ALGEBRAS OF ANALYTIC FUNCTIONS IN THE PLANE

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Let \( X \) be a compact subset of the complex plane and let \( A \) be an algebra of functions analytic near \( X \) which contains the polynomials and is complete in its natural topology. This paper is concerned with determining the spectrum of \( A \) and describing \( A \) in terms of its spectrum. It is shown that the spectrum of \( A \) is formed from the disjoint union of certain compact subsets of \( C \) (suitably topologized) by making certain identifications. \( A \) is closed under differentiation exactly when no identifications need be performed, and then \( A \) admits a simple, complete description. In particular, if \( X \) is connected, then the completion of \( A \) is merely the restriction to \( X \) of the algebra of all functions analytic near the union of \( X \) with some of the bounded components of \( C - X \).

Our principal tool in these investigations is the theory of analytic structure in the spectrum of a function algebra developed by Bishop in [2] and extended by Bjork in [4, 5]. We view the algebra \( A \) as the inductive limit of function algebras and induce analytic structure in the spectrum of \( A \). When \( A \) is closed under differentiation, topological considerations lead quickly to the desired results. In the general case, we pass to the smallest algebra \( B \) containing \( A \) which is closed under differentiation. By introducing differentiation in the spectrum of \( A \), we show that every continuous complex-valued homomorphism of \( A \) may be extended to \( B \). It follows that the spectrum of \( A \) is obtained from the spectrum of \( B \) by making certain identifications. When no identifications need be performed, \( A = B \).

2. Preliminaries. If \( U \) is an open set, we let \( \mathcal{O}(U) \) denote the algebra of functions analytic on \( U \), endowed with the topology of uniform convergence on compact sets. If \( V \) is an open subset of \( U \), we let \( r_{uv}: \mathcal{O}(U) \to \mathcal{O}(V) \) be the restriction. If \( X \) is a compact set, \( \mathcal{O}(X) \) denotes the algebra of functions on \( X \) which have analytic extensions to a neighborhood of \( X \). We view \( \mathcal{O}(X) \) as the inductive limit (in the sense of functions) of the system \( \{\mathcal{O}(U); r_{uv}\} \) and equip \( \mathcal{O}(X) \) with the inductive limit topology; i.e., the finest topology rendering the restriction maps \( r_u: \mathcal{O}(U) \to \mathcal{O}(X) \) continuous.

If \( A \) is a subalgebra of \( \mathcal{O}(X) \) and \( U \) is an open set containing \( X \), we let \( A(U) = \{f \in \mathcal{O}(X): f|_X \in A\} \). Similarly, if \( K \) is a compact set containing \( X \), we let \( A(K) = \{f \in \mathcal{O}(K): f|_X \in A\} \). For compact sets \( K, L \) with \( K \supset L \), we let \( r_{KL}: \mathcal{O} \to \mathcal{O}(L) \) be the restriction. Then it is easy to see that:
\[ A = \text{inductive limit} \{ A(U); r_{uv} \} \]
\[ = \text{inductive limit} \{ A(K); r_{KL} \} \]
and that the inductive limit topologies thus induced on \( A \) coincide with the relative topology from \( \mathcal{O}(X) \). \( A \) is complete in this topology if and only if \( A(U) \) is closed in \( \mathcal{O}(U) \) for each open set \( U \) containing \( X \). Details of the above may be found in §2 of [11].

We also regard a subalgebra \( A \) of \( \mathcal{O}(X) \) as a normed algebra with the norm:
\[ \| f \|_X = \sup \{|f(x)| : x \in X\} . \]

We relate these two topologies in the following proposition.

**Proposition.** Let \( A \) be a subalgebra of \( \mathcal{O}(X) \) containing the constants. Then the norm topology and the inductive limit topology on \( A \) admit the same continuous complex-valued homomorphisms.

**Proof.** If this were not so, there would be a homomorphism \( \phi \) of \( A \) continuous relative to the inductive limit topology, and a function \( f \) in \( A \) such that
\[ \phi(f) = 1 > \| f \|_X . \]

We could then find an open set \( U \) containing \( X \) and a function \( F \) in \( A(U) \) such that \( F|X = f \) and \( |F| < 1 \) on \( U \). Then \( (1 - F) \) would be invertible in the closure of \( A(U) \) in \( \mathcal{O}(U) \). Moreover, \( \phi \cdot r_U \) would be a continuous homomorphism of \( A(U) \), and would thus extend to its closure. Since \( \phi \cdot r_U(1 - F') = \phi(1 - f') = 0 \), we would then have the following contradictory chain of equalities:
\[ 1 = \phi \cdot r_U[(1 - F)(1 - F')^{-1}] = \{\phi \cdot r_U(1 - F')\}(\phi \cdot r_U[(1 - F')^{-1}]) = 0 . \]

This contradiction establishes the proposition.

If \( B \) is a topological algebra, we denote the spectrum of \( B \) (the space of nonzero continuous complex-valued homomorphisms, with the weak* topology) by \( M_B \). We may regard an element \( b \) of \( B \) as a function of \( M_B \) via the Gelfand transform \( \hat{b}(\phi) = \phi(b) \), for each \( \phi \) in \( M_B \). If \( B \) is a normed algebra with identity, then \( M_B \) is compact. Then if \( A \) is a subalgebra of \( \mathcal{O}(X) \) containing the constants, \( M_A \) is compact and a standard argument may be used to show that (see [11]):
\[ M_A = \text{projective limit} \{ M_{A(U)}; r_{UV}^* \} \]
\[ = \text{projective limit} \{ M_{A(K)}; r_{KL}^* \} \]
where \( r_{UV}^* \) and \( r_{KL}^* \) are the adjoints of the restrictions.
We refer to [6] for standard material concerning function algebras. If $B$ is a function algebra with spectrum $M_B$ we denote its Silov boundary by $S_B$. We make use of the techniques developed by Bishop and Bjork in [2, 4, 5] and assume some familiarity with these papers. In particular, if $f$ is an element of $B$ we say that a component $W$ of $C - \hat{f}(S_B)$ is $f$-regular of multiplicity $n$ if for each $w$ in $W$ there are at most $n$ homomorphisms $\zeta$ in $M_B$ for which $\hat{f}(\zeta) = w$; and that for some $w$ there are exactly $n$ such homomorphisms. In that case, there is a discrete subset $E$ of $W$ such that for each $\zeta$ in $M_B$ such that $\hat{f}(\zeta) \in (W - E)$, there is a neighborhood $Q$ of $\zeta$ in $M_B$ mapped homeomorphically by $\hat{f}$ onto a disk, and such that $\hat{g} \circ (\hat{f} | Q)^{-1}$ is analytic for each $g$ in $B$. The neighborhood $Q$ is called an analytic disk about $\zeta$, relative to the function $\hat{f}$.

We conclude this section with a topological lemma.

**Lemma.** Let $M$ be a compact connected real 2-manifold with boundary and let $p$ be a continuous map of $M$ into the 2-sphere $S^2$. If $p$ is locally one-to-one and is one-to-one on the boundary of $M$, then $p$ is one-to-one.

**Proof.** We will reduce to the case of a 2-manifold without boundary. To this end, suppose that $M$ has $k$ boundary components $J_1, \ldots, J_k$. Each $J_i$ is a 1-sphere, so that $p(J_i)$ is a 1-sphere in $S^2$ for each $i$. Hence $S^2 - p(J_i)$ consists of two disjoint connected open sets. A compactness argument, using the fact that $p$ is locally-one-to-one, may be used to show that there is a connected neighborhood of $J_i$ in $M$ on which $p$ is one-to-one. It follows that we may choose a neighborhood $W_i$ of $J_i$ such that $p$ is one-to-one on $W_i$ and $p(W_i)$ does not intersect one of the components of $S^2 - p(J_i)$. It is easy to see that we may attach a disk to $M$ along $J_i$ and extend $p$ to this disk; since $p(W_i)$ lies in only one component of $S^2 - p(J_i)$ this may be effected in such a way that the extension remains locally one-to-one. If we perform this surgery for each boundary component $J_i$ we arrive at a compact connected real 2-manifold $N$ without boundary and a continuous map $q$ of $N$ into $S^2$ which is locally one-to-one. If $q$ is one-to-one then $p$ must certainly be.

For each $x$ in $S^2$, the fiber $q^{-1}(x)$ is compact and discrete (since $q$ is locally one-to-one) and hence finite. Then, using the invariance of domain, we may choose an open set $U$ about $x$ such that $q^{-1}(U)$ consists of open, connected components, each mapped homeomorphically onto $U$ by $q$; thus $q$ is a covering map. Since $S^2$ is its own universal covering space, it follows that $q$, and hence $p$, must be one-to-one, as desired.
3. Main results. If $A$ is a subalgebra of $\mathcal{O}(X)$ that contains the constants and the coordinate function $Z$, we say that $A$ is stable if it is complete in the inductive limit topology and each of the algebras $A(U)$ is closed under differentiation.

In order to see how stable algebras may arise, consider the following construction. Let $X$ be a compact subset of $C$ and let $\{X_\alpha\}$ be a partitioning of $X$ into disjoint closed sets. For each $\alpha$ let $Y_\alpha$ be the union of $X_\alpha$ with some of the bounded components of $C - X_\alpha$. Then let

$$A = \{f \in \mathcal{O}(X) : f|_{X_\alpha} \in \mathcal{O}(Y_\alpha) \mid X_\alpha \text{ for each } \alpha\}.$$  

It is easy to see that $A$ is a stable algebra and that the spectrum of $A$ is the disjoint union of the $Y_\alpha$, suitably topologized. The following theorem shows that this is the only way in which stable algebras may arise.

**Theorem 1.** Let $A$ be a stable subalgebra of $\mathcal{O}(X)$ and let $Y'_\alpha$ be a component of $M_A$. Then $\hat{Z}|_{Y'_\alpha}$ is a homeomorphism. The set $Y_\alpha = \hat{Z}(Y'_\alpha)$ is the union of $X_\alpha = X \cap Y_\alpha$ with some of the bounded components of $C - X_\alpha$. Finally, the collection $\{X_\alpha : Y_\alpha$ is a component of $M_A\}$ is a partitioning of $X$ into disjoint closed sets and $A = \{f \in \mathcal{O}(X) : f|_{X_\alpha} \in \mathcal{O}(Y_\alpha) \mid X_\alpha \text{ for each component } Y'_\alpha \text{ of } M_A\}$.

**Proof.** Let $K$ be a compact set whose interior contains $X$ and whose boundary is the disjoint union of a finite number of smooth, simple closed curves. Let $A(K)^*$ denote the completion of the algebra $A(K)$ in the norm $|| \cdot ||_K$. We proceed by examining the algebra $A(K)^*$ and its spectrum and then passing to the projective limit.

We identify $K$ with a subset of $M_{A(K)^*}$. Clearly, $S_{A(K)^*}$ is contained in the boundary (relative to $C$) of $K$. Let $A$ denote the set of points in $M_{A(K)^*}$, having a neighborhood which is an analytic disk (relative to the function $\hat{Z}$). We show that $M_{A(K)^*} - S_{A(K)^*} - A$ is at most countable. First, a standard argument shows that the unbounded component of $C - \hat{Z}(S_{A(K)^*})$ is $Z$-regular of multiplicity 0. If $T$ is the boundary of this component, then it follows from [5] that there are no points $\zeta$ of $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\zeta)$ belongs to $T$. We conclude from [5] that each component of $C - \hat{Z}(S_{A(K)^*})$ that adjoins the unbounded component is $Z$-regular of multiplicity at most 1. Similarly, if $T'$ denotes the boundary of one of these components, then there is at most one point $\zeta$ in $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\zeta) \in T'$. Then each component of $C - \hat{Z}(S_{A(K)^*})$ that adjoins one of these components is $Z$-regular of multiplicity at most 2. Proceeding inward in this way, we see that each component of $C - \hat{Z}(S_{A(K)^*})$ is $Z$-regular of some multiplicity. Again from [5], it follows that there is a discrete subset
Let \( E \) be a connected component of \( M_{A(K)} \). We assert that \( \hat{Z} \) is a homeomorphism. If this were not so, we could find homomorphisms \( \phi \) and \( \lambda \) in \( L \) such that \( \hat{Z}(\phi) = \hat{Z}(\lambda) \). Since \( M_{A(K)}^* = M_{A(K)} \), we could then find an open set \( U \) containing \( K \) and a function \( f \) in \( A(U) \) such that \( \phi(f|U) \neq \lambda(f|K) \). Since \( A(U) \) is complete, closed under differentiation and contains the polynomials, it follows from a theorem of Bishop [3] that \( M_{A(U)} \) is a 1-dimensional complex analytic manifold and that \( \hat{Z} \) on \( M_{A(U)} \) is a local analytic isomorphism. Let \( \rho: A(U) \rightarrow A(K) \) be the restriction and \( \rho^*: M_{A(K)} \rightarrow M_{A(U)} \) be its adjoint. Then \( \rho^*(L) \) is a compact connected subset of \( M_{A(U)} \). By the invariance of domain theorem, \( \rho^*(A) \) lies in the interior of \( \rho^*(L) \). Hence \( \hat{Z} \) is one-to-one on the boundary of \( \rho^*(L) \). Since \( \hat{Z} \) is a local homeomorphism on \( M_{A(U)} \), we may find a compact connected set \( L' \) containing \( \rho^*(L) \) in its interior such that \( \hat{Z} \) is one-to-one on the boundary of \( L' \) and \( L' \) is a 2-manifold with boundary. Regarding \( C \) as a subset of \( S^2 \), we may then apply the lemma to conclude that \( \hat{Z} \) is one-to-one on \( L' \) and hence on \( \rho^*(L) \). But \( \phi \) and \( \lambda \) restrict to different homomorphisms of \( A(U) \) so that \( \rho^*(\phi) \neq \rho^*(\lambda) \), while \( \hat{Z}[\rho^*(\phi)] = \hat{Z}[\rho^*(\lambda)] \), which is a contradiction. It must be therefore, that \( \hat{Z} \) is a homeomorphism.

From the Silov idempotent theorem, it follows that each component of \( M_{A(K)}^* \) contains a component of the boundary of \( K \). It follows that for each component \( L \) of \( M_{A(K)}^* \), the boundary of \( \hat{Z}(L) \) coincides with \( \hat{Z}(S_{A(K)} \cap L) \), so that \( \hat{Z}(L) \) is formed from \( K \cap \hat{Z}(L) \) by the addition of certain components of \( C - K \cap \hat{Z}(L) \).

Now let us return to \( M_{A} \). For a component \( Y_{a'} \) of \( M_{A} \), and a compact set \( K \) with smooth boundary, containing \( X \) in its interior, let \( r_K: A(K)^* \rightarrow A \) be the restriction and let \( r_K^*: M_{A} \rightarrow M_{A(K)}^* \) be its adjoint. Let \( K_{a'} \) be the component of \( M_{A(K)}^* \) that contains \( r_K(Y_{a}) \). It is clear that

\[
Y_{a'} = \text{projective limit} \{K_{a'}; r_K^*\}.
\]
From the description of $K'_a$ derived above, it follows that $\hat{Z}|_{Y'_a}$ is a homeomorphism and that $\hat{Z}(Y'_a)$ is the union of $X \cap \hat{Z}(Y'_a) = X_a$ with some of the bounded components of $C - X_a$.

If $f'$ belongs to $A$, then it is in $A(K')^*$ for some compact $K$ with smooth boundary containing $X$ in its interior. Since $\hat{Z}$ is a homeomorphism on each component of $M_{A(K')^*}$, it follows that $A$, the set of points in $M_{A(K')^*}$ having neighborhoods which are analytic disks, is all of $M_{A(K')^*} - S_{A(K')^*}$. Now we may see that $f|((K \cap \hat{Z}(L')))$ belongs to $\mathcal{O}(\hat{Z}(L'))|(K \cap \hat{Z}(L'))$ for each component $L'$ of $M_{A(K')^*}$. It follows that $f|((X \cap \hat{Z}(Y'_a)))$ belongs to $\mathcal{O}(\hat{Z}(Y'_a))|(X \cap \hat{Z}(Y'_a))$ for each component $Y'_a$ of $M_A$.

Finally, suppose that $U$ is an open set containing $X$ and that $f$ is a function in $\mathcal{O}(U)$ such that $f|((X \cap \hat{Z}(Y'_a)))$ belongs to $\mathcal{O}(\hat{Z}(Y'_a))|(X \cap \hat{Z}(Y'_a))$

for each component $Y'_a$ of $M_A$. For each such $Y'_a$, choose a compact set $K_a$ with smooth boundary containing $X$ in its interior and such that $Z(L'_a) \subset (U \cup \hat{Z}(Y'_a))$ where $L'_a$ is the component of $M_{A(K')^*}$ that contains $r^*_a(Y'_a)$. If $Y'_a$ is sufficiently close to $Y'_a$, we may choose $K_a$ to be $K_a$. Then the compactness of $M_A$ enables us to choose a single compact set $K'$ with smooth boundary, containing $X$ in its interior, and such that $\hat{Z}(L''_a) \subset (U \cup \hat{Z}(Y'_a))$ for each $\alpha$, where $L''_a$ is the component of $M_{A(K')^*}$ that contains $r^*_{K'}(Y'_a)$. Without loss, we may assume that every component of $K'$ contains a point of $X$. Then for each component $L'$ of $M_{A(K')^*}$, we see that $f|((K' \cap \hat{Z}(L')))$ belongs to $\mathcal{O}(\hat{Z}(L'))|(K' \cap \hat{Z}(L'))$. The Silov idempotent theorem and the Arens-Calderon theorem then imply that $f|K'$ belongs to $A(K')^*$. Since $A$ is complete and $K'$ contains $X$ in its interior, it follows that $f|X$ belongs to $A$, which completes the proof.

The above theorem gives a complete description of stable algebras. In what follows, we use stable algebras to describe the structure of more general subalgebras of $\mathcal{O}(X)$. We let $A$ be a complete subalgebra of $\mathcal{O}(X)$ containing the polynomials and let $A_0$ be the smallest stable algebra containing $A$; $A_0$ is the completion of the algebra generated by the functions in $A$ together with all their derivatives. We let $i: A \to A_0$ be the inclusion and $i^*: M_{A_0} \to M_A$ be its adjoint (the restriction map).

**Theorem 2.** The map $i^*: M_{A_0} \to M_A$ is onto. If $Y$ and $Y'$ are components of $M_{A_0}$ then $i^*|Y$ and $i^*|Y'$ are one-to-one and there are at most finitely many pairs $(\mu, \nu)$ in $Y \times Y'$ such that $i^*(\mu) = i^*(\nu)$. If $f^*$ is a homeomorphism, then $A = A_0$. 

Proof. We show first that $i^*$ is onto. Choose a compact set $K$ with smooth boundary, whose interior contains $X$ and is dense in $K$, and each component of which meets $X$. Let $A_1$ be the (non-complete) subalgebra of $\mathcal{O}(X)$ generated by the functions in $A$ and all their derivatives. Let $i_0: A(K) \to A_1(K)$ be the inclusion and $i_0^* : M_{A_1(K)} \to M_{A(X)}^*$ be its adjoint. If we show that $i_0^*$ is onto for each $K$ belonging to a fundamental system of neighborhoods of $X$, then by passage to the projective limit, it will follow that $i^*$ is onto. So suppose that for a particular choice of $K$, $i_0^*$ is not onto.

Let $A$ be the set of points of $M_{A(K)}^*$ which have a neighborhood which is an analytic disk relative to $Z$. As in the proof of Theorem 1, we see that $M_{A(K)}^* - S_{A(K)}^* - A = E$ is at most countable. In view of the Silov idempotent theorem, no point of $M_{A(K)}^*$ is isolated, so that there is an open subset of $A$ disjoint from $i_0^*(M_{A_1(K)}^*)$. Let $W$ be a component of $A$ containing such an open set. We distinguish two cases.

Regard $K$ as a subset of $M_{A(K)}^*$ and consider first the case in which $W$ contains a point of $K$. Then $W$ is a Riemann surface with the local coordinate $\hat{Z}$. If $f$ is a function in $A(K)$, then $\hat{f}$ is analytic on $W$. Denote the derivative of $\hat{f}$ with respect to the coordinate $\hat{Z}$ by $D\hat{f}$ and the derivative of $f$ with respect to $Z$ by $f'$. If $f$ and $f'$ both belong to $A(K)$, then the connectedness of $W$, together with the fact that $W$ contains a point of $K$ and hence an open subset of $K$, implies that $D\hat{f} = \hat{f}'$ on $W$.

Let $h$ belong to $A(K)$ and let $g = h'$. Define a function $\bar{g}$ on $W$ by $\bar{g}(\zeta) = D\hat{h}(\zeta)$. The analysis of the previous paragraph implies that $\bar{g} = \hat{g}$ if $g = h'$ belongs to $A(K)$. Thus the functions in $A(K)$ together with their first derivatives, extend to be analytic on $W$. By iteration of this process, we may extend each of the functions $g$ in $A_1(K)$ to an analytic function $\bar{g}$ on $W$; since $W$ is connected, this extension is unique.

Thus if $\delta$ is a homomorphism in $W$, $\delta$ extends to a homomorphism of $A_1(K)$ by defining $\bar{\delta}(g) = \hat{g}(\delta)$. If we show that $\bar{\delta}$ is a continuous homomorphism of $A_1(K)$, and thus extends to $A_1(K)^*$, then we will have that $i_0^*(\bar{\delta}) = \delta$ and this contradiction will complete the analysis of this case.

To this end, let us consider the boundary of $W$ in $M_{A(K)}^*$. Since $W$ is a connected component of $A$, no point of $A$ is a boundary point of $W$. Thus the boundary points of $W$ belong either to $K$ or to $E$. If $p$ is a boundary point of $W$ that belongs to $K$, the fact that the interior of $K$ is dense in $K$ and that the boundary of $K$ is smooth implies that there is a half-disk about $p$ belonging to $K$. By enlarging $K$ slightly we may effect a modification of $K$ so that some half-disk around $p$ belongs to $K \cap W$. An argument using the compactness of the part of the boundary of $W$ that lies in $K$ shows that all the
boundary points of $W$ that belong to $K$ may be assumed to have half-disks in $K \cap W$ about them (modifying $K$ as necessary).

Now consider the boundary points of $W$ that belong to $E$. If $q$ is one of these points, then the results of [2] imply that there is a neighborhood $Q$ of $q$ with the property that $Q - q$ lies in $\Lambda$ and consists of finitely many components, each of which is mapped by $\hat{Z}$ homeomorphically onto a disk minus its center. Thus we may cover $W \cup \{q\}$ with a Riemann surface $W_q$ in such a way that the functions in $A(K)$ extend to be analytic on $W_q$. We may certainly do this for each of the boundary points of $W$ lying in $E$. Thus, passing to a covering Riemann surface when necessary, and modifying $W$ as necessary (by enlargement of $K$), we arrive at a Riemann surface $W$ which has a subset of the interior of $K$ as a neighborhood of its boundary, and to which the functions in $A(K)$ extend naturally. As before, we see that the functions in $A_1(K)$ extend to $W$. Hence no function in $A_1(K)$ assumes a larger value on $W$ than on $K$. It follows that $\delta$ is indeed a continuous homomorphism of $A_1(K)$.

We have shown that each compact set $K$ whose interior contains $X$ and is dense in $K$, and all of whose components meet $X$, can be modified slightly to produce another such compact set $K'$ with the property that $i_k^*: M_{A(K')^*} \to M_{A(K)^*}$ is onto. Since the collection of such sets $K'$ forms a fundamental system of neighborhoods of $X$, it follows from a passage to the protective limit that $i^*: M_{A_0} \to M_4$ is onto.

Now let $Y$ and $Y'$ be distinct components of $M_{A_0}$. Considered as a map on $M_{A_0}$, $\hat{Z}|Y$ is one-to-one and $\hat{Z} = \hat{Z} \circ i^*$ so that $i^*$ is certainly one-to-one. If there are infinitely many pairs $(\mu, \nu)$ in $Y \times Y'$ such that $i^*(\mu) = i^*(\nu)$ then some point $(\lambda, \xi)$ is a limit point of such pairs. We may choose a compact set $K$ with smooth boundary, whose interior contains $X$ and is dense in $K$, and such that $i_k^*(Y)$ and $i_k^*(Y')$ belong to different components of $M_{A_1(K)^*}$; say $T$ and $T'$ respectively. If $f$ is in $A(K)$, then $\hat{f}$ is analytic on $T - (T \cap S_{A(K)^*})$ and $T' - (T' \cap S_{A(K)^*})$, and the derivative of $\hat{f}$ may be obtained, as in the first part of the proof, by differentiating with respect to the local coordinate $\hat{Z}$. Then the functions $\hat{g} \circ (\hat{Z}|T)^{-1}$ and $\hat{g} \circ (\hat{Z}|T')^{-1}$ are analytic in a neighborhood of $\hat{Z}(\xi) = \hat{Z}(\lambda)$ and agree to infinite order there. Now it follows that $\hat{g}(i_k^*(\xi)) = \hat{g}(i_k^*(\lambda))$ for each $g$ in $A_1(K)$, since $A_1(K)$ is generated by functions in $A(K)$ and their derivatives. It follows that $i_k^*(\lambda) = i_k^*(\xi)$ which is a contradiction. It follows that only finitely many pairs in $Y \times Y'$ are not separated by $i^*$, as desired.

Finally, suppose that $i^*$ is a homeomorphism, and let $f$ be in $A_0(U)$ for some open set $U$ containing $X$. As in the proof of Theorem 1, we may choose a compact set containing $X$ in its interior such that $f|(K \cap \hat{Z}(L))$ belongs to $\mathcal{O}(\hat{Z}(L))|(K \cap \hat{Z}(L))$ for each component $L$ of $M_{A_1(K)^*}$. As before, we may then conclude that $f$ belongs to $A(K)^*$.
and hence that $f|_X$ belongs to $A$. Since $U$ is arbitrary, this completes the proof.

**COROLLARY 1.** Let $X$ be compact and connected. If $A$ is a complete subalgebra of $\mathcal{O}(X)$ containing the polynomials, then there is a compact connected set $X'$ containing $X$ and such that $X' - X$ is open and $A = \mathcal{O}(X')$.

*Proof.* Let $\mathcal{A}_\circ$ be the stable algebra generated by $A$. The Silov idempotent theorem implies that $\mathcal{M}_A$ is connected and the corollary now follows quickly from Theorems 1 and 2.

The above results have easy applications to questions of approximation on open sets as well. We mention one result that seems particularly striking.

**COROLLARY 2.** Let $U$ be a connected open set and let $g$ be an analytic function on $U$ that admits no analytic extension to the union of $U$ with any of the bounded components of $C - U$. Then every analytic function on $U$ is the limit, uniformly on compact subsets of $U$, of polynomials in $g$ and $Z$.

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