} \\
 & \quad \times P_m^{(\alpha-m, \beta-m)} \left(x + \frac{1}{2} (x^2 - 1)t \right).
 \end{aligned}$$

Note that the last formula (23) is a generalization of the well-known result

$$(24) \quad \sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta-n)}(x) t^n = \left\{ 1 + \frac{1}{2} (x+1)t \right\}^{\alpha} \left\{ 1 + \frac{1}{2} (x-1)t \right\}^{\beta},$$

which follows at once from (23) when $m = 0$.

A comparison of (23) with (1) yields

COROLLARY 6. *If*

$$(25) \quad F[x, t] = \sum_{n=0}^{\infty} \alpha_n P_n^{(\alpha-n, \beta-n)}(x) t^n,$$

then

$$\begin{aligned}
 & \left\{ 1 + \frac{1}{2} (x+1)t \right\}^{\alpha} \left\{ 1 + \frac{1}{2} (x-1)t \right\}^{\beta} \\
 (26) \quad & \times F \left[x + \frac{1}{2} (x^2 - 1)t, yt / \left\{ 1 + \frac{1}{2} (x+1)t \right\} \left\{ 1 + \frac{1}{2} (x-1)t \right\} \right] \\
 & = \sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta-n)}(x) b_n(y) t^n.
 \end{aligned}$$

Next we set $v = 0$ in the bilinear generating relation (18), p. 464 of Srivastava [8]. On replacing α, γ and λ by $1 + \alpha + \beta, 1 + \alpha$ and $1 + \alpha + m$ respectively, it is easy to see that

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha, \beta-m-n)}(x) t^n \\
 (27) \quad & = (1-t)^{\beta-m} \left\{ 1 - \frac{1}{2} (x+1)t \right\}^{-\alpha-\beta-1} P_m^{(\alpha, \beta-m)}(X),
 \end{aligned}$$

where, for convenience,

$$(28) \quad X = \left\{ x - \frac{1}{2} (x+1)t \right\} \left\{ 1 - \frac{1}{2} (x+1)t \right\}^{-1}.$$

We thus obtain

COROLLARY 7. *If*

$$(29) \quad F[x, t] = \sum_{n=0}^{\infty} \alpha_n P_n^{(\alpha, \beta-n)}(x) t^n,$$

then

$$(30) \quad \begin{aligned} & (1-t)^\beta \left\{ 1 - \frac{1}{2}(x+1)t \right\}^{-\alpha-\beta-1} F[X, yt/(1-t)] \\ &= \sum_{n=0}^{\infty} P_n^{(\alpha, \beta-n)}(x) b_n(y) t^n, \end{aligned}$$

where X is given by (28).

Lastly, we recall the following generating relation [9, p. 79, eq. (3.6)]

$$(31) \quad \begin{aligned} & \sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\lambda)}(x, r, p, \alpha) t^n \\ &= (1-\alpha t)^{-m-\lambda/\alpha} \exp [px^r \{1 - (1-\alpha t)^{-r/\alpha}\}] \\ & \quad \times G_m^{(\lambda)}(x(1-\alpha t)^{-1/\alpha}, r, p, \alpha), \end{aligned}$$

where the $G_n^{(\lambda)}(x, r, p, \alpha)$ are polynomials in x^r introduced by us [9] in an attempt to provide an elegant unification of the various recent extensions of the classical Hermite and Laguerre polynomials given, for instance, by Gould and Hopper [4] and others referred to in our earlier paper [9]. A comparison of (31) with (1) would yield the following result:

COROLLARY 8. *If*

$$(32) \quad F[x, t] = \sum_{n=0}^{\infty} \alpha_n G_n^{(\lambda)}(x, r, p, \alpha) t^n,$$

then

$$(33) \quad \begin{aligned} & (1-\alpha t)^{-\lambda/\alpha} \exp [px^r \{1 - (1-\alpha t)^{-r/\alpha}\}] F[x/(1-\alpha t)^{1/\alpha}, yt/(1-\alpha t)] \\ &= \sum_{n=0}^{\infty} G_n^{(\lambda)}(x, r, p, \alpha) b_n(y) t^n. \end{aligned}$$

Corollary 8, which incorporates Corollaries 2 and 5 as its particular cases, was proved earlier by us [9, p. 82, § 6] by using an operational technique.

Now we recall the sequence of functions $\{f_n(x) | n = 0, 1, 2, \dots\}$ defined by Rodrigues' formula

$$(34) \quad f_n(x) = \mu(n)\phi(x) \frac{d^n}{dx^n} \{\Psi(x)\} ,$$

where $\phi(x)$ and $\Psi(x)$ are independent of n . By using Taylor's theorem it is readily seen that the $f_n(x)$ are generated by

$$(35) \quad \sum_{n=0}^{\infty} \frac{\mu(m)}{n!\mu(m+n)} f_{m+n}(x)t^n = \frac{\phi(x)}{\phi(x+t)} f_m(x+t) ,$$

$m = 0, 1, 2, \dots,$

which evidently is of type (1) with

$$(36) \quad A_{m,n} = \frac{\mu(m)}{n!\mu(m+n)}, f = \frac{\phi(x)}{\phi(x+t)}, g = 1, h = x + t .$$

Thus the sequence $\{f_n(x)\}$, considered recently by Saran [6], is merely a proper subset of $\{S_n(x)\}$ defined by the generating relation (1).

Consequently, as a very special case of our theorem we can obtain the following corollary which happens to be the main result of Saran's paper [6]:

COROLLARY 9. For the $f_n(x)$ defined by (34), let

$$(37) \quad F[x, t] = \sum_{n=0}^{\infty} a_n f_n(x) t^n ,$$

where the $a_n \neq 0$ are arbitrary constants.

Then

$$(38) \quad \frac{\phi(x)F[x-t, yt]}{\phi(x-t)} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!\mu(n)} f_n(x)c_n(y) ,$$

where

$$(39) \quad c_n(y) = \sum_{k=0}^n (-n)_k \mu(k) a_k y^k .$$

By comparing (34) with Tricomi's formula (9) it would seem obvious that Corollary 1, involving ultraspherical polynomials, is contained in Corollary 9. However, it may be pointed out that the scope of Corollary 9 is very limited, since Rodrigues formulas of most of the classical polynomials require that the function $\Psi(x)$, involved in (34), depend upon both n and x . Besides, the factor $\mu(n)$ on the right-hand side of (34) is superfluous. Indeed, in equations (34), (35), (37), (38) and (39) one can replace, without any loss of generality, $f_n(x)$ by $\mu(n)f_n(x)$ and a_n by $a_n/\mu(n)$, $n = 0, 1, 2, \dots$.

In conclusion, we remark that by assigning special values to the

arbitrary coefficients a_n it is easy to obtain, from Corollaries 1 to 9, a large variety of bilateral generating functions for the Bessel, Jacobi, Hermite, Laguerre and ultraspherical polynomials, and their generalizations studied earlier. For example, Corollary 2 would lead us fairly easily to a number of extensions of the well-known Hille-Hardy formula given, for instance, by Srivastava [7] and Weisner [10]. The details involved are quite straightforward and are, therefore, omitted.

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Catherine Bandle, <i>Extensions of an inequality by Pólya and Schiffer for vibrating membranes</i>	543
S. J. Bernau, <i>Topologies on structure spaces of lattice groups</i>	557
Woodrow Wilson Bledsoe and Charles Edward Wilks, <i>On Borel product measures</i>	569
Eggert Briem and Murali Rao, <i>Normpreserving extensions in subspaces of $C(X)$</i>	581
Alan Seymour Cover, <i>Generalized continuation</i>	589
Larry Jean Cummings, <i>Transformations of symmetric tensors</i>	603
Peter Michael Curran, <i>Cohomology of finitely presented groups</i>	615
James B. Derr and N. P. Mukherjee, <i>Generalized quasicenter and hyperquasicenter of a finite group</i>	621
Erik Maurice Ellentuck, <i>Universal cosimple isols</i>	629
Benny Dan Evans, <i>Boundary respecting maps of 3-manifolds</i>	639
David F. Fraser, <i>A probabilistic method for the rate of convergence to the Dirichlet problem</i>	657
Raymond Taylor Hoobler, <i>Cohomology in the finite topology and Brauer groups</i>	667
Louis Roberts Hunt, <i>Locally holomorphic sets and the Levi form</i>	681
B. T. Y. Kwee, <i>On absolute de la Vallée Poussin summability</i>	689
Gérard Lallement, <i>On nilpotency and residual finiteness in semigroups</i>	693
George Edward Lang, <i>Evaluation subgroups of factor spaces</i>	701
Andy R. Magid, <i>A separably closed ring with nonzero torsion pic</i>	711
Billy E. Rhoades, <i>Commutants of some Hausdorff matrices</i>	715
Maxwell Alexander Rosenlicht, <i>Canonical forms for local derivations</i>	721
Cedric Felix Schubert, <i>On a conjecture of L. B. Page</i>	733
Reinhard Schultz, <i>Composition constructions on diffeomorphisms of $S^p \times S^q$</i>	739
J. P. Singhal and H. M. (Hari Mohan) Srivastava, <i>A class of bilateral generating functions for certain classical polynomials</i>	755
Richard Alan Slocum, <i>Using brick partitionings to establish conditions which insure that a Peano continuum is a 2-cell, a 2-sphere or an annulus</i>	763
James F. Smith, <i>The p-classes of an H^*-algebra</i>	777
Jack Williamson, <i>Meromorphic functions with negative zeros and positive poles and a theorem of Teichmüller</i>	795
William Robin Zame, <i>Algebras of analytic functions in the plane</i>	811