THE $p$-CLASSES OF AN $H^*$-ALGEBRA

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This paper considers a family of $\ast$-subalgebras of a semisimple $H^\ast$-algebra $A$. For $0 < p \leq \infty$ a nonnegative extended-real value $|a|^p$ is associated with each $a$ in $A$; then the $p$-class $A_p$ is defined to be $(a \in A; |a|^p < \infty)$. If $1 \leq p \leq \infty$, $A_p$ is then a two-sided $\ast$-ideal of $A$ (proper only if $p < 2$), and $(A_p, |\cdot|^p)$ is a normed $\ast$-algebra. $(A_2, |\cdot|_2)$ is $(A, ||\cdot||)$; and for $1 \leq p < 2, (A_p, |\cdot|^p)$ is a Banach $\ast$-algebra, for which structure theorems are given.

1. Introduction. Let $A$ be a semisimple $H^\ast$-algebra with inner product and norm denoted by $(,)$ and $||\cdot||$, respectively. The trace class of $A$, that is, the set $\tau(A) = \{xy; x, y \in A\}$, has been studied by Saworotnow and Friedell [8], who show, first of all, that for any nonzero $a \in A$ there exists a positive element $[a] \in A$ such that $[a]^2 = a^*a$, and $a \in \tau(A)$ if and only if $[a] \in \tau(A)$. An algebra norm $\tau$ is then introduced on $\tau(A)$ by defining $\tau(a) = tr[a]$ for each $a \in \tau(A)$, where in turn the trace functional $tr$ is unambiguously defined on $\tau(A)$ by letting $tr xy = (x, y^*) = \Sigma(xy)p_\omega, p_\omega)$, $\{p_\omega; \omega \in \Omega\}$ being any maximal family of mutually orthogonal nonzero self-adjoint idempotents. With this norm, $\tau(A)$ is actually a Banach algebra [9, Corollary to Theorem 1].

This presentation parallels that of Schatten [10] for $\tau c$, the trace class of $\sigma c$, the Schmidt class of operators on a Hilbert space.

In a somewhat similar sense our central development in $\S 3$ brings over into the present context some of the work of McCarthy [6] on the operator algebras $c_p$. We preface this with a basic spectral theorem established in $\S 2$; in $\S 4$ we study the structure of the Banach $\ast$-algebras $A_p$, where $1 \leq p < 2$. Finally, in $\S 5$ we relate $A_p$ to the class $c_p$ of operators on a Hilbert space [6; 2, ch. XI. 9] and also to $\mathcal{E}_p$ spaces [3, pp. 70 ff.; 5].

2. Preliminary spectral theory. Throughout the remainder of this paper $A$ will continue to denote a semisimple $H^\ast$-algebra. By a projection $p$ in $A$ we shall mean a nonzero self-adjoint idempotent. A projection $p$ is primitive if $p$ cannot be expressed as $p = p_1 + p_2$, where $p_1$ and $p_2$ are orthogonal projections. By a projection base in $A$ we mean a maximal family of mutually orthogonal projections (not necessarily primitive); note that if $a \in A$ and $\{p_\omega; \omega \in \Omega\}$ is a projection base, then $a = \Sigma ap_\omega = \Sigma p_\omega a$ [1, Theorem 4.1, where primitivity of the projections is not needed to establish this point]. Finally, we shall say that an element $a$ in $A$ is positive if $(ax, x) \geq 0$ for every $x \in A$; $a$ is then necessarily self-adjoint.
**Lemma 2.1.** Let $b$ be a nonzero normal element of $A$. There is a well-defined family $\{p_\omega; \omega \in \Omega\}$ of mutually orthogonal projections in $A$, and a well-defined set $\{\alpha_\omega; \omega \in \Omega\}$ of complex numbers, such that

1. $b = \sum \alpha_\omega p_\omega$ 
2. $bp_\omega = p_\omega b = \alpha_\omega p_\omega$ for each $\omega \in \Omega$.

The nonzero $\alpha_\omega$ are precisely the nonzero elements of the spectrum of $b$.

**Proof.** Let $A_0$ be the intersection of all maximal commutative $*$-subalgebras of $A$ containing $b$. $A_0$ is a proper $H^*$-algebra in the inner product and involution of $A$. Let $\{p_\omega; \omega \in \Omega\}$ be the collection of projections of $A_0$ which are primitive in $A_0$; then each $p_\omega A_0$ is a minimal ideal of $A_0$, and if $\omega_i \neq \omega_j$ we have $p_{\omega_i} p_{\omega_j} = 0$ and $(p_{\omega_i}, p_{\omega_j}) = 0$. Also, $A_0 = \sum p_\omega A_0$, the orthogonal direct sum of the minimal ideals $p_\omega A_0$, each of which is one-dimensional and consists of scalar multiples of $p_\omega$ [1, Corollary 4.1]. Therefore $b = \sum \alpha_\omega p_\omega$, where $\{\alpha_\omega; \omega \in \Omega\}$ is a set of complex numbers. Property (2) is immediate from the orthogonality of the $p_\omega$. We shall show that the nonzero $\alpha_\omega$ are the nonzero elements of $sp(b|A_0)$, the spectrum of $b$ relative to $A_0$. Let $\phi$ be any multiplicative linear functional on $A_0$. We have $\phi(p_\omega) = \phi(p_\omega^2) = [\phi(p_\omega)]^2$, and hence the value of $\phi$ at each projection $p_\omega$ must be either 0 or 1. $\phi$ cannot have the value 0 at every $p_\omega$ or else $\phi$ would vanish on $A_0$; nor can we have $\phi(p_\omega) = 1 = \phi(p_\omega^2)$ if $\omega_i \neq \omega_j$, for then $1 = \phi(p_\omega)^2 \neq \phi(p_\omega p_{\omega_j}) = \phi(0) = 0$. Therefore, each multiplicative linear functional on $A_0$ is of the form $\phi(p_\omega) = \delta_\omega$, where $\delta_\omega \in \Omega$. We have, for each $\omega \in \Omega$, $\phi(b) = \sum \alpha_\omega \delta_\omega(p_\omega) = \alpha_\omega = \hat{b}(\delta_\omega)$, where $\hat{b}$ denotes the Gelfand transform of $b \in A_0$. Since the nonzero $\alpha_\omega$ are therefore the nonzero elements of the range of $\hat{b}$, they are by the Gelfand theory precisely the nonzero elements of $sp(b|A_0)$. However, $sp(b|A) = sp(b|A_0)$, since if $c \in A_0$ has a quasi-inverse $c^0$ in $A$, then, as is well-known, $c^0$ belongs to every maximal commutative $*$-subalgebra of $A$ containing $c$, or equivalently, $c^0 \in A_0$. Finally, it is clear that the element $b$ uniquely determines the algebra $A_0$, along with its set of primitive projections $\{p_\omega; \omega \in \Omega\}$ and the corresponding numbers $\alpha_\omega$, since $\alpha_\omega p_\omega$ is the orthogonal projection of $b$ on the closed ideal $p_\omega A_0$ of $A_0$.

**Lemma 2.2.** Let $b$ be a nonzero normal element of $A$, and let $b = \sum \mu_\gamma q_\gamma$, where $\{q_\gamma\}$ is a countable (possibly finite) family of mutually orthogonal projections, and the $\mu_\gamma$ are nonzero complex numbers such that $\mu_m \neq \mu_n$ if $m \neq n$. Let $h$ be any self-adjoint element of $A$ which commutes with $b$. Then for each $n$, $hq_n = q_n h$. 

**Proof.** Extend $\{q_\gamma\}$ to a projection base $\{q_\gamma; \gamma \in \Gamma\}$. For each $\gamma$, if $q_\gamma = q_n$ for some $n$, let $\mu_\gamma = \mu_n$; otherwise, let $\mu_\gamma = 0$. (Note that $b q_\gamma = q_\gamma b = \mu_\gamma q_\gamma$ for each $\gamma \in \Gamma$.) Then for any $q_n$ we have $q_n h =$
\[ \Sigma q_i h q_i. \] Also, since \( b \) and \( h \) commute, \( \mu_n q_i h q_i = q_i b h q_i = q_i h b q_i = \mu_n q_i h q_i. \) If \( q \neq q_i \), then \( \mu_n \neq \mu_i \) and consequently \( q_i h q_i = 0. \) Thus \( q_i h = q_i h q_i. \) Taking adjoints we have \( h q_i = q_i h q_i; \) therefore \( h q_i = q_i h. \)

**Corollary 2.3.** Let \( b, \{\mu_n\}, \) and \( \{q_n\} \) be as in the lemma, and let \( A_o \) be, as before, the intersection of all maximal commutative \( \ast \)-subalgebras of \( A \) containing \( b. \) Then for each \( n, q_n \in A_o. \)

**Proof.** Let \( A_i \) be any maximal commutative \( \ast \)-subalgebra of \( A \) containing \( b. \) Since \( A_i \) is a \( \ast \)-algebra, each \( x \in A_i \) is of the form \( x = h + ik, \) where \( h, k \in A_i, \) and \( h \) and \( k \) are self-adjoint. Therefore, each \( q_n \) commutes with every element of \( A_i. \) and by maximality of \( A_i, q_n \in A_i. \) Therefore, finally, \( q_n \in A_o. \)

**Lemma 2.4.** Let \( b, \{\mu_n\}, \) and \( \{q_n\} \) be as in Lemma 2.2. Then each \( q_n \) is a finite sum of the projections \( p_{\omega} \) of Lemma 2.1.

**Proof.** Each \( q_n \) belongs to \( A_o, \) and therefore, as in the proof of Lemma 2.1, \( q_n = \Sigma \beta_{\omega} p_{\omega} \) for suitable numbers \( \beta_{\omega}. \) Also, \( q_n = q_n^* = \Sigma \beta_{\omega}^* p_{\omega}, \) and therefore each \( \beta_{\omega} \) is either 0 or 1. Only finitely many can be 1, since \( ||q_n||^2 = \Sigma \beta_{\omega}^2 ||p_{\omega}||^2 \geq \Sigma \beta_{\omega}^2. \)

Now let \( q_n = p_{n_1} + \cdots + p_{n_{k(n)}}. \) The orthogonal projection of \( b \) on the closed left ideal \( A_q n \) is \( b q_n = \mu_n q_n = \mu_n (p_{n_1} + \cdots + p_{n_{k(n)}}). \) From Lemma 2.1, since \( b = \Sigma \alpha_{\omega} p_{\omega}, \) this projection of \( b \) is also \( \alpha_{\omega} p_{n_1} + \cdots + \alpha_{n_{k(n)}} p_{n_{k(n)}}. \) Therefore \( \alpha_{\omega} = \mu_n, i = 1, \cdots k(n), \) and in the representation \( b = \Sigma \alpha_{\omega} p_{\omega} \) we may replace the sum \( \alpha_{\omega} p_{n_1} + \cdots + \alpha_{n_{k(n)}} p_{n_{k(n)}} \) by \( \mu_n q_n. \) If this is done for each \( n \) indexing the countable set \( \{q_n\}, \) the procedure evidently replaces the representation \( b = \Sigma \alpha_{\omega} p_{\omega} \) by \( b = \Sigma \mu_n q_n, \) and therefore makes use of every term \( \alpha_{\omega} p_{\omega} \) except those for which \( \alpha_{\omega} = 0. \) We thus have the following spectral theorem.

**Theorem 2.5.** Let \( b \) be a nonzero normal element of \( A. \) Then \( b \) may be represented uniquely (apart from the order of the terms) as a sum

\[ b = \Sigma \lambda_{\omega} e_{\omega}, \]

in which

1. \( \{\lambda_{\omega}\} \) is a countable family of distinct nonzero complex numbers consisting of the nonzero elements of the spectrum of \( b, \) and
2. \( \{e_{\omega}\} \) is a countable family of mutually orthogonal projections.

We have \( b e_{\omega} = e_{\omega} b = \lambda_{\omega} e_{\omega} \) for each \( n; b \) is self-adjoint if and only if each \( \lambda_{\omega} \) is real, and \( b \) is positive if and only if each \( \lambda_{\omega} > 0. \)

**Definition 2.6.** Let \( b \) be a nonzero normal element of \( A. \) A representation \((*)\) of \( b \) having properties (1) and (2) of Theorem 2.5...
will be called a spectral representation of \( b \). If \( b \) is a positive element of \( A \), we shall refer to the spectral representation of \( b \), meaning the one in which \( \lambda_m < \lambda_n \) if \( m > n \). For any nonzero normal element \( b \), the set \( E_b \) of mutually orthogonal projections in a spectral representation of \( b \) will be called the spectral family of \( b \).

**Definition 2.7.** Let \( b \) be a nonzero normal element of \( A \), and let \( E_b \) be its spectral family. A projection base \( \{e_\omega : \omega \in \Omega \} \) containing every \( e_\alpha \) in \( E_b \) will be called a projection base associated with \( b \). (Note that by a simple maximality argument, \( E_b \) can always be extended to a projection base associated with \( b \).)

3. The classes \( A_p \) and their basic properties. We begin this section by recalling some basic results from [8]. Corresponding to each \( a \) in \( A \) there is a unique positive element \([a]\) of \( A \) such that \( [a]^2 = a^*a \). Moreover, there is, for each nonzero \( a \) in \( A \), a well-defined partial isometry \( W \) on \( A \), having initial set \([a]A\) and final set \( aA\), such that \( a = W[a] \), \([a] = W^*a\), and \( \| W \| = 1 \). We shall call \( W \) the partial isometry associated with \( a \). We define a left centralizer on \( A \) to be an operator \( S \) in \( B(A) \) such that \( S(xy) = (Sx)y \) for all \( x,y \in A \). (This terminology, though widely used, is not universal; the type of operator just defined is called a right centralizer in [8] and [9], and elsewhere.) Evidently, each left multiplication operator \( L_a \), \( a \in A \), is a left centralizer on \( A \); also, for any nonzero \( a \) in \( A \), the partial isometry \( W \) associated with \( a \) is a left centralizer (see [8, p. 97]). We note, finally, for fairly frequent use, that for any \( x \in A \), \( \| ax \| = \| [a]x \| \), since \( \| ax \|^2 = (ax, ax) = (a^*ax, x) = ([a]^2x, x) = ([a]x, [a]x) = \| [a]x \|^2 \).

**Definition 3.1.** Let \( a \) be a nonzero element of \( A \), and let \([a] = \sum \lambda_n e_n\) be the spectral representation of \([a]\). We define

\[
|a|_p = (\sum \lambda_n^p \| e_n \|^p)^{1/p} \quad \text{for } 0 < p < \infty ,
\]

\[
|a|_\infty = \lambda_1.
\]

For \( a = 0 \), we define \(|a|_p = 0, \ 0 < p \leq \infty .\)

**Definition 3.2.** For \( 0 < p \leq \infty \), \( A_p = \{a \in A : |a|_p < \infty \} \).

**Remark 3.3.** For \( 0 < p \leq \infty \),

1. \( a \in A_p \) if and only if \([a] \in A_p \), since \([a] = [\{a\}]\) implies \(|a|_p = |[a]|_p \),
2. if \( e \) is a projection, \( e \in A_p \) and \(|e|_p = \|e\|^{p/p} \).
REMARK 3.4. Let \( \{e_\omega: \omega \in \Omega\} \) be a projection base associated with \( [a] \). We shall write \( [a] = \sum \lambda_\omega e_\omega \), always assuming that \( \lambda_\omega = \lambda_{\omega'} \) if \( e_\omega \in E_{(\omega)} \). Then \( |a|_p = (\sum \lambda_\omega ||e_\omega||^p)^{1/p} \) for \( 0 < p < \infty \); and we continue to write \( |a|_\infty = \lambda_1 \), understanding \( \lambda_1 \) to be sup \( \{\lambda_\omega: \omega \in \Omega\} \).

REMARK 3.5. Let \( \{e_\omega: \omega \in \Omega\} \) be a projection base associated with \( [a] \in A \).

(1) \( |a|_1 = ||[a]|| = \sum \lambda_\omega ||e_\omega||^2 = \sum \lambda_\omega ||[a]e_\omega||^2 = \sum ||[a]e_\omega||^2 = ||a||^2 \). Hence \( |a|_1 = ||a|| \) and \( A_1 = A \).

(2) \( |a|_1 = ||[a]|| = \sum \lambda_\omega ||e_\omega||^2 = \Sigma (\lambda_{e_\omega}, e_\omega) = tr[a] = \tau(a) [8, Lemma 3] \). Hence \( |a|_1 = \tau(a) \) and \( A_1 = \tau(A) \), the trace class of \( A \).

DEFINITION 3.6. Let \( b \) be a nonzero positive element of \( A \), with spectral representation \( b = \Sigma \lambda_\omega e_\omega \). For \( 0 < p < \infty \), \( b^p = \Sigma \lambda_\omega^p e_\omega \), provided that this sum exists in \( A \).

REMARK 3.7. From [8, Lemma 3] we have that \( a \in A_p \) if and only if \( [a]^p \in A_1 = \tau(A) \). This occurs if and only if \( [a]^p \) exists in \( A \); we then have \( |a|^p = \sum \lambda_\omega^p ||e_\omega||^2 = \tau([a]^p) = |[a]|_1 = ||[a]^p|| = ||[a]^p||^2 = \Sigma ||a||^p \lambda_{p_{e_\omega}, p_{e_\omega}} \) for any projection base \( \{p_{e_\omega}: \omega \in \Omega\} \).

REMARK 3.8. For \( 0 < p \leq \infty \), clearly \( |a|_p \geq 0 \), and \( |a|_p = 0 \) if and only if \( a = 0 \). Also, since \( [aa] = |a| [a] \) for any complex number \( a \), we have \( |aa|_p = |a|_p |a|_p \).

LEMMA 3.9. For any \( a \in A \) and \( 0 < p < \infty \), \( |a|_\infty \leq |a|_p \).

Proof. For \( a = 0 \) the result is obvious. Otherwise, using the spectral representation of \( [a] \), we have \( |a|_p^2 = \lambda_1^p \leq \Sigma \lambda_\omega^p ||e_\omega||^2 = |a|_p^2 \).

LEMMA 3.10. For any \( a \in A \), \( ||ax|| \leq ||a||_\infty ||x|| \).

Proof. For \( a \neq 0 \), let \( \{e_\omega: \omega \in \Omega\} \) be a projection base associated with \( [a] \). Then \( [a]x = \Sigma \lambda_\omega e_\omega x \) and \( ||[a]x||^2 = \Sigma \lambda_\omega^2 ||e_\omega x||^2 \leq \lambda_1^2 \Sigma ||e_\omega x||^2 = \lambda_1^2 ||x||^2 \). Hence \( ||ax|| = ||[a]x|| \leq ||a||_\infty ||x|| \).

COROLLARY 3.11. For any \( a \in A \), \( |a|_\infty = ||L_a|| \).

Proof. For \( a, x \neq 0 \), \( ||ax||/||x|| \leq |a|_\infty \), by the lemma. But \( ||ae_i||/||e_i|| = ||[a]e_i||/||e_i|| = \lambda_i = |a|_\infty \).

PROPOSITION 3.12. For \( a \in A \) and \( 0 < p < q \leq \infty \), \( |a|_q \leq |a|_p \).
Hence $A_p \subset A_q$, and if $2 \leq p \leq \infty$ then $A_p = A$.

Proof. Using the spectral representation of $[a]$, we have $|a|^p = \sum \lambda_n^p \|e_n\|^2 = \sum \lambda_n^{p-r} \lambda_n^p \|e_n\|^2 \leq \lambda_n^{-p} \sum \lambda_n^p \|e_n\|^2 = |a|^{p-r} |a|_p \leq |a|_p^p$, by Lemma 3.9.

REMARK 3.13. By 3.7, $a \in A_{2p}$ $(0 < p < \infty)$ if and only if $[a]^p$ exists in $A$. For $1 \leq p < \infty$, $A_{2p} = A$ and hence $[a]^p$ is defined.

PROPOSITION 3.14. If $A$ is infinite-dimensional, then for $0 < p < q \leq 2$, $A_q$ is properly larger than $A_p$.

Proof. From the structure theory of $H^*$-algebras [1], we see that if $A$ is infinite-dimensional then $A$ contains a countably infinite set $\{e_n : n \in \mathbb{N}\}$ of mutually orthogonal projections. Choose $r$ such that $p < r < q$; then the series $\sum_{n=1}^{\infty} n^{-r/p} \|e_n\|^{-q/p} e_n$ converges to a positive element of $A$ (since the squares of the norms of its terms have a finite sum). Denoting this element by $a$, we observe that the given series (or one obtained from it by grouping and rearranging terms) is the spectral representation of $a$. Thus $a \in A_q$, since $|a|^q = \sum_{n=1}^{\infty} n^{-q/r} < \infty$; however $a \notin A_p$, since $|a|^p = \sum_{n=1}^{\infty} n^{-p/r} \|e_n\|^{-q/(3p/q)} \geq \sum_{n=1}^{\infty} n^{-p/r} = \infty$.

Some elements of the following lemma appear in [8, p. 96]. For most of it, however, the author is indebted to M. Kervin.

LEMMA 3.15. Let $a$ be any nonzero element of $A$, and let $[a] = \sum \lambda_n^o e_n$ be the spectral representation of $[a]$. For each $n$, let $f_n = \lambda_n^o a e_n a^*$. Then $[a^*] = \sum_n f_n$ is the spectral representation of $[a^*]$, and $\|f_n\| = \|e_n\|$ for each $n$.

Proof. Clearly, the $\lambda_n$ are distinct positive numbers and the $f_n$ are self-adjoint. We recall, first of all, that $[a]^o = \sum \lambda_n^o e_n = a^*a$, and therefore $a^* a e_n = e_n a^* a = \lambda_n^o e_n$. Thus $f_n f^*_n = (\lambda_n^o a e_n a^*)(\lambda_n^o a e_n a^*) = \lambda_n^o \lambda_n^o a e_n a e_n a^* = \lambda_n^2 (\lambda_n^o a e_n a^*) (\lambda_n^o a e_n a^*) = \delta_{mn} f_n$. Therefore, the $f_n$ are mutually orthogonal idempotents. Also, $\lambda_n^o \|f_n\|^2 = \lambda_n^o (a e_n a^*) (a e_n a^*) = \delta_{mn} \|e_n\|^2$, and therefore $\|f_n\| = \|e_n\|$ and the $f_n$ are nonzero. Now we wish to show that $[a^*] = \sum f_n$. We shall show first that $a = \Sigma a e_n$. Extend the family $E_{[a]}$ to a projection base $\{e_n : \omega \in \Omega\}$. Then $a = \Sigma a e_n$ and $a^*a = \Sigma a^* a e_n$. But if $e_n \notin E_{[a]}$ then $a^* a e_n = 0$, since $a^* a = \Sigma a^* a e_n = a^* a e_n$. Therefore, for $e_n \in E_{[a]}$ we have $e_n a^* a e_n = 0 = (a e_n^*) (a e_n)$, and thus $a e_n = 0$ [1, Lemma 2.2]. We conclude that $a = \Sigma a e_n$. Finally, $(\Sigma f_n) = \Sigma \lambda_n^o f_n = \Sigma a e_n a^* = a a^*$, and therefore $\Sigma f_n$ is the (unique) positive square root of $a a^*$; that is, $\Sigma f_n = [a^*]$.
Corollary 3.16. For any \( a \in A \) and \( 0 < p \leq \infty \), \( |a|^p = |a^*|^p \).
Hence \( a \in A_p \) if and only if \( a^* \in A_p \).

In order to arrive at the results announced in our opening synopsis,
we shall need to establish several crucial inequalities. Lemmas 3.17, 3.18, and 3.22 are adapted from [6, Lemmas 2.1, 2.2].

Lemma 3.17. For \( 0 < p < \infty \), let \( b \) be a positive element of \( A^2 \) (so that \( b^p \) exists in \( A \)). Then for any nonzero \( x \in A \),

1. \( (b^p, x) \geq (bx, x)^p \|x\|^{2(1-p)} \) if \( 1 \leq p < \infty \),
2. \( (b^p, x) \leq (bx, x)^p \|x\|^{2(1-p)} \) if \( 0 < p \leq 1 \).

Proof. (1) Suppose \( 1 \leq p < \infty \). Let \( \{e_\omega: \omega \in \Omega\} \) be a projection base associated with \( b \), where, as usual, we take \( \lambda_\omega = \lambda_n \) if \( e_\omega = e_n \in E_0 \), and \( \lambda_\omega = 0 \) if \( e_\omega \in E_b \). We have, by Hölder’s inequality,

\[
(bx, x) = \sum \lambda_\omega (e_\omega x, x) \leq \left[ \sum \lambda_\omega^p (e_\omega x, x) \right]^{1/p} \left[ \sum (e_\omega x, x)^{1-1/p} \right]^{1/p-1} = (bx, x)^p \|x\|^{2(1-p)}.
\]

Hence \( (b^p, x) \geq (bx, x)^p \|x\|^{2(1-p)} \).

(2) Suppose \( 0 < p \leq 1 \). Replace the element \( b \) in (1) by \( b^p \) and the exponent \( p \) by \( 1/p \) to obtain the desired inequality.

Lemma 3.18. Let \( a \in A \), and let \( \{q_\omega: \omega \in \Omega\} \) be a projection base for \( A \). Then

1. \( |a|^p \leq \sum |aq_\omega|^p \|q_\omega\|^{2-p} \) if \( 1 \leq p \leq 2 \),
2. \( |a|^p \leq \sum |aq_\omega|^p \|q_\omega\|^{2-p} \) if \( 2 \leq p < \infty \).

In each case, equality holds if \( \{q_\omega: \omega \in \Omega\} \) is a projection base associated with \( [a] \).

Proof. We note first that \( [a]^p \) exists, since \( p \geq 1 \).

(1) Suppose \( 1 \leq p \leq 2 \). By (2) of Lemma 3.17 we have for each \( q_\omega \),

\[
([a]^p q_\omega, q_\omega) = \left( ([a]^p)^{p/2} q_\omega, q_\omega \right) \leq \left( [a]^p q_\omega, q_\omega \right)^{p/2} \|q_\omega\|^{2-p} = \|aq_\omega\|^p \|q_\omega\|^{2-p}.
\]

Summing over \( \Omega \) gives, by 3.7,

\[
|a|^p = \sum ([a]^p q_\omega, q_\omega) \leq \sum \|aq_\omega\|^p \|q_\omega\|^{2-p}.
\]

If \( \{q_\omega\} \) is a projection base associated with \( [a] \), then by 3.4 we have
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(2) is proved similarly, using (1) of Lemma 3.17.

PROPOSITION 3.19. For \(1 \leq p \leq \infty\), let \(a \in A_p\), and let \(S\) be a left centralizer on \(A\). Then \(Sa \in A_p\), and \(|Sa|_p \leq ||S|| |a|_p^p\).

Proof. The result is standard for \(p = \infty\). Suppose \(1 \leq p \leq 2\); let \(\{e_\omega: \omega \in \Omega\}\) be a projection base associated with \([a]\). By Lemma 3.18 (1), \(|Sa|_p^p \leq \sum ||(Sa)e_\omega||_p^p|e_\omega||^{2-p} = \sum ||S(ae_\omega)||_p^p|e_\omega||^{2-p}\leq ||S||_p^p \sum ||ae_\omega||_p^p|e_\omega||^{2-p} = ||S||_p^p |a|_p^p^p.\) Now suppose \(2 \leq p < \infty\), and this time let \(\{e_\omega: \omega \in \Omega\}\) be a projection base associated with \([Sa]\). We have, using (2) of Lemma 3.18, \(|Sa|_p^p = \sum ||(Sa)e_\omega||_p^p|e_\omega||^{2-p} = \sum ||S(\alpha e_\omega)||_p^p|e_\omega||^{2-p}\leq ||S||_p^p \sum ||ae_\omega||_p^p|e_\omega||^{2-p} \leq ||S||_p^p |a|_p^p^p.\)

COROLLARY 3.20. For \(1 \leq p \leq \infty\), let \(a \in A_p\), \(x \in A\). Then \(xa\) and \(ax\) belong to \(A_p\), and \(|xa|_p \leq |x|_p |a|_p^p\), \(|ax|_p \leq |a|_p^p |x|_p^p\).

Proof. By Corollary 3.11 the statements about \(xa\) are immediate, since \(L_x\) is a left centralizer. We also have, by Corollary 3.16, \(|ax|_p = ||(ax^*)^p\| = |x^*a^*|_p \leq |x^*|_p |a^*|_p = |a|_p |x|_p^p\).

COROLLARY 3.21. For \(1 \leq p \leq \infty\), let \(a, b \in A_p\). Then \(|ab|_p \leq |a|_p^p |b|_p^p\).

In our next lemma we shall make use of a special operator decomposition given by McCarthy [6, p. 250]. Suppose \(T \in B(A)\); then \(T = (TT^*)^{1/4}U(T^*T)^{1/4}\), where \(U\) is a partial isometry with \(||U|| = 1\).

LEMMA 3.22. Suppose \(1 \leq p < \infty\). Let \(a \in A\), and let \(\{e_\omega: \omega \in \Omega\}\) be any projection base for \(A\). Then \(\sum ||(aq_\omega, q_\omega)||^p \leq |a|_p^p\).

Proof. We use the operator decomposition just mentioned: \(L_a = (L_aL_a^*)^{1/4}U(L_a^*L_a)^{1/4} = L^{(1/2)}_{[a]}UL^{(1/2)}_{[a]}\). We have, by two applications of the Schwarz inequality,

\[
\sum ||(aq_\omega, q_\omega)||^p \leq |a|_p^p
\]

\[
\leq \sum ||L^{(1/2)}_{[a]}q_\omega||^p ||L^{(1/2)}_{[a]}q_\omega||^p \leq \sum ||L^{(1/2)}_{[a]}q_\omega||^p \leq \sum ||L^{(1/2)}_{[a]}q_\omega||^p \leq \sum ||L^{(1/2)}_{[a]}q_\omega||^p \leq |a|_p^p
\]
\[ \begin{align*}
&= [\Sigma (L_l^{q}q_\omega, L_l^{q}q_\omega)^p \| q_\omega \|^{1/p}]^{1/p} \Sigma (L_l^{q}q_\omega, L_l^{q}q_\omega)^p \| q_\omega \|^{1/2} \\
&= [\Sigma (L_l^{q}q_\omega, L_l^{q}q_\omega)^p \| q_\omega \|^{2/(1-p)}]^{1/2} \Sigma (L_l^{q}q_\omega, L_l^{q}q_\omega)^p \| q_\omega \|^{2/(1-p)} \\
&\leq [\Sigma (L_l^{q}q_\omega, L_l^{q}q_\omega)^p \| q_\omega \|^{2/(1-p)}]^{1/2} \Sigma [a^* q_\omega, q_\omega]^p \| q_\omega \|^{2/(1-p)} \\
&= [a^*]^p [a^*]^p \text{ by Lemma 3.17 (1)} \\
&= [a^*]^p.
\end{align*} \]

**Proposition 3.23.** For \( 1 \leq p \leq \infty \), let \( a, b \in A_p \). Then \( |a + b|_p \leq |a|_p + |b|_p \).

**Proof.** The result is well-known for \( p = \infty \). For \( 1 \leq p < \infty \), let \( \{e_\omega : \omega \in \Omega\} \) be a projection base associated with \([a + b]\), and let \( W \) be the partial isometry associated with \( a + b \). Then

\[ |a + b|_p = \Sigma (a + b)e_\omega, e_\omega)^p \| e_\omega \|^{2/(1-p)} \]
\[ = \Sigma ((W^*a)e_\omega, e_\omega)^p \| e_\omega \|^{2/(1-p)} \]
\[ = [\Sigma ((W^*a)e_\omega, e_\omega)^p \| e_\omega \|^{2/(1-p)}]^{1/p} + [\Sigma ((W^*b)e_\omega, e_\omega)^p \| e_\omega \|^{2/(1-p)}]^{1/p} \]
\[ \leq \Sigma ((W^*a)e_\omega, e_\omega)^p \| e_\omega \|^{2/(1-p)} + \Sigma ((W^*b)e_\omega, e_\omega)^p \| e_\omega \|^{2/(1-p)} \]

by Minkowski's inequality
\[ \leq W^*a|_p + W^*b|_p \]
\[ \leq \| W^* \| |a|_p + \| W^* \| |b|_p \]
\[ = |a|_p + |b|_p. \]

**Corollary 3.24.** For \( 1 \leq p \leq \infty \), \( A_p \) is a normed linear space. Hence \( A_p \) is a two-sided \(*\)-ideal of \( A \) and \( (A_p, \cdot \cdot_p) \) is a normed algebra.

Now for \( 1 \leq p \leq \infty \) we wish to investigate the relationship between \( A_p \) and the dual space of \( A_q \), where \( (1/p) + (1/q) = 1 \). In what follows we shall omit proofs for the cases \( p = 1, q = \infty \) and \( p = \infty, q = 1 \); these are given in [9].

**Lemma 3.25.** Let \( (1/p) + (1/q) = 1 \), where \( 1 \leq p, q \). Let \( a \in A_p, b \in A_q \). Then \( |tr ab| = |tr ba| \leq |a|_p |b|_q \).

**Proof.** We shall assume with no loss of generality that \( 1 < p \leq 2 \) and hence \( 2 \leq q < \infty \). Let \( \{e_\omega : \omega \in \Omega\} \) be a projection base associated with \( [a] \). Then \( |tr ab| = |tr ba| = \Sigma (bae_\omega, e_\omega) \leq \Sigma |ae_\omega, b^* e_\omega| \leq \Sigma |ae_\omega| \| b^* e_\omega \| = \Sigma |ae_\omega|^p \| e_\omega \|^{-(p-1)/q} \| b^* e_\omega \| \| e_\omega \|^{(2-q)/q} \), since \((2 - p)/p + (2 - q)/q = 0\). By Hölder's inequality, the last sum does not exceed \[ \Sigma \| ae_\omega \|^{r} \| e_\omega \|^{1/r} \| b^* e_\omega \| \| e_\omega \|^{-(r-1)/q}. \] But the first sum in brackets is \( |a|_p^r \), and the second is less than or equal to \( |b^*|_q^r \), by Lemma 3.18
(2). Hence $|\text{tr } ab| \leq |a|_p |b|_q$.

For each $a \in A$, we now define $\phi_a(x) = \text{tr } xa$ for all $x \in A$. From the linearity of $\text{tr}$ on the trace class $\tau(A)$, it is evident that $\phi_a$ is a linear functional on $A$; moreover, $\phi_a$ is bounded and $||\phi_a|| \leq |a|_p$, by Lemma 3.25. We shall show that the opposite inequality holds as well.

**Proposition 3.26.** For $1 \leq p \leq \infty$, the mapping $a \to \phi_a$ is a linear isometry of $A_p$ into $A'_q$, the dual space of $A_q$.

**Proof.** Again using the linearity of $\text{tr}$ on $\tau(A)$ one easily verifies that the mapping is linear. In view of our above remarks, therefore, we need only prove that $|a|_p \leq ||\phi_a||$. Let $[a] = \Sigma \lambda_n e_n$ be the spectral representation of $[a]$, and let $w_n = \Sigma \lambda_n^{-1/2} e_n a \in A_q$. We shall compute $||w_n||_q$.

First of all, $w_n^* w_n = (\Sigma \lambda_n^{-1/2} e_n a^*) (\Sigma \lambda_n^{-1/2} e_n a) = \Sigma \lambda_n^{-1} \lambda_n^{-1} e_n a^* e_n a = \Sigma \lambda_n^{-1} \lambda_n^{-1} e_n a^* e_n a = \Sigma \lambda_n^{-1} \lambda_n^{-1} e_n a^* e_n a$.

Since, by Lemma 3.15, the $\lambda_n$ are mutually orthogonal projections with $||\lambda_n^*|| = ||e_n||$, we have $[w_n] = \Sigma \lambda_n^{-1} \lambda_n^{-1} e_n a^*$ for every $k$. We also have $\Sigma \lambda_n^{-1} ||e_n||^2 = \Sigma \lambda_n^{-1} \lambda_n^{-1} ||e_n||^2 = \Sigma \lambda_n^{-1} \lambda_n^{-1} ||e_n||^2 = \Sigma \lambda_n^{-1} \lambda_n^{-1} ||e_n||^2$. Thus $\Sigma \lambda_n^{-1} ||e_n||^2 \leq ||\phi_a||$ for every $k$, and since $\lambda_n^{-1} ||e_n||^2 \leq ||\phi_a||^p$ for every $k$, we have $|a|_p \leq ||\phi_a||^p$.

**Theorem 3.27.** For $1 \leq p \leq 2$, the mapping $a \to \phi_a$ is a linear isometry of $A_p$ onto $A'_q$.

**Proof.** Let $\phi$ be any bounded linear functional on $A_q$. Then for all $x \in A_{(\cdot)} = A$, $||\phi(x)|| \leq ||\phi|| ||x||$, by Proposition 3.12. Therefore $\phi$ is a bounded linear functional on $A$, and by the Riesz representation theorem there exists $a \in A$ such that $\phi(x) = (x, a^*) = \text{tr } xa$ for all $x \in A$. We need only show that $a \in A_p$. But if we again consider the spectral representation $[a] = \Sigma \lambda_n e_n$ and define $w_n$ as in the preceding proof, the same computations show that $\Sigma \lambda_n^{-1} ||e_n||^2 \leq ||\phi||^p$ for every $k$, and hence $\lambda_n^{-1} ||e_n||^2 < \infty$ and $a \in A_p$.

**Corollary 3.28.** For $1 \leq p \leq 2$, $(A_p, ||\cdot||_p)$ is a Banach $*$-algebra.

We conclude this section with an example to show that if $2 < p \leq \infty$ and $A = A_p$ is infinite-dimensional, then $(A_p, ||\cdot||_p)$ is incomplete. First of all, if $(A_p, ||\cdot||_p)$ is complete, then from the inverse mapping theorem and the fact that $||\cdot||_p$ is dominated by $||\cdot||$, we can conclude that these two norms are equivalent on $A$. But this is not so if $A$ is infinite-dimensional, for if $\{e_n: n \in N\}$ is a countably infinite set of mutually orthogonal projections in $A$ and we let $s_k = \Sigma_{n=1}^k n^{-1/2} ||e_n||^{-1/p} e_n$, then $\{s_k\}$ is a Cauchy sequence in the $||\cdot||_p$-topology but not in the
4. The structure of the Banach $*$-algebras $A_p$. In this section we shall confine our attention mainly to the algebras $A_p$, where $1 \leq p \leq 2$, although some of our results hold for $p > 2$ as well. Unless otherwise indicated, therefore, we shall assume throughout that $1 \leq p < 2$. We begin by observing that for these values of $p$, $A_p$ is a quite special instance of an $IP$-algebra, as introduced and studied by Yood in [12]; hence the entire theory of that paper is at our disposal. Furthermore, it is readily verified that $(A_p, ||\cdot||)$ is a (normed) Hilbert algebra; we shall immediately note some properties of this Hilbert algebra. Our first lemma is a simple consequence of the $||\cdot||$-continuity of multiplication.

**Lemma 4.1.** If $R$ is any right ideal of $A_p$, then $\bar{R}$, the closure of $R$ in $A$, is a closed right ideal of $A$.

**Lemma 4.2.** If $R$ is a right ideal of $A_p$ and $P$ is the orthogonal projection operator of $A$ onto $R$, the closure of $R$ in $A$, then for any $a \in A_p, Pa \in A_p$. In particular, if $R$ is relatively $||\cdot||$-closed in $A_p$ then $Pa \in R$.

**Proof.** This is immediate from Proposition 3.19, inasmuch as $P$ is a left centralizer on $A$.

**Proposition 4.3.** If $R$ is a relatively $||\cdot||$-closed right ideal of $A_p$, then $A_p = R \oplus R^\perp$, where $R^\perp$ is the orthogonal complement of $R$ in $A_p$.

**Proof.** Considering the closures in $A$ of these right ideals, we have, for any $a \in A_p, a = a_1 + a_2$, where $a_1 \in R, a_2 \in R^\perp$. But by Lemma 4.2, $a_1 \in R$ and $a_2 \in R^\perp$.

**Remark 4.4.** For a closed right ideal $R$ in any Hilbert algebra, we have $\mathcal{L}(R) = R^\perp*$, where $\mathcal{L}(R)$ is the left annihilator of $R$. This is readily established by the argument used for an $H^*$-algebra [5, Theorem 12]. Combining this fact with Proposition 4.3 we obtain the following.

**Corollary 4.5.** $(A_p, ||\cdot||)$ is a dual Hilbert algebra.

Our next proposition, along with the known structure theory of $H^*$-algebras [1, Theorem 4.2], enables us to obtain a structure theorem for the Hilbert algebras $A_p$. 

\[\text{THE } p\text{-CLASSES OF AN } H^*\text{-ALGEBRA} 787\]
PROPOSITION 4.6. Let $I$ be a closed two-sided ideal of $A$ (and therefore an $H^*$-algebra). Then $I \cap A_p = I_p$, the $p$-class of $I$.

Proof. If $a \in I_p$ then $[a]$, as an element of the $H^*$-algebra $I$, has a spectral decomposition $[a] = \sum \lambda_n e_n$, where $e_n \in I$ for each $n$, and $\sum \lambda_n \| e_n \|^2 < \infty$. This is therefore the (unique) spectral decomposition of $[a]$ in $A$, and therefore $a \in I \cap A_p$. Conversely, suppose $a \in I \cap A_p$. Since $a \in I$, $[a]$ has a spectral decomposition $[a] = \sum \lambda_n e_n$ in $I$, and again this is its unique spectral decomposition in $A$. Since $a \in A_p$ we have $\sum \lambda_n \| e_n \|^2 < \infty$, and therefore $a \in I_p$.

REMARK 4.7. Let $J$ be a relatively $\| \cdot \|$-closed two-sided ideal of $A_p$. Then $J$ is a minimal closed ideal of $A_p$ if and only if $J_\circ$, the closure of $J$ in $A$, is a minimal closed ideal of $A$. If the latter condition holds (so that $J$ is a topologically simple $H^*$-algebra), then $J$ is a topologically simple Hilbert algebra.

We use these results and Lemma 4.2 to obtain our structure theorem for $A_p$ as a Hilbert algebra.

THEOREM 4.8 The Hilbert algebra $(A_p, \| \cdot \|)$ is the direct topological sum of its minimal closed two-sided ideals, which are mutually orthogonal. Each of these is a topologically simple Hilbert algebra and is the $p$-class of a minimal closed two-sided ideal of $A$.

For the remainder of this section we consider the Banach $*-\$algebras $(A_p, | \cdot |)$. Our aim in the following development is twofold: (1) to investigate the $| \cdot |$-closed right ideals of $A_p$; (2) to obtain a structure theorem for $(A_p, | \cdot |)$ analogous to Theorem 4.8.

LEMMA 4.9. Let $I$ be any $| \cdot |$-closed two-sided ideal of $A$. For any $a \in A$, let $a_i$ denote the orthogonal projection of $a$ on $I$. Then

1. $(a^*)_i = (a_i)^*$,
2. $[a_i] = [a_i]$. 

Proof. Let $a = a_i + a_\perp$, where $a_\perp \in I^\perp$, the orthogonal complement of $I$ in $A$. Then $a^* = a_i^* = (a_i)^* + (a_\perp)^*$. (1) follows readily from the fact that $I$ and $I^\perp$ are closed under the involution. To establish (2), we first note that $a^*a = a_i^*a_i + a_\perp^*a_\perp$. Then, letting $[a] = [a_i] + [a_\perp]$, we have $a^*a = [a]^2 = [a_i]^2 + [a_\perp]^2$, and hence $[a_i]^2 = a_i^*a_i$, by the uniqueness of the decomposition. If we show that $[a_i]$ is positive, then $[a]_i = [a_i]$ by the definition of $[a_i]$. For any $x \in A$, let $x = x_1 + x_\perp$, where $x_1 \in I$, $x_\perp \in I^\perp$. Then $([a]x, x) = ([a]x_1 + [a]x_\perp, x_1 + x_\perp) = ([a]x_1, x_1) = ([a][x_1], x_1) = ([a][x_1] + [a][x_\perp], x_1) = ([a][x_1], x_1) \geq 0$. 

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PROPOSITION 4.10. Let \( \{J_\gamma : \gamma \in \Gamma\} \) be a family of mutually orthogonal relatively ||\( \cdot ||\)-closed two-sided ideals of \( A_\gamma \). Let \( a_\gamma \in J_\gamma \) for each \( \gamma \), and let \( a = \Sigma a_\gamma \) (in the \( ||\cdot||\)-topology). Then \( |a|_p^p = \Sigma |a_\gamma|_p^p \), and hence \( a \in A_\gamma \) if and only if \( \Sigma |a_\gamma|_p^p < \infty \).

Proof. Clearly, each \( a_\gamma \) is the orthogonal projection of \( a \) on \( J_\gamma \), and hence, by the preceding lemma, \( |a|_p^p = \Sigma |a_\gamma|_p^p \). Now for each \( \gamma \), let \( [a_\gamma] = \Sigma \Lambda_\gamma |a_\gamma|_p^p \) be the spectral representation of \( [a_\gamma] \) in the \( H^*\)-algebra \( J_\gamma \), the \( ||\cdot||\)-closure of \( J_\gamma \) in \( A \). Then \( |a_\gamma|_p^p = \Sigma \Lambda_\gamma |a_\gamma|_p^p \). Also, \( [a] = \Sigma \Lambda_\gamma |a_\gamma|_p^p \), and since in this sum there cannot be infinitely many equal coefficients, the spectral representation of \( [a] \) is obtained by merely grouping the terms of the series having the same coefficient, and then rearranging the terms, if necessary. Hence \( |a|_p^p = \Sigma \Lambda_\gamma |a_\gamma|_p^p \).

REMARK 4.11. This proposition also holds for \( 2 \leq p < \infty \). Also, it is easily seen that \( |a|_\infty = \sup \gamma |a_\gamma|_\infty \).

LEMMA 4.12. Let \( a \in A_\gamma \) and let \( \varepsilon \) be any positive number. Then there exist projections \( e \) and \( f \) in \( A_\gamma \) such that \( |a - ae|_p < \varepsilon \) and \( |a - fa|_p < \varepsilon \).

Proof. Let \( A_0 \) be the intersection of all maximal commutative *-subalgebras of \( A \) containing \([a]\). Then, as in Lemma 2.1, we have a representation \([a] = \Sigma \alpha_n p_n \) (each \( \alpha_n \neq 0 \)), which, by grouping and rearranging of terms, yields the spectral representation of \([a]\); hence \( |a|_p^p = \Sigma \alpha_n^p |p_n|_p^p \). (Note that \([a] \in (A_0)_p \).) We may write \([a] = ([a] - \Sigma_{n=1}^k \alpha_n p_n) + (\Sigma_{n=k+1}^\infty \alpha_n p_n)\), where \( \Sigma_{n=k+1}^\infty \alpha_n p_n \) belongs to the relatively \( ||\cdot||\)-closed two-sided ideal \( \Sigma_{n=k+1}^\infty (A_0)_p \) of \((A_0)_p \), and \( [a] - \Sigma_{n=1}^k \alpha_n p_n \) belongs to the orthogonal complement of this ideal in \((A_0)_p \). By Proposition 4.10, \( |a|_p^p = |[a]|_p^p = |[a] - \Sigma_{n=1}^k \alpha_n p_n|_p^p + |\Sigma_{n=k+1}^\infty \alpha_n p_n|_p^p \). But this last term is \( \Sigma_{n=k+1}^\infty \alpha_n p_n \), which has the limit \( |a|_p^p \) as \( k \rightarrow \infty \). We therefore have \( \lim_{k \rightarrow \infty} |[a] - \Sigma_{n=1}^k \alpha_n p_n|_p^p = 0 \). Hence for sufficiently large \( k \) there is a projection \( e = \Sigma_{n=1}^k p_n \) such that \( |[a] - [a]e|_p < \varepsilon \). Taking \( W \) to be the partial isometry associated with \( a \), we have, using Proposition 3.19, \( |a - ae|_p = |W[a] - (W[a])e|_p = |W[a] - W(a)e|_p \leq ||W|| |[a] - [a]e|_p < \varepsilon \). There is likewise a projection \( f \) such that \( |a^* - a^*f|_p < \varepsilon \); hence \( |a - fa|_p = |(a - fa)^*|_p < \varepsilon \).

COROLLARY 4.13. For any \( a \in A_\gamma \), \( a \in \overline{A_\gamma a} \), where the closure is in the \( ||\cdot||_p \)-topology.

We remarked at the beginning of this section that \( A_\gamma \) is a special
case of an IP-algebra, and now that we have established the result of Corollary 4.13, we immediately have the following from [12, Theorems 3.5 and 4.9].

**Corollary 4.14.** \((A_p, \cdot|_p)\) has dense socle, and is the direct topological sum of its minimal closed two-sided ideals.

**Corollary 4.15.** \((A_p, \cdot|_p)\) is a dual algebra.

A simple consequence of Corollary 4.15 is the following.

**Proposition 4.16.** Let \(R\) be a right ideal of \(A_p\). \(R\) is closed in the \(|\cdot|_p\)-topology if and only if \(R\) is relatively closed in the \(||\cdot||\)-topology.

**Proof.** Since \(||a|| \leq |a|_p\) for every \(a \in A_p\), by Proposition 3.12, it is clear that every relatively \(||\cdot||\)-closed subset of \(A_p\) is \(|\cdot|_p\)-closed. Moreover, if the right ideal \(R\) is \(|\cdot|_p\)-closed, then it is an annihilator ideal, by Corollary 4.15, and therefore is relatively \(||\cdot||\)-closed, by the \(||\cdot||\)-continuity of multiplication.

**Remark 4.17.** This result holds for \(2 \leq p \leq \infty\). In this case, \(R\) is clearly \(||\cdot||\)-closed if it is \(|\cdot|_p\)-closed. But if \(R\) is a \(||\cdot||\)-closed right ideal of \(A_p(=A)\), we have \(R = R^{\perp \perp} = \mathcal{L}(R^*)^*, \) by 4.4. By the \(|\cdot|_p\)-continuity of multiplication, \(\mathcal{L}(R^*)\) is \(|\cdot|_p\)-closed.

We combine Proposition 4.16 with Proposition 4.3 to obtain the following.

**Corollary 4.18.** \((A_p, \cdot|_p)\) is a right complemented algebra (in the sense of [11]).

More can be said about the manner in which \(A_p\) is the direct topological sum of its minimal closed two-sided ideals. In order to do so, we obtain a converse of Proposition 4.10, which leads to our final structure theorem.

**Proposition 4.19.** Let \(\{J_\gamma : \gamma \in \Gamma\}\) be a family of mutually orthogonal closed two-sided ideals of \(A_p\). Let \(a_\gamma \in J_\gamma\) for each \(\gamma\), and suppose that \(\Sigma |a_\gamma|^p < \infty\). Then there exists \(a \in A_p\) such that \(a = \Sigma a_\gamma\), where the sum may be taken in the \(|\cdot|_p\)-topology or the \(||\cdot||\)-topology.

**Proof.** Considering only the nonzero \(a_\gamma\), which we denote as \(a_n\), let \(s_k = \Sigma_{n=1}^k a_n\). Then, by Proposition 4.10, for \(k > m\) we have \(|s_k - s_m|^p = \Sigma_{n=m+1}^k |a_n|^p \rightarrow 0\) as \(k, m \rightarrow \infty\). The Cauchy sequence \(\{s_k\}\) thus has a limit \(a\) in the Banach algebra \((A_p, |\cdot|_p)\), and \(a = \Sigma a_\gamma\).
Σaₙ = Σaᵣ in the |·|ᵣ-topology. (A standard argument shows that the limit is independent of the order of summation.) By Proposition 3.12, the sum is the same in the ||·||-topology.

**Theorem 4.20.** The Banach *-algebra \((A_p, |·|_p)\) is the p-direct sum of its minimal closed two-sided ideals \(J_λ\). The \(J_λ\) are mutually orthogonal and each is a topologically simple Banach *-algebra. \(A_p\) is the "p-direct sum" in that it consists precisely of all sums \(Σaₙxₙ\), \(aₙ∈J_λ\), such that \(Σ|aₙ|_p^p < ∞\), where \(a = Σaₙ\) may be understood as a limit in either the |·|ᵣ-topology or the ||·||-topology, and \(|a|_p = (Σ|aₙ|_p^p)^{1/p}\).

5. **Relationship to other systems.** If \(A\) is a topologically simple \(H^*-\)algebra, then there is a *-isomorphism \(x → X\) of \(A\) onto the Schmidt class \(σc\) of operators on the Hilbert space \(H = l_2(Γ)\), where \(Γ\) is the index set of a maximal family \(\{q_λ\}\) of mutually orthogonal primitive projections in \(A\) [1, Theorem 4.3]. Under this isomorphism, \(||x|| = ασ(X)\), where \(σ(X)\) denotes the Schmidt norm of the operator \(X\) and \(α ≥ 1\) is the norm of each of the projections \(q_λ\) (actually, all primitive projections in \(A\) have the same norm [7, Corollary 5.9]). Now if \(x\) is any nonzero element of \(A\) and \([x] = Σλₙeₙ\) is the spectral representation of \([x]\), then we may replace the nonprimitive projections among the \(eₙ\) by finite sums of primitive projections to obtain a new representation

\[\tag{*} [x] = Σμₙpₙ,\]

where \(μₘ ≤ μₖ\) if \(m > k\). For a given coefficient \(μₙ\) in \((*)\), we shall call the number of primitive projections having \(μₙ\) as coefficient the multiplicity of \(μₙ\) in this representation, denoted by \(m(μₙ)\). We have, for \(0 < p < ∞\), \(||x|| = (Σμₙ²||pₙ||_p²)^{1/p} = α^{2/p}(Σμₙ²)^{1/p}\). Also, \(|x|_∞ = μ\). Since the \(μₙ\) are the nonzero elements of the spectrum of \([x]\), and since the corresponding operator \([X]\) is compact, these numbers are the nonzero characteristic values of \([X]\). Now for each \(μₙ\), let \(M(μₙ)\) denote the multiplicity of \(μₙ\) as a characteristic value of the operator \([X]\); that is, the dimension of the subspace of \(H\) spanned by the characteristic vectors of \([X]\) corresponding to \(μₙ\). We shall show that \(m(μₙ) = M(μₙ)\).

**Lemma 5.1.** Let \(p\) be a primitive projection in the topologically simple \(H^*-\)algebra \(A\). Then the corresponding projection \(P\) in \(σc\) is one-dimensional on \(H\).

**Proof.** If \(P\) is not one-dimensional, let \(P = Q + R\), where \(Q\) and \(R\) are projections onto orthogonal nonzero subspaces of \(P(H)\). Letting
q and r be the corresponding elements of A, we see that q and r are orthogonal projections in A with \( p = q + r \). Thus p is not primitive.

**Lemma 5.2.** For any \( \mu_n \) in (\( \ast \)), \( m(\mu_n) = M(\mu_n) \).

**Proof.** Let \( p_{n_1}, \ldots, p_{n_k} \) be the projections in (\( \ast \)) having coefficient \( \mu_n \). Then \( m(\mu_n) = k \). Also, letting \( P_{n_1}, \ldots, P_{n_k} \) be the corresponding projections in \( \sigma \), we have, using the preceding lemma, \( \dim(P_{n_1} + \cdots + P_{n_k})(H) = k \); therefore \( M(\mu_n) \geq k \). Suppose \( M(\mu_n) > k \), and let \( h \) be a nonzero element of \( H \) such that \( [X]h = \mu_nq \) and \( A \) is orthogonal to \( (P_{n_1} + \cdots + P_{n_k})(H) \). Let \( Q \) be the orthogonal projection onto the one-dimensional subspace of \( H \) spanned by \( \{h\} \). \( Q \in \sigma \), and \( [X]Q = \mu_nQ \). Now let \( q \) be the corresponding projection in \( A \), then \( [x]q = \mu_nq \). For \( i = 1, \ldots, k \), \( p_{n_i}q = 0 \) since \( P_{n_i}Q = 0 \); and for \( m \neq n_1, \ldots, n_k \), \( p_m[x]q = \mu_m\mu_nq = \mu_n\mu_mq \), so that \( p_mq = 0 \), since \( \mu_m \neq \mu_n \). Thus \( q \) is orthogonal to all the \( p_{n_i} \), which means that \( [x]q = 0 \), a contradiction. We conclude that \( m(\mu_n) = k = M(\mu_n) \).

Now we observe that the coefficients \( \mu_n \) in (\( \ast \)) are the nonzero characteristic values of \( [X] \) enumerated according to their multiplicity \( M(\mu_n) \). Thus, for \( 0 < p < \infty \), \( |X|_p = (\Sigma \mu_n^p)^{1/p} \) and also \( |X|_\infty = \mu_n \), where \( |\cdot|_p \) here denotes the \( c_p \) norm of \( X \) as an operator on \( H \). Finally, we have \( [x]_p = \alpha^{2/p} |X|_p \) for \( 0 < p \leq \infty \), and therefore the mapping \( x \to X \) is a bicontinuous isomorphism of \( A_p \) into \( c_p(H) \). Since \( c_2 = \sigma \) [2, p. 1093] and \( c_p \subset c_2 \) for \( 0 < p \leq 2 \), the isomorphism is onto \( c_p \) for these values of \( p \).

Now let \( A \) be any proper \( H^* \)-algebra, and let \( \{I_\gamma : \gamma \in A \} \) be the family of minimal closed two-sided ideals of \( A \). Each \( I_\gamma \) is a topologically simple \( H^* \)-algebra and \( A \) is the Hilbert space direct sum \( \Sigma I_\gamma \). For each \( \gamma \in A \), let \( I_\gamma \) be the index set of a maximal family \( \{e_{\gamma i} : \gamma \in I_\gamma \} \) of mutually orthogonal primitive projections in \( I_\gamma \), and let \( a_\gamma \) be the norm \( ||e_{\gamma i}|| \) of each of the \( e_{\gamma i} \) in \( I_\gamma \). For each \( x_{\gamma i} \in I_\gamma \) let \( X_{\gamma i} \) be the corresponding Schmidt class operator on \( H_{\gamma i} = I_\gamma (H) \). Then, as we have noted above, \( |x_{\gamma i}|_p = \alpha^{2/p} |X_{\gamma i}|_p, 0 < p \leq \infty \), where \( |X_{\gamma i}|_p \) is the \( c_p \) norm of the operator \( X_{\gamma i} \). Then, by Proposition 4.10, we have \( |x|_p = (\Sigma |x_{\gamma i}|_p^p)^{1/p} = (\Sigma \alpha^{2/p} |X_{\gamma i}|_p^p)^{1/p} \) for \( 0 < p < \infty \), and, by 4.11, \( |x|_\infty = \sup \{x_{\gamma i} = \sup |X_{\gamma i}|. \) Thus, again, as in Proposition 4.10, \( x \in A_p \) if and only if each \( x_{\gamma i} \in (I_\gamma)_p = I_\gamma \cap A_p \) and \( \Sigma |x_{\gamma i}|_p^p < \infty \). These conditions in turn imply that each corresponding operator \( X_{\gamma i} \in c_p(H_{\gamma i}) \) and \( \Sigma |x_{\gamma i}|_p^p \) < \( \infty \). For \( 1 \leq p \leq 2 \), it has been established that the last-mentioned implication is an equivalence; for these values of \( p \), therefore, in the special situation in which each \( H_{\gamma i} \) is finite-dimensional, we have shown that the algebras \( A_p \) are instances of the \( \mathcal{S}_p \) spaces studied in [3, pp. 70 ff.] and [5].
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