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# THE USE OF MITOTIC ORDINALS IN CARDINAL ARITHMETIC

ALEXANDER ABIAN

**In this paper, based on the properties of mitotic ordinals, some results of the cardinal arithmetic are obtained in a rather natural way.**

In what follows, any reference to order among ordinal numbers is made with respect to their usual order. Thus, if $u$ and $v$ are ordinals then $u \leqq v$ if and only if $u \subseteqq v$ if and only if "$u \in v$ *or* $u = v$".

DEFINITION. *A nonzero ordinal $w$ is called mitotic if and only if it can be partitioned into $\overline{\overline{w}}$ pairwise disjoint subsets each of type $w$. Such a partition is called a mitotic partition of $w$.*

For instance, $\omega$ is a mitotic ordinal since $\omega$ can be partitioned into denumerably many pairwise disjoint denumerable subsets $R_i$ with $i = 0, 1, 2, \cdots$, where the elements of $R_i$ are precisely the ordinals appearing in the $i$-th row of the following table:

$$\begin{matrix} 0 & 1 & 3 & 6 & \cdot & \cdot & \cdot \\ 2 & 4 & 7 & \cdot & \cdot & \cdot & \cdot \\ 5 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} .$$

Clearly, each $R_i$ is of type $\omega$.

LEMMA 1. *Let $w$ be a mitotic ordinal. Then $w$ is a limit ordinal. Moreover, for every element $S_i$ of a mitotic partition $(S_i)_{i \in w}$ of $w$ we have*:

$$\cup S_i = \sup S_i = w . \tag{1}$$

*Proof.* Since $S_i$ is of type $w$ we see that $S_i$ is similar to $w$. Let $f_i$ be a similarity mapping from $w$ onto $S_i$. But then by [1, p. 302] we have $x \leqq f_i(x)$ for every $x \in w$. Now, assume on the contrary that $w$ is not a limit ordinal and let $k$ be the last element of $w$. But then clearly, $k = f_i(k)$ and therefore $k \in S_i$. However, since 1 is not a mitotic ordinal, we see that the mitotic partition of $w$ must have at least two distinct elements, $S_0$ and $S_1$. But then $k \in S_0$ and $k \in S_1$ which contradicts the fact that $S_0$ is disjoint from $S_1$. Thus, our assumption is false and $w$ is a limit ordinal.

Next, since the similarity of $w$ to $S_i$ implies the existence of a one-to-one mapping $f_i$ from $w$ onto $S_i$ such that $x \leqq f_i(x)$ for every $x \in w$, we see that $\cup w \leqq S_i$ and therefore $\cup w = \cup S_i$ since $S_i \subseteqq w$. On the other hand, since $w$ is a limit ordinal by [1, p. 323] we have $\cup w = w$. Hence, (1) is established.

Based on the *natural expansion* [1, p. 355] of ordinals we prove the following lemma.

LEMMA 2. *Let $w$ be a mitotic ordinal and let $\omega^e n$ be the last term of the normal expansion of $w$. Then*

$$\overline{\overline{w}} = \overline{\overline{\omega^e n}} \tag{2}$$

*Proof.* Let $w = u + \omega^e n$ and let $(S_i)_{i \in w}$ represent a mitotic partition of $w$. From (1) it follows that for every $i \in w$, we must have $(u + v) \in S_i$ for some $v < \omega^e n$. But then (2) follows from the fact that $(S_i)_{i \in w}$ is a family of pairwise disjoint elements $S_i$.

LEMMA 3. *For every nonzero ordinal $e$ the ordinal $\omega^e$ is mitotic.*

*Proof.* Since $\omega < \omega^e$ we see that there is a mitotic ordinal of type $\omega^h$ such that $h \leqq e$. Let $P$ be the set of all mitotic partitions of mitotic ordinals of type $\omega^h$ which are less than or equal to $\omega^e$. Partial order $P$ by $\leqq^*$ as follows:

$$(S_{u_i})_{i \in \omega^u} \leqq^* (S_{v_i})_{i \in \omega^v}$$

if and only if $S_{u_i} \subseteqq S_{v_i}$ for every $i \in (\omega^u \cap \omega^v)$.

Let $((S_{u_i})_{i \in \omega^u})_{u \in A}$ be a simply ordered subset of $(P, \leqq^*)$. But then it is easy to verify that $(\bigcup_{u \in \cup A} S_{u_i})_{i \in \omega^{\cup A}}$ is a mitotic partition of the ordinal $\omega^{\cup A}$. Hence every simply ordered subset of the nonempty partially ordered set $(P, \leqq^*)$ has a least upper bound. Consequently, $(P, \leqq^*)$ has a maximal element $(M_i)_{i \in \omega^k}$ where $\omega^k$ is a mitotic ordinal such that $k \leqq e$.

Let $(M_i)$ denote the mitotic partition $(M_i)_{i \in \omega^k}$ of $\omega^k$, i.e.,

$$(M_i) = (M_i)_{i \in \omega^k} \,. \tag{3}$$

To prove the lemma it is sufficient to show that $k = e$. Assume on the contrary that $k < e$. Thus $\omega^k \omega \leqq \omega^e$.

For every $n \in \omega$, let $(M_i)n$ denote the mitotic partition given by (3) where each entry is augmented on the left by $\omega^k n$. But then

$$\begin{array}{cccc}
(M_i)0 & (M_i)1 & (M_i)3 & \cdot \\
(M_i)2 & (M_i)4 & \cdot & \cdot \\
(M_i)5 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{array}$$

is clearly a mitotic partition of $\omega^k\omega = \omega^{k+1}$. But since $\omega^k \leqq \omega^k\omega < \omega^{k+1} \leqq \omega^e$ we arrive at a contradiction. Thus, our assumption is false and $k = e$.

LEMMA 4. *The sum of finitely many pairwise equipollent mitotic ordinals is a mitotic ordinal.*

*Proof.* Obviously, it is sufficient to prove that the sum of two equipollent mitotic ordinals is a mitotic ordinal. Let $(R_i)_{i\in\bar{\bar{u}}}$ and $(S_i)_{i\in\bar{\bar{v}}}$ represent respectively mitotic partitions of mitotic ordinals $u$ and $v$ where $\bar{\bar{u}} = \bar{\bar{v}} = c$. Now, let

$$R_i = (r_0, r_1, r_2, \cdots) \text{ and } S_i = (s_0, s_1, s_2, \cdots) .$$

Consider

$$H_i = (r_0, r_1, r_2, \cdots, (\cup R_i) + s_0, (\cup R_i) + s_1, (\cup R_i) + s_2, \cdots) .$$

Clearly, $H_i \subseteqq (u + v)$ and $H_i$ is of type $u + v$ for every $i \in c$. But then observing that $\overline{\overline{u+v}} = c$ we see that $(H_i)_{i\in c}$ is a mitotic partition of the ordinal $u + v$. Thus, $u + v$ is mitotic, as desired.

THEOREM 1. *An infinite ordinal is mitotic if and only if it is equipollent to the last term of its normal expansion.*

*Proof.* Let $w$ be an infinite ordinal. Without loss of generality we may assume that the normal expansion of $w$ has two terms and is given by:

$$w = \omega^a m + \omega^e n . \tag{4}$$

Now, if $w$ is mitotic then by (2) we see that $w$ is equipollent to the last term of its normal expansion. Conversely, let $w$ be equipollent to the last term of its normal expansion. But then clearly,

$$\bar{\bar{w}} = \overline{\overline{\omega^a m}} = \overline{\overline{\omega^e n}} . \tag{5}$$

However, since $\omega^a m$ is a finite sum of summands each equal to $\omega^a$, in view of Lemmas 3 and 4, we see that $\omega^a m$ is mitotic. Similarly, $\omega^e n$ is mitotic. But then again, from (5), (4) and Lemma 4, we see that $w$ is mitotic, as desired.

From Theorem 1 it follows that each of the following ordinal numbers is mitotic:

$$\omega^\omega, \omega^\omega + \omega, \omega_1^\omega + \omega_1, \omega_2^\omega + \omega_2\omega_1\omega, \cdots .$$

Also, since the normal expansion of every infinite cardinal has one term, from Theorem 1, we have:

COROLLARY 1. *Every infinite cardinal is mitotic.*

Next, based on the properties of mitotic ordinals we derive some results pertaining to the cardinal arithmetic.

THEOREM 2. *Let $w$ be a mitotic ordinal and $(c_i)_{i \in w}$ a nondecreasing sequence of type $w$ of cardinals $c_i$. Then*

$$\prod_{i \in w} c_i = (\prod_{i \in w} c_i)^{\overline{\overline{w}}} . \tag{6}$$

*Proof.* Let $(S_i)_{i \in w}$ be a mitotic partition of $w$. Since $(c_i)_{i \in w}$ is nondecreasing, we have

$$\prod_{i \in w} c_i \leqq \prod \{c_i \mid c_i \in S_j\} \text{ for every } j \in w$$

and since the right side of the above inequality is a subproduct of the left side, we have

$$\prod_{i \in w} c_i = \prod \{c_i \mid c_i \in S_j\} \text{ for every } j \in w . \tag{7}$$

On the other hand, in view of the general commutativity and associativity of the infinite product of cardinal numbers, we have

$$\prod_{i \in w} c_i = \prod_{j \in w} (\prod \{c_i \mid c_i \in S_j\} . \tag{8}$$

But then (6) follows readily from (7) and (8).

Based on Theorem 2, we prove a theorem which extends a result of Tarski-Hausdorff [2, p. 14] to the case of a nondecreasing sequence of cardinals.

THEOREM 3. *Let $w$ be a mitotic ordinal and $(c_i)_{i \in w}$ a nondecreasing sequence of type $w$ of nonzero cardinals $c_i$. Then*

$$\prod_{i \in w} c_i = (\sup_{i \in w} c_i)^{\overline{\overline{w}}} . \tag{9}$$

*Proof.* Since $c_i \leqq \sup_{i \in w} c_i$ for every $i \in w$, we have

$$\prod_{i \in w} c_i \leqq (\sup_{i \in w} c_i)^{\overline{\overline{w}}} . \tag{10}$$

On the other hand, for establishing (9), we may assume without loss of generality, that $c_i > 1$ for every $i \in w$. But then we have:

$$(\sup_{i \in w} c_i)^{\overline{\overline{w}}} \leqq (\sum_{i \in w} c_i)^{\overline{\overline{w}}} \leqq (\sum_{i \in w} c_i)^{\overline{\overline{w}}} \tag{11}$$

and then (9) follows readily from (6), (10) and (11).

Thus, Theorem 3 is proved.

Let us observe that the formula analogous to (9) for the sum of an (not necessarily nondecreasing) infinite sequence $(c_i)_{i\in v}$ of type $v$ (not necessarily mitotic) of nonzero cardinals $c_i$ is given by:

$$\sum_{i\in v} c_i = \overline{\overline{v}} \sup_{i\in v} c_i \,. \tag{12}$$

REMARK. In the arithmetic of ordinal numbers infinite sums and products of ordinals are respectively equal to the limit of their partial sums and partial products. In fact, in ordinal arithmetic, evaluation of the result of an infinite operation as the limit of those of partial ones is a general method. In contrast to this, in the arithmetic of cardinal numbers infinite sums and products of cardinals are not equal, in general, to the limit of their partial sums and the limit of their partial products respectively. However, as shown below, in cardinal arithmetic, infinite sums of cardinals and products of nondecreasing cardinals are respectively equal to the sum of their partial sums and to the product of their partial products (this, in general, is not true in ordinal arithmetic).

The statement concerning an infinite sum of cardinals can be given as a corollary of (12).

COROLLARY 2. *Let $(c_i)_{i\in v}$ be an infinite sequence of type $v$ of nonzero cardinals $c_i$. Then*

$$\sum_{i<v} c_i = \sum_{u<v} \Big(\sum_{i\leqq u} c_i\Big) \,. \tag{13}$$

*Proof.* From (12) it follows:

$$\sum_{u<v} \Big(\sum_{i\leqq u} c_i\Big) = \sum_{u<v} \overline{\overline{u}}\cdot c_u = \overline{\overline{v}}\cdot\overline{\overline{v}} \sup c_i = \overline{\overline{v}} \sup c_i = \sum_{i<v} c_i \,.$$

Next, based on the properties of mitotic ordinals we prove the following theorem.

THEOREM 4. *Let $u$ be limit ordinal and $(c_i)_{i\in u}$ a nondecreasing sequence of type $u$ of cardinals $c_i$. Then*

$$\prod_{i<u} c_i = \prod_{j<u} \Big(\prod_{i<j} c_i\Big) \,. \tag{14}$$

*Proof.* Without loss of generality, we may assume that the normal expansion of $u$ has two terms and is given by

$$u = \omega^e p + \omega^h q \,.$$

Hence, by Lemma 3, without loss of generality, we may assume that $u$ is a sum of two mitotic ordinals $w$ and $r$, i.e.

$$u = w + r \text{ with } \overline{\overline{w}} \geqq \overline{\overline{r}} \geqq \aleph_0 . \tag{15}$$

Thus, to prove (14), it is enough to show that

$$\prod_{i<w+r} c_i = \prod_{i<w+r} (\prod_{i<j} c_i) . \tag{16}$$

However, since $u$ is a limit ordinal and $c_j \leqq \prod_{i<j+1} c_i$ for every $j < u$, we see that the left side of the equality sign in (16) is less than or equal to the right side. Thus, it is enough to show that the right side is less than or equal to the left side.

Since $w$ and $r$ are both mitotic ordinals, in view of (15) and (9) we have:

$$\begin{aligned}
\prod_{j<w+r} (\prod_{i<j} c_i) &= \prod_{i<w} (\prod_{i<j} c_i) \cdot \prod_{j<r} (\prod_{i<w+j} c_i) \\
&\leqq (\sup_{i<w} c_i)^{\overline{\overline{w}} \cdot \overline{\overline{w}}} \cdot \prod_{j<r} (\prod_{j<w} c_i \cdot \prod_{i<j} c_{w+i}) \\
&\leqq (\sup_{i<w} c_i)^{\overline{\overline{w}}} \cdot (\sup_{i<w} c_i)^{\overline{\overline{w}} \cdot \overline{\overline{r}}} \cdot (\sup_{i<r} c_{w+i})^{\overline{\overline{r}} \cdot \overline{\overline{r}}} \\
&= (\sup_{i<w} c_i)^{\overline{\overline{w}}} \cdot (\sup_{i<r} c_{w+i})^{\overline{\overline{r}}} \\
&= \prod_{i<w} c_i \cdot \prod_{i<r} c_{w+i} = \prod_{i<w+r} c_i
\end{aligned}$$

as desired.

Finally, based on (14) we obtain the formula analogous to (13) for the product of cardinals.

THEOREM 5. *Let $(c_i)_{i<v}$ be an infinite nondecreasing sequence of type $v$ of cardinals $c_i$. Then*

$$\prod_{i<v} c_i = \prod_{j<v} (\prod_{i \leq j} c_i) . \tag{17}$$

*Proof.* As the proof indicates, without loss of generality we may assume $v = u + 1$ where $u$ is a limit ordinal. But then from (14) it follows:

$$\prod_{i<u+1} c_i = (\prod_{i<u} c_i) c_u = \prod_{j<u} (\prod_{i<j} c_i) \cdot c_u = \prod_{j<u+1} (\prod_{i \leq j} c_i) .$$

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IOWA STATE UNIVERSITY

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# FILTRATIONS AND VALUATIONS ON RINGS

HELEN E. ADAMS

**The concept of a multiplicative filtration on a ring is generalized so as to include among filtered rings, rings with valuation, pseudovaluation and semivaluation. The generalized filtration induces a topology on the ring, and it is shown that the Hausdorff completion of the resulting topological ring can be described by an inverse limit. The paper finishes with an example illustrating the theory.**

1. Definitions and immediate consequences. In this section we define a generalized filtration and generalized pseudovaluation on a ring and show that a pseudovaluation induces a filtration on a ring.

If $A$ and $B$ are subsets of a ring we shall write $AB$ to mean the set $\{xy: x \in A, y \in B\}$. By an *ordered semigroup* we mean a semigroup which is partially ordered as a set such that the ordering relation is compatible with the semigroup operation. A *directed semigroup* is an ordered semigroup which is directed above as an ordered set; and a *quasi-residuated semigroup* (Blyth and Janowitz [2]) is an ordered semigroup $T$ with the property: given any $s, t \in T$, there exists $u \in T$ such that $ut \geqq s$ and $tu \geqq s$.

Let $R$ be a ring and let $S$ be a directed semigroup with the property:

(1.1) given any $s \in S$, there exists $t \in S$ such that $t^2 \geqq s$.

A *filtration* on $R$ over $S$ is a set of additive subgroups $\{P_s\}_{s \in S}$ of $R$, indexed by $S$, with the following properties:

(1.2) if $s, t \in S$ such that $s \geqq t$, then $P_s \subseteqq P_t$;

(1.3) for any $s, t \in S$, $P_sP_t \subseteqq P_{st}$;

(1.4) given $x \in R$, $s \in S$, there exists $t \in S$ such that $xP_t \subseteqq P_s$ and $P_tx \subseteqq P_s$.

Note that $\bigcap_{s \in S} P_s$ is a two-sided ideal of $R$. For a treatment of the classical multiplicative filtration on a ring, see Atiyah and Macdonald [1] and Northcott [6].

The following lemma gives a less general form of a filtration which will be shown to arise from a pseudovaluation on a ring. The proof of the lemma is straightforward.

LEMMA 1.1. *Let $S$ be a quasi-residuated, directed semigroup. Let $\{P_s\}_{s \in S}$ be a set of additive subgroups of a ring $R$ such that* (1.2), (1.3) *hold, and* (1.4′) $\bigcup_{s \in S} P_s = R$.

*Then $\{P_s\}_{s \in S}$ is a filtration on $R$.*

The following definition of a pseudovaluation on a ring allows us to treat at the same time Manis [5] valuations and pseudovaluations (Mahler [4]) on commutative rings, and semivaluations (Zelinsky [7]) on fields.

Let $S$ be a quasi-residuated, directed semigroup, and let $S_0$ be the disjoint union of $S$ and a zero element $O_S$ with the properties: $O_S O_S = O_S$; and, for any $s \in S$, $O_S > s$ and $sO_S = O_S = O_S s$. A *pseudovaluation* on a ring $R$ into $S_0$ is a map $\varphi$ of $R$ into $S_0$ such that: for all $a, b \in R$,

(1.5) $\varphi(ab) \geqq \varphi(a)\varphi(b)$;

(1.6) if $s \in S$ such that $s \leqq \varphi(a), \varphi(b)$, then $\varphi(a - b) \geqq s$;

(1.7) $\varphi(0) = O_S$;

(1.8) the set $\varphi(R) \backslash \{O_S\}$ is nonempty.

Let $\varphi\colon R \rightarrow S_0$ be a pseudovaluation on a ring $R$. Define, for any $s \in S$,

(1.9) $P_s = \{x\colon x \in R, \varphi(x) \geqq s\}$.

Then, from Lemma 1.1:

PROPOSITION 1.1. *The family of subsets* $\{P_s\}_{s \in S}$ *of* $R$, *defined in* (1.9), *is a filtration on* $R$.

**2. The completion of a ring with respect to a filtration.** Throughout this section, $R$ is a ring with filtration $\{P_s\}_{s \in S}$. It will be shown that the filtration $\{P_s\}_{s \in S}$ induces a topology $\mathscr{T}$ on $R$ compatible with the ring structure of $R$, and the completion of $(R, \mathscr{T})$ will be explicitly defined both algebraically and topologically.

From Bourbaki [3, III §1.2, example], the set $\{P_s\}_{s \in S}$ is the fundamental system of neighbourhoods of the zero for a uniquely determined topology $\mathscr{T}$ on $R$, addition in $(R, \mathscr{T})$ is continuous, and $\mathscr{T}$ is Hausdorff if and only if $\bigcap_{s \in S} P_s = \{0\}$. Further, multiplication in $(R, \mathscr{T})$ is continuous by the definition of a filtration and [3, III §6.3, $(\mathrm{AV_I})$ and $(\mathrm{AV_{II}})$]. Hence $(R, \mathscr{T})$ is a topological ring and, as such, admits a Hausdorff completion.

Now the Hausdorff completion of a topological ring is just the Hausdorff completion of the ring considered as an additive topological group [3, III §6.5]. Multiplication is then defined on the completion by a continuous extension of multiplication on the associated Hausdorff ring, in this case the factor ring $R/\bigcap_{s \in S} P_s$.

But in this case we already have, from [3, III §7.3, Proposition 2, Corollary 2], that the Hausdorff completion of the additive topological group $(R, \mathscr{T})$ is isomorphic, both algebraically and topologically, to the

Hausdorff group $(\tilde{R}, \tilde{\mathscr{T}})$ where $\tilde{R} = \varprojlim R/P_s$ and $\tilde{\mathscr{T}}$ is the usual topology induced on $\tilde{R}$ by the topology $\mathscr{T}$ on $R$. Hence the Hausdorff completion of the topological ring $(R, \mathscr{T})$ is isomorphic to the Hausdorff ring $(\tilde{R}, \tilde{\mathscr{T}}, \times)$ where $\times$ denotes the multiplication constructed on $\tilde{R}$ by means of a continuous extension of multiplication in $R/\bigcap_{s \in S} P_s$. The main aim of this section is to define explicitly the multiplication $\times$. This is not a straightforward task since each factor group $R/P_s$, $s \in S$, in the direct product $\prod_{s \in S} R/P_s$, is not a ring.

For reference we define the topological group $(\tilde{R}, \tilde{\mathscr{T}})$ explicitly [3, III §7]. Now $\tilde{R} = \{\{\xi_s\}_{s \in S} \in \prod_{s \in S} R/P_s$: for all $s, t \in S$ such that $s \leqq t$, $\xi_t \subseteqq \xi_s\}$. That is, the elements of $\tilde{R}$ are sets of subsets of $R$, indexed by $S$, and written $\{\xi_s\}_{s \in S}$ where: for each $s \in S$, $\xi_s \in R/P_s$; and, for any $s, t \in S$ such that $s \leqq t$, $\xi_t \subseteqq \xi_s$. Note that, for each $x \in R$, $\{X + P_s\}_{s \in S} \in \tilde{R}$. Equality and addition in $\tilde{R}$ are defined as follows: Let $\{\xi_s\}_{s \in S}$, $\{\eta\}_{s \in S} \in \tilde{R}$. Then $\{\xi_s\}_{s \in S} = \{\eta_s\}_{s \in S}$ if and only if, for each $s \in S$, $\xi_s = \eta_s$; and $\{\xi_s\}_{s \in S} + \{\eta_s\}_{s \in S} = \{\xi_s + \eta_s\}_{s \in S}$. When there is no risk of ambiguity, $\{\xi_s\}_{s \in S}$ will be written as $\{\xi_s\}$.

The topology $\tilde{\mathscr{T}}$ is defined on $\tilde{R}$ by inducing the usual quotient topology on each $R/P_s$, $s \in S$, then inducing the usual product topology on $\prod_{s \in S} R/P_s$, and finally restricting this topology to $\tilde{R}$, considered as a subspace of $\prod_{s \in S} R/P_s$.

Let $t \in S$ and let $f_t$: $\tilde{R} \to R/P_t$ be the canonical projection defined thus: For any $\{\xi_s\}_{s \in S} \in \tilde{R}$, $f_t(\{\xi_s\}_{s \in S}) = \xi_t$. Since $R/P_t$ is discrete [3, III §7.3], the set $\tilde{P}_t = f_t^{-1}(P_t) = \{\{\xi_s\}_{s \in S} \in \tilde{R}$: $\xi_t = P_t\}$ is an open set in $(\tilde{R}, \tilde{\mathscr{T}})$, containing the zero $\{P_s\}_{s \in S}$ of $\tilde{R}$.

Further, it is easily checked that, for each $t \in S$, $\tilde{P}_t$ is a subgroup of $\tilde{R}$. Hence the set of subgroups $\{\tilde{P}_t\}_{t \in S}$ of $\tilde{R}$ forms a fundamental system of neighbourhoods of the zero of $(\tilde{R}, \tilde{\mathscr{T}})$ and thus, by [3, I §2.3, Example 3], defines the topology $\tilde{\mathscr{T}}$ on $\tilde{R}$.

Next we define a multiplication "$*$" in $\tilde{R}$, and show that $*$ is in fact the required multiplication $\times$. When there is no risk of ambiguity, we shall omit the multiplication sign $*$. Note that if each of the subgroups $P_s$, $s \in S$, were a two-sided ideal of $R$, then multiplication in $\tilde{R}$ would be as simple to define as addition: but this is not the case.

Let $\{\xi_s\}_{s \in S}$, $\{\eta_s\}_{s \in S} \in \tilde{R}$. Let $\{\xi_s\}_{s \in S} * \{\eta_s\}_{s \in S} = \{\Omega_s\}_{s \in S}$ where $\{\Omega_s\}_{s \in S}$ is defined as follows: Let $s \in S$. Then by (1.1) there exists $t \in S$ such that $t^2 \geqq s$. Choose $x_1 \in \xi_t$, $y_1 \in \eta_t$. From (1.4) there exist $u, v \in S$ such that $x_1 P_u \subseteqq P_s$, $P_v y_1 \subseteqq P_s$. Let $w \in S$ be such that $w \geqq t, u, v$. Define $\Omega_s = xy + P_s$ where $x \in \xi_w$, $y \in \eta_w$. The following two lemmas show that $\Omega_s$ is well-defined and independent of the particular choice of $w$.

LEMMA 2.1. *With $w$ chosen, the coset $\Omega_s$ does not depend upon the choice of $x$ and $y$.*

*Proof.* Let $x, x' \in \xi_w$; $y, y' \in \eta_w$. Now

$$xy - x'y' = (x - x')y_1 + x_1(y - y') + (x - x')(y - y_1) + (x' - x_1)(y - y') . \tag{2.2}$$

It is easily checked that each of the summands of (2.2) belongs to $P_s$. Hence $xy - x'y' \in P_s$ and the lemma follows.

LEMMA 2.2. *Let the notation be as above. Let $f, g \in S$ such that, for all $a', a'' \in \xi_f$ and for all $b', b'' \in \eta_g$, $a'b' - a''b'' \in P_s$. Then $\Omega_s = ab + P_s$ for any $a \in \xi_f$, $b \in \eta_g$.*

*Proof.* Let $a \in \xi_f$, $b \in \eta_g$. Let $h \in S$ such that $h \geqq w, f, g$. Let $c \in \xi_h$, $d \in \eta_h$. Then, by Lemma 2.1, $\Omega_s = cd + P_s$ since $c \in \xi_w$, $d \in \eta_w$. But $ab - cd \in P_s$ since $a, c \in \xi_f$ and $b, d \in \eta_g$. Hence $\Omega_s = ab + P_s$.

COROLLARY. *The definition of $\Omega_s$ is independent of the particular choice of $w$.*

*Proof.* Let $w' \in S$ be another possible choice for $w$ (with possibly different $t, u, v, x_1, y_1$). Then, by Lemma 2.1, Lemma 2.2 holds for $f = g = w'$, and the corollary follows.

LEMMA 2.3. *In the above notation, $\{\Omega_s\}_{s \in S} \in \tilde{R}$.*

*Proof.* By the definition, for each $s \in S$, $\Omega_s \in R/P_s$. Let $\lambda, \mu \in S$ such that $\lambda \geqq \mu$. Then, by Lemma 2.1, there exist $m, n \in S$ such that $\Omega_\lambda = x'y' + P_\lambda$ for any $x' \in \xi_m$, $y' \in \eta_m$; and $\Omega_\mu = x''y'' + P_\mu$ for any $x'' \in \xi_n$, $y'' \in \eta_n$. Let $q \in S$ such that $q \geqq m, n$; and let $x \in \xi_q$, $y \in \eta_q$. Then $\Omega_\lambda = xy + P_\lambda$ and $\Omega_\mu = xy + P_\mu$. Hence $\Omega_\lambda \subseteqq \Omega_\mu$ since $P_\lambda \subseteqq P_\mu$. Therefore $\{\Omega_s\}_{s \in S} \in \tilde{R}$.

PROPOSITION 2.1. *With the multiplication defined above, $\tilde{R}$ is a ring which is commutative if $R$ is commutative and has identity $\{1 + P_s\}_{s \in S}$ if $R$ has identity 1.*

*Proof.* We already have that $\tilde{R}$ is an additive Abelian group.
(i) Using the definition of multiplication in $\tilde{R}$ and the directed

property of $S$, it is a straightforward task to show that multiplication in $\tilde{R}$ is associative and that both distributive laws hold. Hence $\tilde{R}$ is a ring which, by the definition of multiplication, is commutative if $R$ is commutative.

(ii) Let $R$ have identity 1. As noted before, $\{1 + P_s\}_{s \in S} \in \tilde{R}$. Again, using the directed property of $S$ and the fact that, for each $s \in S$, $1 \in 1 + P_s$, it is a straightforward task to show that $\{1 + P_s\}_{s \in S}$ is the identity of $\tilde{R}$.

Next we show that $\{\tilde{P}_s\}_{s \in S}$, the fundamental system of neighbourhoods of the zero of $(\tilde{R}, \tilde{\mathscr{T}})$, is in fact a filtration on $(\tilde{R}, *)$ which defines the topology $\tilde{\mathscr{T}}$ as at the beginning of §2; and hence the multiplication $*$ is continuous in $(\tilde{R}, \tilde{\mathscr{T}}, *)$. We need the following preliminary result.

LEMMA 2.4. *Let $x \in R$, $t \in S$. Then there exists $u \in S$ such that $\{x + P_s\}_{s \in S} * \tilde{P}_u \subseteqq \tilde{P}_t$ and $\tilde{P}_u * \{x + P_s\}_{s \in S} \subseteqq \tilde{P}_t$.*

*Proof*. By (1.4) there exists $v \in S$ such that $xP_v \subseteqq P_t$; and by (1.1) there exists $w \in S$ such that $w^2 \geqq t$. Let $u \in S$ such that $u \geqq v, w$. Let $\{\eta_s\}_{s \in S} \in \tilde{P}_u$; that is, $\eta_u = P_u$. Let $x_1 \in x + P_u$, $y_1 \in P_u$. Then $x_1 y_1 \in P_t$ since $P_u \subseteqq P_v \cap P_w$, and so $xP_u \subseteqq P_t$, $P_u P_u \subseteqq P_t$. Therefore, for all $x', x'' \in x + P_u$ and for all $y', y'' \in \eta_u$, $x'y' - x''y'' \in P_t$. Hence, by Lemma 2.2, with $f = g = u$, $s = t$, $a = x_1$ and $b = y_1$, $\{x + P_s\}\{\eta_s\} = \{\Omega_s\}$ where $\Omega_t = P_t$: that is, $\{x + P_s\}\{\eta_s\} \in \tilde{P}_t$. Similarly $\tilde{P}_u\{x + P_s\} \subseteqq \tilde{P}_t$.

PROPOSITION 2.2. *$\{\tilde{P}_s\}_{s \in S}$ is a filtration on $\tilde{R}$ which defines the topology $\tilde{\mathscr{T}}$.*

*Proof*. (i) $S$ is a directed semigroup with property (1.1) and, as noted, each $\tilde{P}_s$, $s \in S$, is an additive subgroup of $\tilde{R}$.

(ii) Let $t, u \in S$ such that $u \geqq t$. It is easily checked that $\tilde{P}_u \subseteqq \tilde{P}_t$.

(iii) Let $t, u \in S$. Again, it is easily checked that $\tilde{P}_t \tilde{P}_u \subseteqq \tilde{P}_{tu}$.

(iv) Let $\{\xi_s\} \in \tilde{R}$, $t \in S$. We must show that there exists $r \in S$ such that $\{\xi_s\}\tilde{P}_r \subseteqq \tilde{P}_t$ and $\tilde{P}_r\{\xi_s\} \subseteqq \tilde{P}_t$. Let $w \in S$ such that $w^2 \geqq t$ and let $x \in \xi_w$. Then $\{\xi_s\} - \{x + P_s\} \in \tilde{P}_w$. By Lemma 2.4 there exists $u \in S$ such that $\{x + P_s\}\tilde{P}_u \subseteqq \tilde{P}_t$. Let $r \in S$ such that $r \geqq u, w$; and let $\{\zeta_s\} \in \tilde{P}_r$. Now $\{\xi_s\}\{\zeta_s\} = (\{\xi_s\} - \{x + P_s\})\{\zeta_s\} + \{x + P_s\}\{\zeta_s\}$; $(\{\xi_s\} - \{x + P_s\})\{\zeta_s\} \in \tilde{P}_w \tilde{P}_r \subseteqq \tilde{P}_{wr} \subseteqq \tilde{P}_t$ by (ii) and (iii); and $\{x + P_s\}\{\zeta_s\} \in \{x + P_s\}\tilde{P}_u \subseteqq \tilde{P}_t$ since $r \geqq u$. Hence, by (i), $\{\xi_s\}\{\zeta_s\} \in \tilde{P}_t$. Similarly $\tilde{P}_r\{\xi_s\} \subseteqq \tilde{P}_t$. This completes the proof.

THEOREM 2.1. *The Hausdorff completion of* $(R, \mathscr{T})$ *is isomorphic to* $(\tilde{R}, \tilde{\mathscr{T}}, *)$.

*Proof.* By [3, III §7.3, Proposition 2], the mapping $i: R \to \tilde{R}$ given by: for all $x \in R$, $i(x) = \{x + P_s\}_{s \in S}$, has an image which is dense in $(\tilde{R}, \tilde{\mathscr{T}})$. From [3, III §6.5 and III §7.3, Proposition 2, Corollary 1], the mapping $i: R \to (\tilde{R}, \times)$ is a ring homomorphism. Hence $i(xy) = i(x) \times i(y)$. But

$$i(xy) = \{xy + P_s\}_{s \in S} = \{x + P_s\}_{s \in S} * \{y + P_s\}_{s \in S} = i(x) * i(y) .$$

Thus the multiplications $*$ and $\times$, which are continuous in $\tilde{\mathscr{T}}$, agree on the dense subset $i(R)$ of $(\tilde{R}, \tilde{\mathscr{T}})$. Therefore, by the principle of extension of identities [3, I §8.1], $*$ and $\times$ agree on $\tilde{R}$. Thus $(\tilde{R}, \tilde{\mathscr{T}}, *)$ is the Hausdorff completion of $(R, \mathscr{T})$.

3. **Example.** In this section we illustrate our theory with a semivaluation on the field $Q$ of rational numbers (Zelinsky [7]).

We shall reserve the sign "$\geqq$" for the usual ordering on $Q$ and shall denote the usual absolute value of the rational number $x$ by $|x|$. Define $S = \{x: x \in Q, x > 0\}$. Order $S$ as follows: For all $a, b \in S$, $a \geqq b$ if and only if $ab^{-1} \in I$ (the set of natural numbers). Then $(S, \geqq)$ is a quasi-residuated, directed semigroup under multiplication. Define a mapping $\varphi: Q \to S_0$ as follows: For all $x \in Q \backslash \{0\}$, $\varphi(x) = |x|$; and $\varphi(0) = O_S$. Then it can easily be checked that $\varphi: Q \to S_0$ is a pseudovaluation on $Q$. (In fact, $\varphi$ is a semivaluation on $Q$, from Zelinsky [7]).

PROPOSITION 3.1. *The completion of* $Q$ *with respect to* $\varphi$ *is isomorphic to the ring of formal series* $\sum_{i=1}^{\infty} i!\, a_i$ *where* $a_i \in Q$, $0 \leqq a_1 < 2$, *and, for each* $i \in I \backslash \{1\}$, $a_i \in \{0, 1, \cdots, i\}$.

*Proof.* We shall use the notation of §§1 and 2 throughout. Now, for each $s \in S$,

$$P_s = \{x: x \in Q, \varphi(x) \geqq s\} = \{ms: m \in Z\} .$$

We shall use the fact that, for all $p, q \in I$, $p! \geqq p \geqq p/q$ and $p! \geqq (p-1)!$: that is, for all $\{\xi_s\}_{s \in S} \in \tilde{Q}$, $\xi_{p!} \subseteqq \xi_p \subseteqq \xi_{p/q}$ and $\xi_{p!} \subseteqq \xi_{(p-1)!}$.

(i) Let $\{\xi_s\}_{s \in S} \in \tilde{Q}$.

Let $x_1 \in \xi_2$. Then there exists a unique $a_1 \in Q$ such that $0 \leqq a_1 < 2$ and $x_1 - a_1 \in P_2$. Suppose that $x_1' \in \xi_2$ and $a_1' \in Q$ such that $0 \leqq a_1' < 2$ and $x_1' - a_1' \in P_2$. Then

$$a_1 - a_1' = (x_1' - a_1') - (x_1 - a_1) + (x_1 - x_1') \in P_2 .$$

Hence $a_1 = a_1'$, and so $a_1$ is independent of $x_1$. Then $\xi_2 = a_1 + P_2$.

Let $x_2 \in \xi_{3!} - (a_1 + P_{3!})$. Since $\xi_{3!} \subseteqq \xi_2$ and $P_{3!} \subseteqq P_2$, we have $x_2 \in \xi_2 - (a_1 + P_2) = P_2$. Hence $x_2/2$ is an integer. Let $a_2 \in \{0, 1, 2\}$ such that $a_2 \equiv x_2/2 \bmod 3$. Then $\xi_{3!} = a_1 + 2a_2 + P_{3!}$.

Next, suppose $k \in I \backslash \{1, 2\}$ such that $\xi_{k!} = a_1 + \sum_{i=2}^{k-1} i!\ a_i + P_{k!}$ where $a_i \in \{0, 1, \cdots, i\}$ for each $i \in \{2, 3, \cdots, k-1\}$. As before, we can show that there exists $a_k \in \{0, 1 \cdots, k\}$ such that $\xi_{(k+1)!} = a_1 + \sum_{i=2}^{k} i!\ a_i + P_{(k+1)!}$. Further, each $a_i$ is unique.

Let $s \in S$. Then there exist unique $p, q \in I$ such that $s = p/q$ and $(p, q) = 1$. Now $\xi_{p!} \subseteqq \xi_{p/q}$. Hence $\xi_s = \sum_{i=1}^{p-1} i!\ a_i + P_s$.

Suppose that $\{\xi_s\}_{s \in S}$ and $\{\eta_s\}_{s \in S} \in S$ define the same set of $a_i$, $i \in I$. Then, for each $s \in S$, $\xi_s = \eta_s$. Hence $\{\xi_s\}_{s \in S}$ defines a unique set of $a_i$, $i \in I$.

(ii) Let $\{a_i\}_{i \in I}$ be given such that $a_1 \in Q$, $0 \leqq a_1 < 2$ and, for each $i \in I \backslash \{1\}$, $a_i \in \{0, 1, \cdots, i\}$. Let $s \in S$. Then, as before, there exists a unique $p \in I$ such that $p \geqq s$. Define $\xi_s = \sum_{i=1}^{p-1} i!\ a_i + P_s$. It is a straightforward task to show that $\{\xi_s\}_{s \in S} \in \tilde{Q}$.

Thus far we have established a one-one correspondence between the elements of $\tilde{Q}$ and formal power series $\sum_{i=1}^{\infty} a_i\ i!$ where $a_1 \in Q$, $0 \leqq a_1 < 2$, and, for each $i \in I \backslash \{1\}$, $a_i \in \{0, 1, \cdots, i\}$.

(iii) Let $\{\xi_s\}_{s \in S}, \{\eta_s\}_{s \in S} \in \tilde{Q}$ correspond to the series $\sum_{i=1}^{\infty} i!\ a_i, \sum_{i=1}^{\infty} i!\ b_i$ respectively. Now $\{\xi_s\}_{s \in S} + \{\eta_s\}_{s \in S} = \{\xi_s + \eta_s\}_{s \in S}$. Hence we can define addition of the series as would be hoped: $\sum_{i=1}^{\infty} i!\ a_i + \sum_{i=1}^{\infty} i!\ b_i = \sum_{i=1}^{\infty} i!\ (a_i + b_i)$ where at the $i$th stage $a_i + b_i$ is reduced modulo $(i+1)$ and the integral part of $(a_i + b_i)/(i + 1)$ carried on.

Let $\{\Omega_s\} = \{\xi_s\}\{\eta_s\}$. Let $s \in S$. Then there exists $t \in S$ such that, for all $x \in \xi_t$, $y \in \eta_t$, $\Omega_s = xy + P_s = \sum_{i=1}^{k-1} i!\ a_i \sum_{i=1}^{k-1} i!\ b_i + P_s$ for some $k \in I$. Hence we can define multiplication of the series in the usual way, taking care to correct each term as described for the addition. This proves the proposition.

Remark. The above example illustrates that the definition of multiplication $*$ in $\tilde{R}$ in §2 cannot be obviously simplified. For example, if $\{\xi_s\}_{s \in S} = \{5 + P_s\}_{s \in S}$ and $\{\eta_s\}_{s \in S} = \{3 + P_s\}_{s \in S}$, then $\{\Omega_s\}_{s \in S} = \{\xi_s\}_{s \in S}\{\eta_s\}_{s \in S} = \{15 + P_s\}_{s \in S}$. Now $\xi_2 = 1 + P_2 = \eta_2$, but $\Omega_4 = 3 + P_4$:

that is, it would not have been sufficient to choose the $w$ of §2 such that $w^2 \geqq s$.

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MONASH UNIVERSITY, CLAYTON, VICTORIA, 3168, AUSTRALIA
AND
MOUNT HOLYOKE COLLEGE

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# GEOMETRIC ASPECTS OF PRIMARY LATTICES

BENNO ARTMANN

**The incidence structure derived from a primary lattice with a homogeneous basis of three $n$-cycles is a Hjelmslev plane of level $n$. A desarguesian Hjelmslev plane $H(R)$ is of level $n$ if and only if $R$ is completely primary and uniserial of rank $n$.**

**Introduction.** The classical correspondence between vector spaces, projective spaces and complemented modular lattices was extended to finitely generated modules over completely primary and uniserial rings and primary lattices by Baer [5], Inaba [7] and, recently, by Jónsson and Monk [8]. In these extensions, however, an analogue to the classical projective space is missing. It is shown in the present paper, that the appropriate concept is that of a Hjelmslev space as defined by Klingenberg [9], [10] and by Lück [11]. To be correct, this is only shown for the case of a plane geometry, namely Hjelmslev planes of level $n$, corresponding to primary lattices with homogeneous basis of three $n$-cycles, and to free modules $R^3$. Also, we have the complete correspondence only in the desarguesian case. The restriction to this case is justified, as the author believes, by the fact it is well known to be typical for higher dimensional spaces in the classical theory.

In the non desarguesian case, there is a coordinatization theory for Hjelmslev planes of level $n$ given by Drake [6], but this does not seem to lead to a construction of a lattice from the plane. Every primary lattice with a homogeneous basis of three $n$-cycles, however, leads to a Hjelmslev plane of level $n$ (Theorem 2.13). Planes of level 1 (ordinary projective planes) and of level 2 (uniform Hjelmslev planes) can be shown to be obtainable from lattices. For uniform planes, this was done by the author in [2]. A combination of Theorem 2.13 with results of [4] shows that a desarguesian Hjelmslev plane $\mathscr{H}(\mathscr{R})$ is of level $n$ if and only if $\mathscr{R}$ is completely primary and uniserial of rank $n$.

## 0. Definitions.

0.1. Let $\mathscr{H} = (\mathfrak{p}, \mathfrak{G}, I)$ be an incidence structure consisting of a set $\mathfrak{p}$ of points, a set $\mathfrak{G}$ of lines and an incidence relation $I \subseteqq \mathfrak{p} \times \mathfrak{G}$. We say that two points $p, q$ of $\mathscr{H}$ are neighbors, $p \sim q$, if there are two different lines $G, H$ such that $p, q\ IG, H$. Neighborhood for lines is defined dually. A mapping $\varphi: \mathscr{H} \rightarrow \mathscr{H}^*$ is a morphism of incidence structures, if it maps points on points, lines on lines and

$pIG$ implies $\varphi pI\varphi G$.

An incidence structure $\mathscr{H}$ is called a projective Hjelmslev plane, short $H$-plane, if it satisfies the axioms [9, Def. 0]:

(i) For all points $p, q$ of $\mathscr{H}$ there exists a line $G$ of $\mathscr{H}$ such that $p, qIG$.

(ii) For all lines $G, H$ of $\mathscr{H}$ there exists a point $p$ of $\mathscr{H}$ such that $pIG, H$.

(iii) There exists an ordinary projective plane $\mathscr{P}$ and an epimorphism $\alpha: H \to \mathscr{P}$ such that $\alpha p = \alpha q$ is equivalent to $p \sim q$, and $\alpha G = \alpha H$ is equivalent to $G \sim H$.

Using (iii), we see that neighborhood is an equivalence relation and the factor structure $\mathscr{H}/\sim = \mathscr{H}'$ is a projective plane isomorphic to $\mathscr{P}$. We call $\mathscr{H}'$ the canonical epimorphic image and the projection $\varphi: \mathscr{H} \to \mathscr{H}'$ the canonical epimorphism of $\mathscr{H}$. In [9] it is shown that this set of axioms is equivalent to the ones used in [1] to define $H$-planes.

0.2. We deal with modular lattices with universal bounds $N$ and $U$. The lattice operations are denoted by $\vee, \wedge$ and we make the convention that $\wedge$ shall bind closer than $\vee$, that is $a \vee b \wedge c = a \vee (b \wedge c)$. $L(a, b)$ is the interval of elements $x$ such that $a \leqq x \leqq b$. We use $a \dot{\vee} b$ to denote independent join, i.e. to indicate $a \wedge b = N$. A cycle $a \in \mathscr{L}$ is an element such that $L(N, a)$ is a chain. A cycle of dimension $k$ is a $k$-cycle.

Definition [8, Def. 4.2 and Def. 6.1]: A lattice $\mathscr{L}$ is said to be primary, if:

(i) $\mathscr{L}$ is modular of finite dimension.

(ii) Every element of $\mathscr{L}$ is the join of cycles and the meet of dual cycles.

(iii) Every interval in $\mathscr{L}$ that is not a chain contains at least three atoms.

Furthermore, we make the assumption

(iv) There are three independent $n$-cycles $a_1, a_3, a_3$ such that $U = a_1 \dot{\vee} a_2 \dot{\vee} a_3$ for the greatest element $U$ of $\mathscr{L}$. This means that $\mathscr{L}$ is of type $(0, \cdots, 0, 3)$ in the sense of [8, Def. 4.10]. By [8, Lemma 6.4] it follows, that the $a_i$ are pairwise perspective. Hence they form a homogeneous basis of order three of $\mathscr{L}$ (for a definition of that concept, see [1, Def 1]). Since the dual $\overline{\mathscr{L}}$ of a primary lattice $\mathscr{L}$ is again primary [8, Cor. 6.2], and the type of $\overline{\mathscr{L}}$ is equal to the type of $\mathscr{L}$ [8, Cor. 4.11], we may use duality in deriving results from (i)—(iv).

For the rest of this paper, $\mathscr{L}$ will always denote a lattice satisfying (i)—(iv), i.e. a primary lattice with a homogeneous basis of $n$-cycles $a_1, a_2, a_3$. For $\{i, j, k\} = \{1, 2, 3\}$ we put $A_i = a_j \dot{\vee} a_k$. Since the geometric dimension of $\mathscr{L}$ [8, Def. 5.1] is three, $\mathscr{L}$ may be non-arguesian.

## 1. The $H$-plane $\mathscr{H}(\mathscr{L})$.

1.1. Points and lines in $\mathscr{L}$. Let $\mathfrak{q}$ be the set on $n$-cycles of $\mathscr{L}$, and

$$\mathfrak{p} = \{p \in \mathscr{L} \mid \text{there is } i \in \{1, 2, 3\} \text{ such that } p \mathbin{\dot{\vee}} A_i = U\} .$$

Every $p \in \mathfrak{p}$ is perspective to some $a_i$, hence is $n$-cycle. For an $n$-cycle $q$, assume $q \cap A_i \neq N \neq q \cap A_k$. Then we have $q \wedge A_i \wedge A_k = q \wedge a_j \neq N$ since $q$ is a cycle, and by the same reason $q \wedge A_j = N$. Therefore $L(q, q \vee A_j)$ has dimension $n$ and $q \mathbin{\dot{\vee}} A_j = U$. Hence we have $\mathfrak{p} = \mathfrak{q}$.

By duality, we get: The set of dual cycles of $\mathscr{L}$ of codimension $n$ is equal to the set

$$G = \{G \in \mathscr{L} \mid \text{ there is } i \in \{1, 2, 3\} \text{ such that } G \mathbin{\dot{\vee}} a_i = U\} .$$

We call $\mathfrak{p}$ the set of points of $\mathscr{L}$ and $\mathfrak{G}$ the set of lines of $\mathscr{L}$.

1.2. Geometric elements. Every Element of $\mathscr{L}$ which is the join of independent points is said to be geometric [8, Def. 5.1]. By definition, $a_1, a_2, a_3$ and $A_1, A_2, A_3$ are geometric. From [8, Thm. 5.2] we derive $(FC)$ (a) For every $b \in \{a_1, a_2, a_3, A_1, A_2, A_3\}$ and every

$$x \in \mathscr{L} \text{ with } x \wedge b = N, \text{ there exists } y \geqq x \text{ such that } y \mathbin{\dot{\vee}} b = U.$$

Since the dual (b) of (a) is true as well, $\mathscr{L}$ satisfies the condition (FC) of [1, p. 77].

Let $G$ be a line of $\mathscr{L}$, say $G \mathbin{\dot{\vee}} a_i = U$, and $r = G \wedge A_k$ and $s = G \wedge A_j$. We claim that $r$ and $s$ are points such that $G = r \mathbin{\dot{\vee}} s$. Obviously we have $a_i \wedge (r \vee s) = N$. Then, $a_i \vee r = a_i \vee G \wedge A_k = (a_i \vee G) \wedge A_k = A_k$, so that $r$ and $a_j$ are perspective with center $a_i$. Hence $r$ and $s$ are points. From $a_i \vee (r \vee s) = A_k \vee A_j = U$ and $r \vee s \leqq G$ we get $r \vee s = G$ by the indivisibility of complements.

In particular, every line of $\mathscr{L}$ is geometric.

Since the independent join of three points is $U$, and it is easy to see that the independent join of two points is always a line (by $(FC)$ and [1, Lemma 8]), points and lines make up all geometric elements of $\mathscr{L}$ except for $N$ and $U$.

1.3. For a line $G$ and a point $p \leqq G$, the interval $L(p, G)$ is a chain. Proof: Consider two points $r, s$ such that $r \mathbin{\dot{\vee}} s = G$. For at least one of them, say $r$, we have $r \wedge p = N$. Then $r \mathbin{\dot{\vee}} p = G$ and we have $L(p, G) \cong L(N, r)$, the assertion.

1.4. Neighbors of $p$ on $G$. Again let $p$ be a point, $G$ a line and

$p \leqq G$. We use $\ll$ to denote the covering relation in $\mathscr{L}$. Let $N = z_0 \ll z_1 \ll \cdots \ll z_n = p$ be the chain of elements less than or equal to $p$, and let $p = y_0 \leqq \cdots \leqq y_n = G$ be the chain of elements between $p$ and $G$.

LEMMA. *For every $i \in \{0, 1 \cdots, n\}$ there exists a point $c_i \leqq G$ such that $y_i = p \vee c_i$ and $z_{n-i} = p \wedge c_i$.*

*Proof.* For every $i$, $p$ is a maximal cycle contained in $y_i$ [8, Cor. 4.7]. By [8, Thm. 4.8] $p$ has a relative complement $x_i$ in $L(N, y_i)$ and by [8, Lemma 6.4] there exists a cycle $c_i$ such that $y_i = p \dot{\vee} x_i = p \vee c_i = x_i \dot{\vee} c_i$. Since $c_i$ and $p$ are perspective, $c_i$ is an $n$-cycle, hence a point. Counting the relative dimensions shows $p \wedge c_i = z_{n-i}$.

1.5. Let $G$ and $H$ be two lines and $p$ a point such that $p \leqq G \wedge H$. By the last lemma, there is a point $q \leqq G$ such that $p \vee q = G \wedge H$. This and the dual statement yield

(S) (a) For points $p, q$ of $\mathscr{L}$ and a line $G$ with $p \vee q \leqq G$ there exists a line $H$ such that $p \vee q = G \wedge H$.

(b) For lines $G, H$ of $\mathscr{L}$ and a point $p \leqq G \wedge H$ there exists a point $q$ such that $p \vee q = G \wedge H$.

1.6. In [1, p. 77/78] it was defined: A modular lattice with a homogeneous basis of order three consisting of cycles is called an $H$-lattice, if it satisfies $(FC)$ and $(S)$. By 1.2 and 1.5, $\mathscr{L}$ is an $H$-lattice. From an $H$-lattice an incidence structure $(\mathfrak{p}, \mathfrak{G}, I)$ is derived by defining $\mathfrak{p}$ and $\mathfrak{G}$ as in 1.1 and incidence by the ordering of the lattice. Using Theorem 1 of [1], we can now state:

PROPOSITION. *$\mathscr{L}$ is an $H$-lattice and the incidence structure $\mathscr{H} = \mathscr{H}(\mathscr{L}) = (\mathfrak{p}, \mathfrak{G}, I)$ derived from $\mathscr{L}$ is a projective $H$-plane. Two points $p, q$ of $\mathscr{L}$ are neighbors in $\mathscr{H}$ if and only if $p \wedge q > N$, two lines $G, H$ are neighbors if and only if $G \vee H < U$.*

More information about $\mathscr{H}$ will be given in the next section.

## 2. $\mathscr{H}(\mathscr{L})$ is of level $n$.

DEFINITION 2.1. (cf. [3] and [6]) Let $\mathscr{H}$ and $\mathscr{H}^*$ be $H$-planes with canonical epimorphisms $\varphi: \mathscr{H} \rightarrow \mathscr{H}'$ and $\kappa: \mathscr{H}^* \rightarrow (\mathscr{H}^*)'$ onto ordinary projective planes. Let $\psi: \mathscr{H} \rightarrow \mathscr{H}^*$ be an epimorphism and $\lambda: (\mathscr{H}^*)' \rightarrow \mathscr{H}'$ an isomorphism. If $\varphi = \lambda\kappa\psi$ we say $\mathscr{H}$ has a refined neighbor property defined by $\psi: \mathscr{H} \rightarrow \mathscr{H}^*$. We define $p \equiv q$ by $\psi p = \psi q$ and $G \equiv H$ by $\psi G = \psi H$. Then $\equiv$ is called a refined neighbor relation in $\mathscr{H}$.

We say $\equiv$ is minimal provided the following conditions hold:

(M) Let $p, q$ be points on $G$ and $p$ on $H$.

(a) If $p \equiv q$ and $G \sim H$, then $q$ is on $H$.

(b) If $p \sim q$ and $G \equiv H$, then $q$ is on $H$.

(c) There exist distinct points $a$ and $b$ and distinct lines $A$ and $B$ such that $a \equiv b$ and $A \equiv B$.

DEFINITION 2.2. The ordinary projective planes make up the class of projective $H$-planes of height 1. Suppose $\mathscr{H}$ is an $H$-plane with a minimal neighborhood defined by $\psi\colon \mathscr{H} \to \mathscr{H}^*$, where $\mathscr{H}^*$ is of height $n-1$. Then one calls $\mathscr{H}$ an $H$-plane of height $n$.—It is suitable to denote an $H$-plane of height $n$ by $\mathscr{H}_n$ and by $\mathscr{H}_{n-1}$ the plane and by $\psi_{n-1}, \varphi_{n-1}, \lambda_{n-1}$ the maps which define the minimal neighborhood in $\mathscr{H}_n$. Proceeding thus we obtain, for every $H$-plane of height $n$, the following commutative diagram

$$\begin{array}{ccccccc} \mathscr{H}_n & \xrightarrow[\psi_{n-1}]{} & \mathscr{H}_{n-1} & \xrightarrow[\psi_{n-2}]{} & \cdots & \xrightarrow[\psi_1]{} & \mathscr{H}_1 \\ \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n-1}} & & & & \downarrow{\scriptstyle\varphi_1} \\ \mathscr{H}_n' & \xleftarrow[\lambda_{n-1}]{} & \mathscr{H}_{n-1}' & \xleftarrow[\lambda_{n-2}]{} & \cdots & \xleftarrow[\lambda_1]{} & \mathscr{H}_1' . \end{array}$$

We set $\mu_k = \psi_k \cdots \psi_{n-2}\psi_{n-1}$ and take $\mu_n$ to be the identity on $\mathscr{H}_n$. We denote by $(\sim k)$ the refined neighborhood defined by $\mu_k\colon \mathscr{H}_n \to \mathscr{H}_k$ in $\mathscr{H}_n$.

DEFINITION. 2.3. If $\mathscr{H}_n$ is an $H$-plane of height $n$, then the $H$-planes $\mathscr{H}_i$ in the defining sequence of $\mathscr{H}_n$ are of height $i$. The notion of $(\sim k)$-neighborhood is defined in $\mathscr{H}_i$ as in $\mathscr{H}_n$. A $k$-segment in $\mathscr{H}_i$ is the nonempty intersection of a line with a class of $(\sim k)$-neighbor points. An $H$-plane $\mathscr{H}_n$ of height $n$ is called of level $n$, if the following axiom of reciprocal segments holds in every plane $\mathscr{H}_i$ of the defining sequence of $\mathscr{H}_n$:

(RS) (a) For all lines $G, H$ of $\mathscr{H}_i$, the set of common points of $G$ and $H$ is a $k$-segment, for some $k \in \{1, 2, \cdots, i\}$.

(b) $G(\sim k)H$ if and only if the set of common points of $G$ and $H$ contains an $(i-k)$-segment.

REMARK. For the change of (N) [3, p. 175] to (RS), see [4].

2.4. If the cycles $a_i$ of $\mathscr{L}$ are of dimension 1, then $\mathscr{H}(\mathscr{L})$ is an ordinary projective plane (an $H$-plane such that two points $p, q$ are neighbors if and only if $p = q$), hence an $H$-plane of level 1. If the $a_i$ are bicycles, that is of dimension 2, then by 1.5 every point of $\mathscr{H}(\mathscr{L})$ has at least one proper neighbor and by [2, Satz 3], $\mathscr{H}(\mathscr{L})$ is a uniform $H$-plane, that is of level 2 [3, p. 179].

We are going to apply induction to show that $\mathscr{H}(\mathscr{L})$ is of level $n$ if the $a_i$ are $n$-cycles. We may assume $n > 2$. First we have to show that $\mathscr{H}(\mathscr{L})$ is of height $n$.

2.5. Let $a_i$ cover $b_i$ and $B = b_1 \vee b_2 \vee b_3$. Then $b_1, b_3, b_3$ form a homogeneous basis of $\mathscr{L}^* = L(N, B)$ (cf. [8, Cor. 4.13]). By [8, Cor. 4.4] $\mathscr{L}^*$ satisfies (i) and (ii) of Def. 0.2. Moreover, every interval of $\mathscr{L}^*$ is an interval of $\mathscr{L}$, so $\mathscr{L}^*$ satisfies (iii) as well. Hence $\mathscr{L}^*$ is a primary lattice with the homogeneous basis $b_1, b_2, b_3$ of three $(n-1)$-cycles. Let the derived $H$-plane be $\mathscr{H}^* = \mathscr{H}(\mathscr{L}^*) = (\mathfrak{p}^*, \mathfrak{G}^*, I)$.

Let $p$ be a point of $\mathscr{H} = \mathscr{H}(\mathscr{L})$ and $G$ be a line of $\mathscr{H}$. We define

$$\psi\colon \mathscr{H} \to \mathscr{H}^*$$

by

$$\psi p = p \wedge B \text{ and } \psi G = G \wedge B\,.$$

In the following paragraphs, we will show that $\psi$ is an epimorphism.

If $p \leqq G$, then $p \wedge B \leqq G \wedge B$, so the fact that $\psi$ preserves incidence is trivial.

2.6. Let $p$ be a point of $\mathscr{H}$, say $p \dot{\vee} A_i = U$, and let $B_i = b_j \vee b_k$. Then $(p \wedge B) \vee B_i = B$, and $\psi$ maps $\mathfrak{p}$ into $\mathfrak{p}^*$. We want to show that it is onto. Let $p^*$ be a point of $\mathscr{H}^*$, say $p^* \vee B_i = B$. Then $p^* \wedge A_i = N$, and by [8, Thm. 5.2], $p^*$ is contained in some complement $p$ of $A_i$. It follows $p \in \mathfrak{p}$ and $\psi p = p^*$.

2.7. Let $G$ be a line of $\mathscr{H}$, say $G \dot{\vee} a_i = U$, and $G \wedge A_j = s$ and $G \wedge A_k = r$ as in 1.3. We have $b_i \vee r \geqq b_j$ and $b_i \vee s \geqq b_k$, hence $b_i \vee (G \wedge B) = (b_i \vee G) \wedge B = B$. Since $G \wedge B \wedge b_i = N$, $\psi$ maps $\mathfrak{G}$ into $\mathfrak{G}^*$. Again we have to show that it is onto. Let $G^*$ be a line of $\mathscr{H}^*$ and $G^* = r^* \dot{\vee} s^*$ for two points of $\mathscr{H}^*$. There exist points $r, s$ of $\mathscr{H}$ such that $r \wedge B = r^*$ and $s \wedge B = s^*$. For $G = r \vee s$ we have $\psi G = G^*$.

2.8. Since $p \sim q$ in $\mathscr{H}$ means $p \wedge q > N$ in $\mathscr{L}$, we have $p \sim q$ in $\mathscr{H}$ if and only if $\psi p \sim \psi q$ in $\mathscr{H}^*$.

We want to show that the same is true for lines. Assume $G \sim H$ in $\mathscr{H}$. We know that this means $G \wedge H > p$ for some common point $p$ of $G$ and $H$. Let $x$ be a cycle $\leqq G$ such that $G \wedge H = p \dot{\vee} x$. We may assume $G \neq H$, hence the dimension of $x$ is at most $n - 1$. Therefore $x \leqq G \wedge B$ and $x \leqq H \wedge B$, and we have

$$\begin{aligned} G \wedge H \wedge B &= (p \dot{\vee} x) \wedge B \\ &= p \wedge B \vee x \\ &= \psi p \vee x \,, \end{aligned}$$

and from $x > N$ we deduce $\psi G \sim \psi H$.

Now let $G \not\sim H$ in $\mathscr{H}$, then $G \wedge H = p$ for a unique point $p$. There are points $r, s$ of $\mathscr{H}$ such that $G = p \dot{\vee} r$ and $H = p \dot{\vee} s$. From this we derive $\psi p \vee \psi r \vee \psi s \leqq \psi G \vee \psi H$, and since $\psi p$, $\psi r$, $\psi s$ are three independent $(n-1)$-cycles, it follows $G \wedge B \vee H \wedge B = B$, hence $\psi G \not\sim \psi H$.

Thus we have arrived at: $G \sim H$ in $\mathscr{H}$ if and only if $\psi G \sim \psi H$ in $\mathscr{H}^*$.

2.9. By 2.5 – 2.8 we know:
$\psi\colon \mathscr{H} \to \mathscr{H}^*$ is an epimorphism and

$$\begin{aligned} &p \sim q \text{ if and only if } \psi p \sim \psi q \,, \\ &G \sim H \text{ if and only if } \psi G \sim \psi H \,. \end{aligned}$$

Now, for $n - 1 > 1$, we may repeat the procedure and, changing notation to $\mathscr{H} = \mathscr{H}_n$, $\mathscr{H}^* = \mathscr{H}_{n-1}$ and $\psi = \psi_{n-1}$, get a sequence

$$\mathscr{H}_n \xrightarrow[\psi_{n-1}]{} \mathscr{H}_{n-1} \xrightarrow[\psi_{n-2}]{} \cdots \xrightarrow[\psi_1]{} \mathscr{H}_1 \,,$$

where the final incidence structure $\mathscr{H}_1$ is an ordinary projective plane. The mapping

$$\mu_1 = \psi_1 \cdots \psi_{n-1}\colon \mathscr{H}_n \to \mathscr{H}_1$$

is an epimorphisms such that

$$(*) \qquad \begin{aligned} &p \sim q \text{ in } \mathscr{H} \text{ if and only if } \mu_1 p = \mu_1 q \text{ in } \mathscr{H}_1 \text{ and} \\ &G \sim H \text{ in } \mathscr{H} \text{ if and only if } \mu_1 G = \mu_1 H \text{ in } \mathscr{H}_1. \end{aligned}$$

Now the canonical epimorphism $\varphi_n\colon \mathscr{H}_n \to \mathscr{H}_n'$ is universal with the property (*), hence we have a unique isomorphism $\phi\colon \mathscr{H}_n' \to \mathscr{H}_1$ such that $\mu_1 = \theta\varphi_n$. By the same reasoning for $\mathscr{H}_{n-1}$ and $\nu_1\colon \mathscr{H}_{n-1} \to \mathscr{H}_1$ we get the following commutative diagram

$$\begin{array}{ccccc} & \mathscr{H}_n & \xrightarrow[\psi_{n-1}]{} & \mathscr{H}_{n-1} & \\ {}^{\varphi_n}\swarrow & {}^{\mu_1}\searrow & & \swarrow{}^{\nu_1} & \searrow{}^{\psi_{n-1}} \\ \mathscr{H}_n' & \xrightarrow[\theta]{} & \mathscr{H}_1 & \xleftarrow[\eta]{} & \mathscr{H}_{n-1}' \,. \end{array}$$

If we put $\lambda_{n-1} = \theta^{-1}\eta$, we have $\varphi_n = \lambda_{n-1}\varphi_{n-1}\psi_{n-1}$ and $\psi_{n-1}\colon \mathscr{H}_n \to \mathscr{H}_{n-1}$ defines a refined neighborhood in $\mathscr{H}_n$. Clearly, the same is true for all $\psi_i: \mathscr{H}_{i+1} \to \mathscr{H}_i$ $(1 \leqq i < n)$. Thus we arrive at a commutative diagram as required in Definition 2.2 We did not yet show that the refined neighborhood defined by $\psi_{n-1}: \mathscr{H}_n \to \mathscr{H}_{n-1}$ is minimal. Without knowing this, we define $\mu_i$ and $(\sim i)$ as in 2.2.

2.10. In order to prove the axioms (M) and (RS) of Definitions 2.1 and 2.3, it is useful to have an alternative description $p(\sim i)q$ and $G(\sim i)H$ in $\mathscr{H} = \mathscr{H}(\mathscr{L})$.

(i) Let $N = p_0 \ll p_1 \ll \cdots p_{n-1} \ll p_n = p$ and $N = q_0 \ll \cdots \ll q_n = q$ be the chains of elements below the points $p$ and $q$. We have $\psi_{n-1}p = p \wedge B = p_{n-1}$, hence $\psi_{n-1}p = \psi_{n-1}q$ if and only if $p_{n-1} = q_{n-1}$. Repeating the argument we obtain $\mu_i p = p_i$, which yields $\mu_i p = \mu_i q$ if and only if $p_i = q_i$.

(ii) Let $G, H$ be lines of $\mathscr{H}(\mathscr{L})$ and $p, r, s$ be points such that $G = p \dot{\vee} r$ and $H = p \dot{\vee} s$. Let $r_i, s_i$ be defined like $p_i$ in (i), and $p = x_0 \ll x_1 \ll \cdots \ll x_n = G$. If $\mu_i G = \mu_i H$, then

$$r_i = \mu_i r \leqq \mu_i G = \mu_i H$$

and

$$s_i = \mu_i s \leqq \mu_i H = \mu_i G .$$

Hence $p \dot{\vee} r_i = x_i \leqq G \wedge H$ and from Lemma 1.4 we get
(+) There exists a point $q$ such that $p_{n-i} = q_{n-i}$ and

$$p \vee q = x_i \leqq G \wedge H .$$

Conversely, assume (+). There exists a cycle $r_i$ such that $p \vee q = p \dot{\vee} r_i = x_i$ and points $r, s$ such that $r_i \leqq r \leqq G$ and $r_i \leqq s \leqq H$ [8, Thm. 4.8]. From this we derive $G = p \dot{\vee} r$ and $H = p \dot{\vee} s$ and

$$\mu_i G = p_i \dot{\vee} r_i = \mu_i H .$$

Letting $G = g_0 \ll g_1 \ll \cdots \ll g_n = U$ and $H = h_0 \ll \cdots \ll h_n = U$ we may equivalently say

$$\mu_i G = \mu_i H \text{ if and only if } g_{n-i} = h_{n-i} .$$

Or, using $p = y_0 \ll \cdots \ll y_n = H$:

$$\mu_i G = \mu_i H \text{ if and only if } x_i = y_i .$$

2.11. We are now ready to verify that $\psi_{n-1}: \mathscr{H}_n \to \mathscr{H}_{n-1}$ defines a minimal neighborhood in $\mathscr{H}_n$.

(Ma) From $p \wedge B = q \wedge B$ it follows that $p$ and $q$ cover $p \wedge q$. Hence $p \vee q$ covers $p$ and $q$. Now if $G \vee H < U$, then $G \wedge H > p$ and since $L(p, G)$ is a chain, we have

$$p \ll p \vee q \leqq G \wedge H ,$$

hence $q \leqq H$.
(Mb) Let $x_i$ and $y_i$ be as in 2.10. By 2.10 (ii) we know $x_{n-1} = y_{n-1}$. Now if $p \sim q$, then $p \vee q < G$, hence $p \vee q \leqq x_{n-1} = y_{n-1}$ which implies $q \leqq H$.
(Mc) Taking $i = 1$ in 1.5 we get points with the desired property. By duality, we have lines $G \neq H$ such that $G \vee H$ is a cocycle of codimension $n - 1$, hence $\psi_{n-1}G = \psi_{n-1}H$.

2.12. The axiom of reciprocal segments. By 2.10 (i) an $i$-segment is a set of points on a line $G$ such that $p_i = q_i$ for any two points $p, q$ of the set.
(RSa) Let $p \leqq G \wedge H$ and $p_i$, $x_i$ as before. Assume $G \wedge H = x_{n-i}$. Then for every point $q \leqq H$ we have that

$p \wedge q \geqq p_i$ implies $q \leqq G$, and

$p \wedge q < p_i$ implies $q \nleqq G$, since otherwise $G \wedge H > x_{n-i}$.

Hence the set of points incident with both $G$ and $H$ is an $i$-segment.
(RSb) By 2.10 (ii), $\mu_i G = \mu_i H$ if and only if $G$ and $H$ have (at least) an $i$-segment in common.

THEOREM 2.13. *The H-plane $\mathscr{H}(\mathscr{L})$ derived from a primary lattice L with a homogeneous basis of three n-cycles is an H-plane of level n.*

*Proof*. By 2.9, $\psi_{n-1}: \mathscr{H}_n \to \mathscr{H}_{n-1}$ defines a refined neighborhood in $\mathscr{H}_n$ which is minimal by 2.11. By 2.12, the axiom (RS) of reciprocal segments holds in $\mathscr{H}_n$. Since $\mathscr{H}_{n-1}$ is derived from a primary lattice with a homogeneous basis of $(n - 1)$-cycles, we may assume that $\mathscr{H}_{n-1}$ is of level $n - 1$. But then $\mathscr{H}_n$ is of level $n$.

## 3. Desarguesian $H$-planes of level $n$.

DEFINITION 3.1. [8, Def. 6.6]. A ring $\mathscr{R}$ (associative with unit) is said to be completely primary and uniserial if there is a two-sided ideal $\mathscr{A}$ of $\mathscr{R}$ such that every left or right ideal of $\mathscr{R}$ is of the form $\mathscr{A}^k$ (where $\mathscr{A}^0 = \mathscr{R}$). The rank of such a ring is the smallest integer $k$ such that $\mathscr{A}^k = (0)$.

It is a simple exercise to verify that a completely primary and uniserial ring is an $H$-ring in the sense of [9 Def. 9].

DEFINITION 3.2. Let $\mathscr{R}$ be a completely primary and uniserial ring of rank $n$. The lattice $\mathscr{L}(\mathscr{H}^3)$ of all submodules of the ($\mathscr{R}$-

left) module $\mathscr{R}^3$ is primary [8, Thm. 6.7] and has the homogeneous basis $a_1 = \mathscr{R}(1, 0, 0)$, $a_2 = \mathscr{R}(0, 1, 0)$, $a_3 = \mathscr{R}(0, 0, 1)$ of $n$-cycles. Let $\mathscr{H}(\mathscr{R}) = \mathscr{H}(\mathscr{L}(\mathscr{R}^3))$ be the $H$-plane derived from $\mathscr{L}(\mathscr{R}^3)$. It is easy to check that this plane is essentially the same as defined by Klingenberg [9 Def. 10] via homogeneous coordinates. An $H$-plane $\mathscr{H}$ is called desarguesian if there exists an $H$-ring $\mathscr{R}$ such that $\mathscr{H}$ is isomorphic to $\mathscr{H}(\mathscr{R})$, the latter defined as in [9].

THEOREM 3.3. *If $\mathscr{R}$ is a completely primary and uniserial ring of rank $n$, then the H-plane $\mathscr{H}(\mathscr{R})$ is of level $n$.*

*Proof.* Theorem 2.13 and Definition 3.2.

3.4. In [4] it is shown: If $\mathscr{H} = \mathscr{H}(\mathscr{R})$ is a desarguesian $H$-plane of level $n$, then $\mathscr{R}$ is a completely primary and uniserial ring of rank $n$. We combine this with 3.3:

COROLLARY. *A desarguesian H-plane $\mathscr{H}(\mathscr{R})$ is of level $n$ if and only if $\mathscr{R}$ is completely primary and uniserial of rank $n$.*

3.5. Since the lattice $\mathscr{L}(\mathscr{R}^3)$ defined in 3.2 is arguesian, we have a correspondence between completely primary and uniserial rings of rank $n$, arguesian primary lattices with a homogeneous basis of three $n$-cycles and desarguesian $H$-plane of level $n$ as in the classical theory of projective spaces. With the appropriate definitions, it should be not too hard to verify the analogues correspondences for finite dimensional $H$-spaces. The coordinatization theorems relevant for this can be found in [7] and [8] for lattices and in [10] and [11] for Hjelmslev spaces.

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MATHEMATISCHES INSTITUT
ARNDTSTR. 2
63 GIESSEN, GERMANY

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# DETERMINING A POLYTOPE BY RADON PARTITIONS

MARILYN BREEN

**In an extension of the classical Radon theorem, Hare and Kenelly have introduced the concept of a primitive partition, allowing a reduction to minimal subsets which still possess the necessary intersection property.**

**Here it is proved that primitive partitions in the vertex set $P$ of a polytope reveal the subsets of $P$ which give rise to faces of conv $P$, thus determining the combinatorial type of the polytope. Furthermore, the polytope may be reconstructed from various subcollections of the primitive partitions.**

2. Preliminary results. Throughout, $|P|$ denotes the cardinality of $P$. If $P$ is a set of points in $R^d$, $A \cup B$ is a *Radon partition* for $P$ iff $P = A \cup B$, $A \cap B = \emptyset$, and conv $A \cap$ conv $B \neq \emptyset$. Each of $A$ and $B$ is called half a partition for $P$ and each element of $A$ is said to *oppose* $B$ in the partition. The Radon theorem says that for $P \subseteq R^d$ having at least $d + 2$ points, there exists a Radon partition for $P$. When $P$ is in general position in $R^d$ and $P$ has exactly $d + 2$ elements, the partition is unique.

In [2], Hare and Kenelly introduce the concept of a primitive partition: For $P \subseteqq R^d$, $A \cup B$ is a Radon partition *in* $P$ iff $A \cup B$ is a Radon partition for a subset $S$ of $P$. We say that the Radon partition $A \cup B$ *extends* the Radon partition $A' \cup B'$ iff $A' \subseteqq A$ and $B' \subseteqq B$. Finally, $A \cup B$ is called a *primitive partition* in $P$, or simply a *primitive*, provided it is a Radon partition in $P$ and $A \cup B$ extends the Radon partition $A' \cup B'$ iff $A' = A$ and $B' = B$. It is proved that each Radon partition extends a primitive partition having cardinality at most $d + 2$.

Theorem 1 follows immediately from the results of Hare and Kenelly.

THEOREM 1. *Let $P$ denote a set of $d + 2$ points in $R^d$ and let $A \cup B$ be a primitive for $P$. Then $|A| + |B| = d + 2$ iff $P$ is in general position.*

COROLLARY 1. *If $A \cup B$ is a primitive for $P$, $P \subseteqq R^d$, then $A \cup B$ is in general position in $R^k$ for some $k \leqq d$, and $|A| + |B| = k + 2$ for this $k$.*

THEOREM 2. *If $P \subseteqq R^1$ and $A \cup B$ is a primitive for $P$, then dim (conv $A \cap$ conv $B$) $= 0$.*

*Proof.* By the corollary to Theorem 1, $A \cup B$ is in general position in $R^k$ for some $k \leqq d$.

Recall that dim (aff $A \cap$ aff $B$) = dim aff $A$ + dim aff $B$ − dim (aff $A$ + aff $B$). Letting $j = |A|$ and $l = |B|$, for points in general position, this is equal to $(j-1)+(l-1)-k = j+l-k-2$. Also, for $k+2$ points in general position, the partition is unique, and so $j+l = k+2$, and the above is zero.

**3. Reconstructing polytopes.** Our goal is to establish the relationship between faces of conv $P$ and primitive partitions for $P$. Throughout, $P$ denotes the vertex set of a convex polytope in $R^d$, and $|P| = n$.

THEOREM 3. *If $S \subseteqq P$ and conv $S$ is a face of conv $P$, then $S$ is not half a Radon partition for $P$.*

*Proof.* Assume conv $S$ is a proper face, for otherwise the result is trivial. Let $H$ be a supporting hyperplane to conv $P$ for which $H \cap \text{conv}\, P = \text{conv}\, S$. Assume $P \subseteqq \text{cl}\,(H_+)$, the closure of the open half-space $H_+$. Then $P \sim S \subseteqq H_+$, and conv $(P \sim S) \cap$ conv $S = \varnothing$.

The following definitions are useful in obtaining a converse to Theorem 3.

DEFINITION. *Let $S \subseteqq P$.* Then we say conv $S$ *cuts* conv $P$ (or $S$ *cuts* conv $P$) iff one of the following is true: Either (1) dim aff $S = d$ or (2) dim aff $S \leqq d-1$ and any hyperplane containing $S$ cuts conv $P$.

DEFINITION. If $S \subseteqq P$ and conv $S$ cuts conv $P$, then a subset $T$ of $S$ is said to be a *minimal cutting subset of $S$ for $P$* iff conv $T$ cuts conv $P$ and no subset of $S$ of cardinality less than $|T|$ cuts conv $P$.

THEOREM 4. *If $|P| = n \geqq d+1$, and $S \subsetneqq P$, then the following is true: conv $S$ is a face for conv $P$ iff for $A \subseteqq S$, $A$ is half a primitive for $P$ only in case all the elements opposing $A$ in the primitive are also in $S$.*

*Proof.* If conv $S$ is a face for conv $P$, then by Theorem 3, $S$ cannot be half a Radon partition for $P$. Thus if $A \subseteqq S$ and $A$ is half a primitive for $P$, some of the elements opposing $A$ must lie in $S$. We must show that all the elements opposing $A$ lie in $S$:

Suppose not, and let $A \cup B$ be a primitive for $P$ with $A \subseteqq S$, $B \cap$

$S \neq \emptyset$, and $B \cap (P \sim S) \neq \emptyset$. Since $A \cup B$ is a primitive, $\operatorname{conv} A \cap \operatorname{conv}(B \cap S)$ is empty. Thus any point in $\operatorname{conv} A \cap \operatorname{conv} B$ cannot lie in $\operatorname{conv} S$. Yet $A \subseteqq S$, so $\operatorname{conv} A \subseteqq \operatorname{conv} S$, and we have a contradiction. Our supposition is false, and all members of $B$ lie in $S$.

Conversely, suppose $S \subsetneqq P$ has the property that for $A \subseteqq S$, $A$ is half a primitive only in case all the elements opposing $A$ in the primitive come from $S$.

Let $x \in P \sim S \neq \emptyset$.

First we assert that $x \notin \operatorname{aff} S$. If $x \in \operatorname{aff} S$, then reduce $S$ to a $(k+1)$-subset $T \subseteqq S$ such that $\operatorname{aff} T = \operatorname{aff} S$, where $k = \dim \operatorname{aff} S$. Then $\operatorname{conv} T$ is necessarily a simplex. Since $T \cup \{x\}$ is a $(k+2)$-subset of $R^k = \operatorname{aff}(T \cup \{x\})$, there is a Radon partition for $T \cup \{x\}$. Let $A_0 \cup B_0$ be a primitive for $T \cup \{x\}$. Necessarily $x$ appears, since $T$ is a simplex. Assume $x \in B_0$. Then $A_0$ is a subset of $T$ (and thus a subset of $S$) which is half a primitive for $P$. Yet $x$ opposes $A_0$ and $x$ is not in $S$, contradicting our hypothesis. Thus we have proved that for $x$ in $P \sim S$, $x \notin \operatorname{aff} S$. Also, this implies that $S = P \cap \operatorname{aff} S$ and $\dim \operatorname{aff} S \leqq d-1$.

We assert that $S$ lies in a proper face of $\operatorname{conv} P$. Assume that $S$ does not lie in a proper face of $\operatorname{conv} P$ to reach a contradiction. Let $x \in P \sim S$. If $S$ does not lie in a face of $\operatorname{conv} P$, then $\operatorname{conv} S$ necessarily cuts $\operatorname{conv} P$. Choose $S' \subseteqq S$ to be a minimal cutting subset of $S$ for $P$. Let $p$ be in $\operatorname{conv} S'$ and interior to $\operatorname{conv} P$. We will show that a subset $A$ of $S'$ is half a primitive partition $A \cup B$ for $P$, where $B \nsubseteqq S$:

Consider the ray from $x$ through $p$. Since $p$ is interior to conv $P$, this ray intersects bdry conv $P$ at a point $v$ beyond $p$. Clearly $v \notin \operatorname{aff} S$, or else $x \in \operatorname{aff}(S \cup \{v\}) = \operatorname{aff} S$, a contradiction since $x \notin \operatorname{aff} S$. Now $v$ lies in a facet $F$ of conv $P$. Choose exactly $d$ vertices $T$ in $F$ such that $v \in \operatorname{conv} T$ and $T$ determines a simplex.

Let $Q \equiv T \cup S' \cup \{x\}$. Consider the polytope conv $Q$. We will show that $S'$ is half a partition for $Q$:

By minimality of $|S'|$, it follows that $\operatorname{aff} S' \cap \operatorname{conv} P = \operatorname{conv} S'$. For otherwise, $\operatorname{conv} S'$ is not in a face for the polytope $\operatorname{aff} S' \cap \operatorname{conv} P$ (since the dimensions are the same), and some proper subset of $S'$ must cut $\operatorname{aff} S' \cap \operatorname{conv} P$. Thus a proper subset of $S'$ cuts our original polytope conv $P$, contradicting minimality of $S'$. This implies also that $\operatorname{aff} S' \cap \operatorname{conv} Q = \operatorname{conv} S'$.

To show that $\operatorname{conv} S' \cap \operatorname{conv}(Q \sim S') \neq \emptyset$, it suffices to show that $\operatorname{aff} S' \cap \operatorname{conv}(Q \sim S') \neq \emptyset$. Assume that the intersection is empty to reach a contradiction. If the intersection is empty, then strictly separate aff $S'$ from $\operatorname{conv}(Q \sim S')$ by a hyperplane $H$. Since $H \cap \operatorname{aff} S' = \emptyset$, $H$ must be parallel to aff $S'$. Let $J$ be a hyperplane parallel to $H$ and containing aff $S'$. Clearly $J \cap \operatorname{conv}(Q \sim S') = \emptyset$, so $J$ is a supporting hyperplane for conv $Q$ such that $J \cap \text{conv } Q = \text{conv } S'$, and conv $S'$ is a face for conv $Q$. However, this is a contradiction, for the segment $[x, v]$ intersects conv $S'$ at $p$. Our assumption is false, conv $S' \cap \text{conv}(Q \sim S')$ is not empty, and $S'$ is half a partition for $Q$.

Let $A \cup B$ be a primitive inside $S' \cup (Q \sim S')$. We claim that $x$ necessarily appears in $B$, for otherwise we have $B \subseteq T$, but conv $T$ is a face for conv $Q$ so by the first part of this theorem, $A \subseteq T$ also. But we chose $T$ to be a simplex, so there is no primitive for $T$; we have a contradiction, and $x$ must appear.

Recall that $x \notin S$. Thus $B \nsubseteq S$ since $x \in B$. At last we have contradicted our hypothesis, for $A \cup B$ is a primitive such that $A \subseteq S$ and $B \nsubseteq S$. Our assumption that $S$ does not lie in a face of conv $P$ is false, and $S$ does indeed lie in a face.

To complete the proof, it remains to show that conv $S$ is a full face of conv $P$. Select a face $F$ of conv $P$ having minimal dimension for which $S \subseteq F$. Clearly $S$ cannot lie in a proper face of the polytope $F$. Thus, $F \subseteq \text{aff } S$, so $P \cap F \subseteq P \cap \text{aff } S = S$, and vert $F = S$, finishing the proof.

COROLLARY 1. *For a simplicial polytope conv $P$ and $S \subseteq P$, conv $S$ is a face for conv $P$ iff no subset of $S$ is half a primitive for $P$.*

The proof to Theorem 4 required a construction which we will need again, and for this reason we list it as a corollary:

COROLLARY 2. *Let $S \subseteq P, x \in P \sim \text{aff } S \neq \emptyset$. If $S$ does not lie in a face of conv $P$, let $S'$ be a minimal cutting subset of $S$ for $P$. Then aff $S' \cap \text{conv } P = \text{conv } S'$. Moreover, $S'$ is half a Radon partition for a subset $Q$ of $P$ where $x \in Q$, and $Q$ may be chosen so that $Q \sim [S' \cup \{x\}]$ is a simplex and lies in a facet of conv $P$. For any primitive $A \cup B$ inside $S' \cup [Q' \sim S']$ with $A \subseteq S', x \in B$.*

COROLLARY 3. *If $P$ is in general position, $S$ half a Radon partition for $P, x \in P \sim S$, and $S'$ a minimal cutting subset of $S$ for $P$, then $S'$ is half a primitive for $P$, and this primitive may be selected so that $x$ still appears.*

DEFINITION. We say that it is possible to *reconstruct* the polytope conv $P$ iff for each face $F$ of conv $P$ we can determine the unique subset $S$ of $P$ such that conv $S = F$.

The author wishes to thank the referee for the following observation: Let $\mu$ determine the collection of all sets $S \subseteq P$ for which conv $S$ is a face for conv $P$. Since $\mu$ is a complete lattice under inclusion, and each maximal chain in $\mu$ is of length $d + 2$, beginning with $\emptyset$

and ending with $P$, we can determine the dimension of each face $\operatorname{conv} S$ from its position in any maximal chain. The lattice $\mu$ also determines all inclusion relations between faces and hence gives the combinatorial type of $\operatorname{conv} P$.

Therefore, when the definition of reconstruct is satisfied, the combinatorial type of the polytope is revealed.

DEFINITION. Let $P_1, P_2$ be vertex sets for two polytopes $\operatorname{conv} P_1$, $\operatorname{conv} P_2$, and let $R_1$, $R_2$, denote the set of primitive partitions for $P_1$, $P_2$ respectively. We say that $R_1$ is *isomorphic* to $R_2$ iff there is a one-to-one map $\psi$ of $P_1$ onto $P_2$ having the following property: $A \cup B$ is a primitive for $P_1$ iff $\psi(A) \cup \psi(B)$ is a primitive for $P_2$.

The following corollary is a direct consequence of Theorem 4.

COROLLARY 4. *Let $P_1, P_2$ be vertex sets for polytopes, $R_1$, $R_2$ their respective primitive partitions. If $R_1$ is isomorphic to $R_2$, then conv $P_1$ is combinatorially equivalent to conv $P_2$. Thus it is possible to determine the combinatorial type of a polytope from the Radon partitions of its vertex set.*

The following example shows that the converse is false. That is, two polytopes may be combinatorially equivalent although their vertex sets have non-isomorphic Radon partitions.

EXAMPLE 1. Let $\{1, 2, 3, 4\}$ be the vertex set for a square which is base for two distinct bipyramids $\operatorname{conv} P_1$ and $\operatorname{conv} P_2$. Let $\{5, 6\}$ be the remaining vertices for $\operatorname{conv} P_1$, and let the segment $[5, 6]$ pass through the center of the square. The primitives for $P_1$ are

$$\{1, 3\} \cup \{2, 4\} ,$$
$$\{1, 3\} \cup \{5, 6\} ,$$
$$\{2, 4\} \cup \{5, 6\} .$$

Now let $\{7, 8\}$ be the remaining vertices for $\operatorname{conv} P_2$, where the segment $[7, 8]$ intersects the base within $[2, 4] \cap \operatorname{rel\,int} \operatorname{conv} \{1, 2, 3\}$. The primitives for $P_2$ are

$$\{1, 3\} \cup \{2, 4\}$$
$$\{1, 2, 3\} \cup \{7, 8\}$$
$$\{2, 4\} \cup \{7, 8\} .$$

The primitives for $P_1, P_2$ are not isomorphic, yet the map $\psi$ from $P_1$ onto $P_2$ defined as the identity on $\{1, 2, 3, 4\}$, $\psi(5) = 7$, $\psi(6) = 8$, sets up a one-to-one correspondence between faces and is inclusion preserving.

Even for points in general position, combinatorial equivalence of conv $P_1$, conv $P_2$ does not imply that $R_1$ is isomorphic to $R_2$. However, in case we have exactly $d + 2$ points in general position in $R^d$, the implication does hold.

COROLLARY 5. *For $i = 1, 2$, let conv $P_i$ be a simplicial polytope having $d + 2$ vertices, and let $R_i$ be the unique Radon partition for $P_i$. Then combinatorial equivalence of conv $P_1$, conv $P_2$ implies that $R_1$ is isomorphic to $R_2$.*

It is interesting that Corollary 5 may be used to obtain the following familiar result.

COROLLARY 6. *Consider the collection $\mathscr{P}$ of all sets $P$ in $R^d$ consisting of $d + 2$ points in general position with no point of $P$ interior to conv $P$. Then there are exactly $[d/2]$ possible Radon partitions for $P$ in $\mathscr{P}$ and each one determines a distinct polytope conv $P$. Therefore, there are exactly $[d/2]$ simplicial polytopes having $d + 2$ vertices.*

**4. Reductions.** Of major interest is the problem of obtaining a minimal subcollection of primitive partitions for $P$ which will determine the combinatorial type of conv $P$. The following theorems are concerned with one kind of reduction.

For $x \in P$, let $\mathscr{C}_x$ denote the subcollection of primitive partitions for $P$ defined in the following manner: $A \cup B$ belongs to $\mathscr{C}_x$ iff either (1) $x$ appears in $A \cup B$ or (2) $|A| + |B| \leqq d + 1$.

Theorems 5 and 6 show that conv $P$ may be reconstructed from $\mathscr{C}_x$.

THEOREM 5. *For $x \in P$ and $S \subseteqq P \sim \{x\}$, conv $S$ is not a face for conv $P$ iff there is some member $A \cup B$ of $\mathscr{C}_x$ such that $A \subseteqq S$, $B \nsubseteq S$.*

*Proof.* By Theorem 4, if a subset $A$ of $S$ is half a primitive $A \cup B$ for $P$, and $B \nsubseteq S$, conv $S$ cannot be a face for conv $P$.

Conversely, suppose that $x$ is a specified point in $P$, $S \subseteqq P \sim \{x\}$, and conv $S$ is not a face for conv $P$. We consider cases:

*Case* 1. If $S$ lies in a facet $F$ of conv $P$, then by a fundamental property of polytopes, conv $S$ cannot be a face for $F$. Using Theorem 4, since conv $S$ is not a face for the polytope $F$, a subset $A$ of $S$ must be half a primitive $A \cup B$ for vert $F$, with $B \nsubseteq S$. Moreover, since $F$ is $(d-1)$-dimensional, $|A| + |B| \leqq d + 1$, and Condition (2) is satisfied.

*Case* 2. If $S$ does not lie in a facet and if $x \in$ aff $S$, then as in the proof of Theorem 4, let dim aff $S = k \leqq d$ and reduce $S$ to a

$(k+1)$-subset $T$ of $S$ such that aff $T =$ aff $S$. Conv $T$ is necessarily a simplex. Since $T \cup \{x\}$ is a $(k+2)$-subset of $R^k =$ aff $(T \cup \{x\})$, there is a Radon partition for $T \cup \{x\}$. Let $A \cup B$ be a primitive corresponding to this partition. Necessarily $x$ appears since conv $T$ is a simplex. Assume $x \in B$. Then $A \subseteqq T \subseteqq S$, and Condition (1) is satisfied.

*Case* 3. If $S$ does not lie in a facet and if $x \notin$ aff $S$, then we may call on the technical corollary following Theorem 4 to obtain a subset $S'$ of $S$ and a subset $Q$ of $P$ having the property that $S' \cup (Q \sim S')$ is a Radon partition for $Q$. Moreover, if $A \cup B$ is a primitive inside $S' \cup (Q \sim S')$, then $x$ appears in $B$. Thus $A \subseteqq S$, $B \nsubseteq S$, and $x$ opposes a subset of $S$ in this primitive. We have satisfied Condition (1) and completed the proof of the theorem.

For $x$ in $P$, Theorem 5 allows us to recognize all faces of conv $P$ not containing $x$ by listing the primitives in which $x$ appears plus the primitives having $\leqq d+1$ points. Our next problem, of course, is recognizing the faces containing $x$, and we would like to be able to do this from the same collection of primitives. Happily, the next theorem shows that this is possible.

THEOREM 6. *For $T \subsetneqq P$ and $x$ in $T$, conv $T$ is not a face for conv $P$ iff there is some member $A \cup B$ of $\mathscr{C}_x$ such that $A \subseteqq T$, $B \nsubseteq T$.*

*Proof*. Certainly if there is a primitive $A \cup B$ with $A \subseteqq T$ and $B \nsubseteq T$, then by Theorem 4, conv $T$ cannot be a face for conv $P$.

Conversely, assume that conv $T$ is not a face for conv $P$ and $x \in T$. Again, we must consider cases:

*Case* 1. Now if $T$ lies in a facet $F$ of conv $P$, repeating the argument in Case 1 of Theorem 5 shows that Condition (2) is satisfied.

In the remaining cases, assume that $T$ does not lie in a facet for conv $P$. Let $S \equiv T \sim \{x\}$:

*Case* 2. If $S$ is contained in a facet $F$ but conv $S$ is not a face for conv $P$, then by repeating the argument in Case 1 of Theorem 5, Condition (2) holds.

*Case* 3. Suppose $S$ is contained in a facet and conv $S$ is a face for conv $P$. Recall $T \equiv S \cup \{x\}$ is not a face, for we are assuming that $T$ does not lie in a facet. By Theorem 4, there is a primitive $A \cup B$ for $P$ with $A \subseteqq S \cup \{x\} \equiv T$ and $B \nsubseteq S \cup \{x\}$. Moreover, since conv $S$ is a face for conv $P$, a subset $C$ of $S$ is half a primitive $C \cup D$ for $P$ iff $D \subseteqq S$. This implies that $x$ must appear in $A$, for otherwise we would have $A \subseteqq S$ and $B \nsubseteq S$, a contradition. Thus $A \subseteqq T$, $B \nsubseteq T$, and $x$ appears, satisfying Condition (1).

*Case* 4. If $\operatorname{conv} S$ is not in a facet for $\operatorname{conv} P$ and $x$ is in aff $S$, then unfortunately it is necessary to consider subcases:

(4a) If dim aff $S = d$, then since $T \neq P$, there is some $y \in P \sim T$ and necessarily $y$ is in aff $S$. Let $T'$ be the vertex set for a $d$-dimensional simplex, $x \in T' \subseteqq T \equiv S \cup \{x\}$. Then $T' \cup \{y\}$ is a set having $d + 2$ points in $R^d$, so there is a primitive $A \cup B$ for $T' \cup \{y\}$. Certainly $y$ appears (since $T'$ is a simplex). Assume $y \in B$. Then $A \subseteqq T' \subseteqq T$, and $B \nsubseteq T$. Now if $|A| + |B| = d + 2$, then $x$ appears and Condition (1) holds. If $|A| + |B| \leqq d + 1$, then Condition (2) holds.

(4b) Similarly, if dim aff $S = k < d$ and if there is some $y$ in $(P \cap \text{aff } S) \sim T$, let $T'$ be the vertex set for a $k$-dimensional simplex, $x \in T' \subseteqq T$, and repeat the above proof.

(4c) If dim aff $S = k < d$ and if $(P \cap \text{aff } S) \sim T = \emptyset$, then select a point $y \in P \sim \text{aff } S$. (This is possible since $T \neq P$.) Again, let $T'$ be the vertex set for a $k$-dimensional simplex, $x$ in $T' \subseteqq T$.

Now we want to use our old friend, the corollary following Theorem 4, but first we must make a few adjustments.

Let $\operatorname{conv} R$ be a new polytope, where $R \equiv P \sim (\text{aff } T \sim T')$. We have thrown away the vertices in aff $T$ except for those in $T'$. Notice that $x$ remains. Also $y$ remains since $y \notin \text{aff } S = \text{aff } T$.

We assert that $T'$ does not lie in a face of $\operatorname{conv} R$: If $T'$ is in a face, then let the hyperplane $H$ support $\operatorname{conv} R$ with $T' \subseteqq H$. Then aff $T' \subseteqq H$. But aff $T' = \text{aff } T$, so aff $T \subseteqq H$, and $H$ supports conv $P \equiv \operatorname{conv}(R \cup T)$ with $T \subseteqq H$. But $T$ does not lie in a face of conv $P$ by hypothesis. We have a contradiction, and $T'$ does not lie in a face of $\operatorname{conv} R$.

We are ready for the corollary to Theorem 4. $T'$ does not lie in a face of $\operatorname{conv} R$, and $y$ is in $R \sim \text{aff } T'$. Thus there is a subset $T''$ of $T'$ which appears as half a Radon partition for a subset $Q$ of $R$, where $y \in Q$. Moreover, $Q$ may be chosen so that $Q \sim (T'' \cup \{y\})$ is a simplex and lies in a facet of $\operatorname{conv} R$. For any primitive $A \cup B$ inside $T'' \cup (Q \sim T'')$ with $A \subseteqq T''$, $y \in B$.

Now if $x$ is in $T''$, and if $x \in A$, then we have $A \subseteqq T$, $B \nsubseteq T$ (since $y \in B$), and $x$ appears in the primitive, satisfying Condition (1). If $x$ is in $T''$ but $x$ is not in $A$, then by our minimality condition of $T''$, no proper subset of $T''$ may cut $\operatorname{conv} R$, so conv $A$ cannot cut conv $R$, and likewise, conv $A$ cannot cut conv $Q$. Then conv $A$ must lie in some face of conv $Q$, and certainly conv $A \cap$ conv $B$ must lie in the boundary of conv $Q$. By Theorem 1, Corollary 1, necessarily $|A| + |B| \leqq d + 1$, satisfying Condition (2).

We still need to examine what happens in case $x$ does not appear in $T''$. Again by the corollary to Theorem 4, aff $T'' \cap$ conv $R =$ conv $T''$. Now conv $T'$ is a simplex, $T'' \subseteqq T'$, and $x \in T'$. If $x$ is not in $T''$, then $x \notin$ conv $T''$, and so $x \notin$ aff $T''$. By the very choice of $T''$, conv $T''$ cuts conv $R$, and so conv $T''$ does not lie in a face of conv $R$. Also $x \in R \sim$ aff $T''$, so there is a subset $T^{(3)}$ of $T''$ which is half a partition for a subset of $R$ (by the corollary). Let $C \cup D$ be a corresponding primitive. Then $C \subseteqq T^{(3)}$ and $x \in D$. Not all of $D$ can lie in $T'$, for if it did, we would have a primitive $C \cup D$ in the vertex set of the simplex $T'$, and this is ridiculous. Thus, $D \nsubseteqq T'$, but $D \subseteqq R$, and the only points of $T$ in $R$ are those in $T'$. Thus, $D \nsubseteqq T$. To review, $C \subseteqq T, D \nsubseteqq T$, and $x$ appears in $D$, satisfying Condition (1), and completing Case 4c.

*Case* 5. If $S$ is not in a face and $x$ is not in aff $S$, then as in Case 4c, reduce conv $P$ to a new polytope conv $R$, where $R \equiv P \sim$ (aff $S \sim S'$), and where $S'$ is the vertex set for a $k$-dimensional simplex with $k =$ dim aff $S$. By our earlier argument, $S'$ does not lie in a face of conv $R$. Also, $x \in R$ and $x \notin$ aff $S'$. Then by the corollary to Theorem 4, a subset $S''$ of $S'$ appears as half a partition for a subset $Q$ of $R$. Let $A \cup B$ be a corresponding primitive. Then by the corollary, $A \subseteqq S''$ and $x \in B$. Moreover, $B \nsubseteqq T \equiv S \cup \{x\}$, for if $B \subseteqq T$, we would have $A \subseteqq S'$, $B \subseteqq T \cap Q \equiv S' \cup \{x\}$. But $S'$ determines a simplex and $x \notin$ aff $S'$, so $S' \cup \{x\}$ determines a simplex and has no primitives. Thus $A \subseteqq T, B \nsubseteqq T$, and $x$ appears in $B$, satisfying Condition (1) and finishing Case 5.

This completes the proof of Theorem 6.

At last we have obtained a reduction in the number of partitions necessary to reconstruct an arbitrary polytope. Combining Theorems 5 and 6, we have the following corollaries:

COROLLARY 1. *The combinatorial type of conv* $P$ *is determined by* $\mathscr{C}_x$ *for any* $x \in P$.

COROLLARY 2. *For* $P$ *in general position and* $x \in P$, *the combinatorial type of conv* $P$ *is determined by the primitive partitions for* $P$ *which contain* $x$.

**5. Locating points.** Another approach to the problem of obtaining a minimal collection of primitive partitions which determine conv $P$ leads to the method of reconstructing a polytope by locating vertices, one at a time.

DEFINITION. Let $P \cup \{x\}$ be the vertex set for a polytope in $R^d$ and assume that we have reconstructed conv $P$. We say that we *locate* $x$ relative to conv $P$ iff we are able to reconstruct conv $(P \cup \{x\})$.

DEFINITION. Let $P$ be the vertex set for a polytope in $R^d$ and let $x$ be a point not in $P$. For $F$ a facet of conv $P$, we say $x$ is *beyond* $F$ iff $x$ is in the open halfspace of $H_F$ not containing $P$ (where $H_F$ is the hyperplane determined by $F$). For $E$ a face of conv $P$, we say $x$ is *beyond* $E$ iff $x$ is beyond $F$ for every facet $F$ containing $E$.

To reconstruct conv $P$ by locating vertices, one at a time, first select a $(d+1)$-subset $S$ of $P$ for which there is no primitive. (Clearly $S$ determines a simplex.) The following theorem describes the procedure for locating additional points.

THEOREM 7. *Let $P \cup \{x\}$ be the vertex set for a polytope, and assume that we have reconstructed conv $P$. Then to reconstruct conv $(P \cup \{x\})$, it is sufficient to consider the primitives $A \cup B$ for $P \cup \{x\}$ such that $A$ lies in a face of conv $P$, $x \in B$, and $x$ opposes no proper subset of $A$ in a primitive.*

*Proof.* Using Theorem 5.2.1 of Grünbaum [1], we see that to establish the faces for conv $(P \cup \{x\})$, it suffices to examine the faces for conv $P$.

For $S \subseteqq P$ and conv $S$ a face for conv $P$, $S$ determines a face for conv $(P \cup \{x\})$ iff no subset $A$ of $S$ appears as half a primitive $A \cup B$ with $x$ in $B$. Also, $S \cup \{x\}$ determines a face for conv $(P \cup \{x\})$ iff for every primitive $A \cup B$ with $A \subseteqq S$ and $x$ in $B$, then $B \subseteqq S \cup \{x\}$.

However, if there is one primitive $A_0 \cup B_0$ with $A_0 \subseteqq S$, $x \in B_0$, and $B_0 \subseteqq S \cup \{x\}$, then by general position of the points involved, $x \in \operatorname{aff} S$, $x$ lies in every face containing $S$, and $S \cup \{x\}$ determines a face for conv $(P \cup \{x\})$. Therefore, if one primitive with $A_0 \subseteqq S$ and $x$ in $B_0$ satisfies $B_0 \subseteqq S \cup \{x\}$, then every primitive with $A \subseteqq S$ and $x$ in $B$ satisfies $B \subseteqq S \cup \{x\}$, and it is easy to determine all faces of conv $(P \cup \{x\})$ from those listed.

As the following example illustrates, the construction in Theorem 7 allows us to locate $x$ relative to conv $P$ but does not allow us to locate $x$ relative to conv $Q$, where $Q \subsetneqq P$.

EXAMPLE 2. Let $\{1, 2\} \cup \{3, 4, 5\}$ be the primitive partition for the set $P = \{1, 2, 3, 4, 5\}$ in $R^3$, and let 6 lie beyond the face conv $\{1, 4, 5\}$. This does not determine the location of 6 relative conv $Q$, $Q = \{1, 2, 3, 4\}$, for 6 may or may not lie beyond the edge $[1, 2]$ of conv $Q$.

REMARK. It is easy to find examples for which the subcollection of primitive partitions described in Theorem 7 is minimal. Moreover, at each stage of the construction at least one primitive is required to locate an additional vertex. Thus at least $n - (d + 1)$ primitive partitions are needed to reconstruct conv $P$. This lower bound is always attained for simplicial polytopes having $d + 2$ vertices.

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CLEMSON UNIVERSITY
AND
UNIVERSITY OF OKLAHOMA

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# DERIVED ALGEBRAS IN $L_1$ OF A COMPACT GROUP

DAVID S. BROWDER

**Let $G$ be a compact topological group. In this paper, it is shown that the derived algebra $D_p$ of $L_p(G)$ (for $1 \leqq p < \infty$) is contained in the ideal $S_p$ of functions in $L_p(G)$ with unconditionally convergent Fourier series. It is also noted that this inclusion can be strict if $G$ is nonabelian. Finally, it is shown that the derived algebra of the center of $L_p(G)$ is always equal to the center of $S_p$, generalizing a known result that $D_p = S_p$ when $G$ is compact and abelian.**

In general, let $(A, \| \ \|_A)$ be a Banach algebra which is an essential left Banach $L_1(G)$-module in $L_1(G)$ under convolution. For convenience and with no loss of generality it is assumed that

$$\| f \|_A \geqq \| f \|_1 \qquad \text{for every } f \in A .$$

This paper investigates the relationship between the derived algebra of $A$ and the ideal in $A$ of functions with unconditionally convergent Fourier series. Bachelis has shown in [1] that in case $G$ is abelian and $A$ is equal to $L_p(G)$, for $1 \leqq p < \infty$, the two algebras coincide.

Bachelis' result is generalized to the derived algebra of the center of $L_p(G)$ and it is shown that for the compact group $\mathscr{S}_3^\infty$ and $A = L_p(\mathscr{S}_3^\infty)$ with $p \neq 2$, the derived algebra is strictly contained in the ideal of functions in $L_p(\mathscr{S}_3^\infty)$ whose Fourier series converge unconditionally.

Notation throughout will be as in [4]. $\Sigma$ will denote the dual object of $G$, the set of equivalence classes of continuous irreducible unitary representations of $G$. For each $\sigma \in \Sigma$, $H_\sigma$ will denote the representation space of $\sigma$ (of finite dimension $d_\sigma$) and $\mathscr{E}(\Sigma)$ will denote the product space $\prod_{\sigma \in \Sigma} B(H_\sigma)$. Important subspaces of $\mathscr{E}(\Sigma)$ referred to in the text include:

(i) $\mathscr{E}_0(\Sigma) = \{E = \{E_\sigma\}: \| E_\sigma \|_{op}$ is small off finite sets$\}$

(ii) $\mathscr{E}_1(\Sigma) = \{E = \{E_\sigma\}: \| E \|_1 = \sum_{\sigma \in \Sigma} d_\sigma \| E_\sigma \|_{\phi_1} < \infty\}$

(iii) $\mathscr{E}_2(\Sigma) = \{E = \{E_\sigma\}: \| E \|_2^2 = \sum_{\sigma \in \Sigma} d_\sigma \| E_\sigma \|_{\phi_2}^2 < \infty\}$.

For $f \in L_1(G)$, $f$ has Fourier series $f \sim \sum_{\sigma \in \Sigma} d_\sigma \, tr(A_\sigma U^{(\sigma)})$ where $A_\sigma \in B(H_\sigma)$, $U^{(\sigma)} \in \sigma$. The Fourier transform $\hat{f}$ of $f$ has the property that $\hat{f}(\sigma) = A_\sigma^t$ and hence:

$$\| \hat{f} \|_\infty = \sup_{\sigma \in \Sigma} \| A_\sigma \|_{op} .$$

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**1. The derived algebra.** We begin by defining the derived algebra $D_A$ for an essential left Banach $L_1(G)$-module $A$, and noting a few of its properties.

DEFINITION 1.1. If $f \in A$, we define

$$\|f\|_{D_A} = \sup_{g \in A} \frac{\|f * g\|_A}{\|\hat{g}\|_\infty}$$

and let

$$D_A = \{f \in A : \|f\|_{D_A} < \infty\}\ .$$

The following facts are easy to check.

PROPOSITION 1.2. (i) *$(D_A, \|\ \|_{D_A})$ is a Banach algebra and a left Banach $L_1(G)$-module in $L_1(G)$ under convolution.*

(ii) *$\|f\|_A \leqq \|f\|_{D_A}$ for every $f \in A$.*

(iii) *If we denote the set of trigonometric polynomials by $T(G)$ then we have*

$$\|f\|_{D_A} = \sup_{g \in T(G)} \frac{\|f * g\|_A}{\|\hat{g}\|_\infty} \qquad \textit{for every } f \in A\ .$$

We next give a characterization of $D_A$ which is due essentially to Helgason ([3], Theorem 2).

THEOREM 1.3. (Helgason)

$$D_A = \{f \in A : \hat{f}E \in \hat{A},\ \textit{for every } E \in \mathscr{E}_0(\Sigma)\}\ .$$

*Proof.* Suppose $f \in A$ and that for $E \in \mathscr{E}_0(\Sigma), \hat{f}E = \hat{g}_E$ for some $g_E \in A$. Then the linear map $E \to g_E$ of $\mathscr{E}_0(\Sigma)$ into $A$ has closed graph and is therefore continuous. In particular, there exists a constant $k > 0$ such that

$$\|f * h\|_A \leqq k\|\hat{h}\|_\infty \qquad \text{for every } h \in A\ .$$

Consequently, $f$ belongs to $D_A$.

Conversely, if $f \in D_A$ then the continuous map $\hat{g} \to f * g$ of $\hat{A}$ into $A$ extends to a continuous map $E \to h_E$ of $\mathscr{E}_0(\Sigma)$ into $A$. Then the element $\hat{f}E = \hat{h}_E$ belongs to $\hat{A}$ for every $E \in \mathscr{E}_0(\Sigma)$.

This characterization of $D_A$ gives two more properties of $D_A$.

COROLLARY 1.4. *(i) $D_A$ is an ideal in $L_1(G)$ and*
*(ii) $\hat{D}_A$ is a right ideal in $\mathscr{C}_0(\Sigma)$.*

We denote by $C(G)$ the algebra of continuous complex valued functions on $G$, and by $K(G)$ the algebra of functions on $G$ with absolutely convergent Fourier series (see [4], Sect. 34).

For $1 \leqq p < \infty$, the derived algebra of $L_p(G)$ is denoted by $D_p$.

EXAMPLES 1.5. *(i) $D_{K(G)} = K(G)$,*
*(ii) $D_{C(G)} = K(G)$, and*
*(iii) $D_p = L_2(G)$ for $1 \leqq p \leqq 2$.*

*Proof.* First we show (i). Let $f$ belong to $K(G)$ and $g$ to $T(G)$. Then $\|f * g\|_K = \|\hat{f}\hat{g}\|_1 \leqq \|\hat{f}\|_1 \|\hat{g}\|_\infty = \|f\|_K \|\hat{g}\|_\infty$. Hence, by (1.2), $f$ belongs to $D_{K(G)}$.

To see (ii), observe that since $\|\ \|_u \leqq \|\ \|_{K(G)}$ on $K(G)$, it follows that $K(G) = D_{K(G)} \subset D_{C(G)}$. Conversely, let $f \in D_{C(G)}$ with Fourier series given by

$$f \sim \sum_{\sigma \in \Sigma} d_\sigma tr(A_\sigma U^{(\sigma)}) .$$

For each $\sigma \in \Sigma$, let $V_\sigma$ be the unitary matrix such that $V_\sigma A_\sigma = |A_\sigma|$. For $F \subset \Sigma$, a finite set, define:

$$g = \sum_{\sigma \in F} d_\sigma tr(V_\sigma U^{(\sigma)}) .$$

Then $g \in T(G)$, $\|\hat{g}\|_\infty = 1$ and we have:

$$\sum_{\sigma \in F} d_\sigma \|A_\sigma\|_{\phi_1} = \sum_{\sigma \in F} d_\sigma tr|A_\sigma| = |f * g(e)| \leqq \|f * g\|_u \leqq \|f\|_{D_C} .$$

Hence $\|f\|_{K(G)} \leqq \|f\|_{D_{C(G)}}$ and $f \in K(G)$.

To prove (iii), we use the facts (see [4], 36.10, 36.12) that $D_1 = L_2(G)$ and

$$2^{-1/2}\|f\|_2 \leqq \|f\|_{D_1} \leqq \|f\|_2 \qquad \text{for every } f \in L_2(G) .$$

It $1 < p \leqq 2$ and $f \in L_2(G)$, then for $g \in T(G)$ we see that

$$\|f * g\|_p \leqq \|f * g\|_2 = \|\hat{f}\hat{g}\|_2 \leqq \|\hat{f}\|_2 \|\hat{g}\|_\infty = \|f\|_2 \|\hat{g}\|_\infty .$$

Hence, we conclude that $\|f\|_{D_p} \leqq \|f\|_2$ and

$$\|f\|_{D_p} \geqq \|f\|_{D_1} \geqq 2^{-1/2}\|f\|_2 .$$

## 2. The ideal in $A$ of functions with unconditionally con-

vergent Fourier series. Let $\mathscr{F}$ denote the family of all nonvoid finite subsets of $\Sigma$. For $F \in \mathscr{F}$, let $D(F) = \sum_{\sigma \in F} d_\sigma \chi_\sigma$. For $f$ in $L_1(G)$, $f * D(F)$ is the finite partial sum of the Fourier series of $f$ consisting only of terms involving elements of $F$. We say that $f$ in $A$ has unconditionally convergent Fourier series in $A$ whenever

$$\lim_{F \in \mathscr{F}} \| f - f * D(F) \|_A = 0 .$$

We denote by $S_A$ the family of all functions in $A$ with this property. If we also define

$$\| f \|_{S_A} = \sup_{F \in \mathscr{F}} \| f * D(F) \|_A ,$$

then the following facts are easily verified.

PROPOSITION 2.1. (i) *If $f \in S_A$, then $\| f \|_{S_A} < \infty$.*

(ii) *$(S_A, \| \; \|_{S_A})$ is a Banach algebra.*

(iii) *$\| f \|_A \leqq \| f \|_{S_A}$ for every $f \in A$.*

(iv) *If $f \in S_A$, then $\lim_{F \in \mathscr{F}} \| f - f * D(F) \|_{S_A} = 0$.*

(v) *$S_A$ is an essential left Banach $L_1(G)$-module in $L_1(G)$ under convolution.*

Since $S_A$ satisfies the conditions we have postulated for $A$, we may compute its derived algebra.

THEOREM 2.2. (i) *$D_{S_A} = D_A \cap S_A$ and $\| f \|_{D_{S_A}} = \| f \|_{D_A}$ for $f \in D_{S_A}$.*

(ii) *$S_{S_A} = S_A$ (isometry).*

*Proof.* Suppose $f$ belongs to $D_{S_A}$. Then for $f \in S_A$ and $g \in T(G)$ we have

$$\frac{\| f * g \|_A}{\| \hat{g} \|_\infty} \leqq \frac{\| f * g \|_{S_A}}{\| \hat{g} \|_\infty} \leqq \| f \|_{D_{S_A}} .$$

Hence we have $\| f \|_{D_A} \leqq \| f \|_{D_{S_A}} < \infty$, and thus $f$ belongs to $D_A \cap S_A$.

Conversely, if $f \in D_A \cap S_A$ then for $g \in T(G)$ and $F \in \mathscr{F}$, we have

$$\frac{\| f * g * D(F) \|_A}{\| \hat{g} \|_\infty} \leqq \frac{\| f * g * D(F) \|_A}{g * D(F)} \leqq \| f \|_{D_A} .$$

Thus it follows that $\| f \|_{D_{S_A}} \leqq \| f \|_{D_A} < \infty$, and $f$ belongs to $D_{S_A}$.

Part (ii) follows immediately from (2.1, iv).

**3. Central derived algebras.** Let $A^z$ denote the center of $A$. Then $A^z = L_1^z(G) \cap A$ and $(A^z, \| \; \|_A)$ is an essential Banach $L_1^z$-module in $L_1^z(G)$ under convolution. Before we investigate the derived algebra of $A^z$, we prove a useful proposition.

PROPOSITION 3.1. *For $E \in \mathscr{E}_\infty(\Sigma)$, define a function $\varphi_E$ on $\Sigma$ by: $\varphi_E(\sigma) = 1/d_\sigma\, tr(E_\sigma)$ for every $\sigma \in \Sigma$. The map $E \to \varphi_E$ is an isometric isomorphism of*

(i) $\mathscr{E}_\infty^z(\Sigma)$ *onto* $l_\infty(\Sigma)$,

(ii) $\mathscr{E}_0^z(\Sigma)$ *onto* $c_0(\Sigma)$, *and*

(iii) $\mathscr{E}_{00}^z(\Sigma)$ *onto* $c_{00}(\Sigma)$.

*For $f \in L_1^z(G)$, let $\overset{\circ}{f}(\sigma) = 1/d_\sigma\, tr(\hat{f}(\sigma)) = \varphi_{\hat{f}}(\sigma)$, so that $f$ has Fourier series $\sum_{\sigma\in\Sigma} d_\sigma \overset{\circ}{f}(\sigma)\chi_\sigma$. Then the map $f \to \overset{\circ}{f}$ is the Gel'fand transform $A^z$, $\Sigma$ is the maximal ideal space of $A^z$, and*

(iv) $\|\overset{\circ}{f}\|_\infty = \|\hat{f}\|_\infty$ *for every* $f \in L_1^z(G)$.

*Proof.* Let $E$ belong to $\mathscr{E}_\infty^z(\Sigma)$. By Schur's lemma we have

$$(1) \qquad E_\sigma = \varphi_E(\sigma) I_{d_\sigma} \qquad \text{for } \sigma \in \Sigma .$$

It follows that

$$(2) \qquad \|E\|_\infty = \|\varphi_E\|_\infty .$$

Clearly the map $E \to \varphi_E$ is linear and carries $\mathscr{E}_\infty^z(\Sigma)$ isometrically onto $l_\infty(\Sigma)$. By (1), $E \to \varphi_E$ is multiplicative. By (2), the image of $\mathscr{E}_0^z(\Sigma)$ is $c_0(\Sigma)$, and the image of $\mathscr{E}_{00}^z(\Sigma)$ is $c_{00}(\Sigma)$. The rest of the proof is analogous to ([4], 28.71).

DEFINITION 3.2. For $f$ in $A^z$, let

$$\|f\|_{\mathscr{D}_A} = \sup_{g\in A^z} \frac{\|f * g\|_A}{\|\overset{\circ}{g}\|_\infty} .$$

The derived algebra $\mathscr{D}_A$ of $A^z$ is defined as

$$\mathscr{D}_A = \{f \in A^z\colon \|f\|_{\mathscr{D}_A} < \infty\} .$$

The following properties of $\mathscr{D}_A$ are easily proved.

PROPOSITION 3.3. (i) *$(\mathscr{D}_A, \|\ \|_{\mathscr{D}_A})$ is a Banach algebra and an $L_1^z(G)$-module under convolution.*

(ii) $\|f\|_A \leqq \|f\|_{\mathscr{D}_A}$ *for every* $f \in A^z$.

(iii) $\|f\|_{\mathscr{D}_A} = \sup_{g\in T^z(G)} \|f * g\|_A / \|\overset{\circ}{g}\|_\infty$ *for every* $f \in A^z$.

(iv) $D_A^z \subset \mathscr{D}_A$.

Helgason's characterization (1.3) has an analogue in the central case. We omit the proof since it is exactly like that of (1.3).

THEOREM 3.4. (Helgason)

$$\mathscr{D}_A = \{f \in A^z : \overset{\circ}{f}\varphi \in (A^z)^\circ \ \textit{for every} \ \varphi \in c_0(\Sigma)\} .$$

We next prove that the center $S_A^z$ of $S_A$ is always contained in $\mathscr{D}_A$. To do so, we use the following well known fact which follows from a theorem of Seever ([6]).

FACT 3.5. *Let $X$ be a discrete topological space and $M$ a Banach space. If $T: M \to l_\infty(X)$ is a bounded linear map whose image contains the characteristic function of every subset of $X$, then $T$ is onto.*

We also use the following lemma which states that every element of $l_\infty(\Sigma)$ is a multiplier for $S_A^z$.

LEMMA 3.6. *If $f \in S_A^z$ and $\varphi \in l_\infty(\Sigma)$, then there exists $g \in S_A^z$ such that $\overset{\circ}{g} = \varphi\overset{\circ}{f}$.*

*Proof.* Let $f$ belong to $S_A^z$, and denote by $M$ the collection of all $\varphi \in l_\infty(\Sigma)$ such that $\varphi\overset{\circ}{f} \in (S_A^z)^0$. Then $M$ is a Banach space under the norm

$$\| \varphi \| = \| \varphi \|_\infty + \| g \|_{S_A} \ \text{where} \ \overset{\circ}{g} = \varphi\overset{\circ}{f} .$$

To show $M = l_\infty(\Sigma)$, it suffices by (3.5) to show that for $\Delta \subset \Sigma$, the characteristic function $\varphi$ of $\Delta$ is an element of $M$. To establish this, we note that the net $\{f * D(E): E^{\text{finite}} \subset \Delta\}$ is Cauchy in $S_A^z$, so there is a function $g$ in $S_A^z$ such that

$$\lim_{E^{\text{finite}} \subset \Delta} \| g - f * D(E) \|_{S_A} = 0 .$$

We conclude that $\overset{\circ}{g} = \varphi\overset{\circ}{f}$ and hence, $\varphi$ belongs to $M$.

THEOREM 3.7. $S_A^z \subset \mathscr{D}_A$.

*Proof.* Suppose $f$ belongs to $s_A^z$. Then for $\varphi \in c_0(\Sigma) \subset l_\infty(\Sigma)$, $\varphi\overset{\circ}{f}$ belongs to $(S_A^z)^0$ and hence to $(A^z)^0$ by (3.6). Therefore $f \in \mathscr{D}_A$ by (3.4).

We now restrict our attention to the case of $A = L_p(G)$ for $1 \leqq p < \infty$. As before we write $D_A = D_p$; we also write $S_A = S_p$ and $\mathscr{D}_A = \mathscr{D}_p$. To compare $D_p$ and $S_p$ we use the following.

LEMMA 3.8. *Let* $1 \leqq p < \infty$. *If* $f \in L_p(G)$ *and* $\|f\|_{S_p} < \infty$, *then* $f \in S_p$.

*Proof.* Let $f$ belong to $L_p(G)$ with $\|f\|_{S_p} < \infty$. Suppose $f$ has Fourier series

$$f \sim \sum_{j=1}^{\infty} d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)}) .$$

For $\varphi \in L_p(G)^*$ and any nonvoid finite $F \subset Z^+$, we have

$$\left| \sum_{j \in F} \varphi(d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)})) \right| \leqq \|f\|_{S_p} \|\varphi\|_{op} .$$

Hence, we see

$$\sup_{F \text{finite} \subset Z^+} \left| \sum_{j \in F} \varphi(d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)})) \right| < \infty ,$$

which implies

$$\sum_{j=1}^{\infty} | \varphi(d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)})) | < \infty .$$

Thus the Fourier series of $f$ is weakly subseries Cauchy and, since $L_p(G)$ is weakly complete, the series is weakly subseries convergent. Therefore, by the Orlicz-Pettis theorem ([2], p. 60, or [6], p. 19) it is norm convergent and unconditionally convergent to some $g \in L_p(G)$. Comparing transforms, we see that $f = g$ and consequently, $f$ belongs to $S_p$.

Finally, we state the main result of this section, generalizing the abelian result of Bachelis.

THEOREM 3.9. *Let* $1 \leqq p < \infty$. *Then we have*

(i) $D_p \subset S_p$, *and*

(ii) $\mathscr{D}_p = S_p^z$.

*Proof.* Observe that $\|f\|_{S_p} \leqq \|f\|_{D_p}$ for every $f \in D_p$, and that $\|f\|_{S_p} \leqq \|f\|_{\mathscr{D}_p}$ for every $f \in \mathscr{D}_p$. The theorem now follows from (3.8).

**4. $\mathscr{S}_3^\infty$ as a source of examples.** Throughout this section $G$ will denote $\mathscr{S}_3^\infty = \prod_{\aleph_0} \mathscr{S}_3$, where $\mathscr{S}_3$ is the symmetric group on three symbols. Using this group we demonstrate that Bachelis' result does not extend to the non-abelian case.

THEOREM 4.1. *Let* $G = \mathscr{S}_3^\infty$ *and* $1 \leqq p < \infty$. *Then*

(i) $D_p = S_p$ *if and only if* $p = 2$, *and*

(ii) $D_p = L_p$ *if and only* $p = 2$.

*Proof.* By (1.5, iii) and (3.9), we have

$$L_2(G) = D_2 \subset S_2 \subset L_2(G) .$$

Suppose $p \neq 2$. Observe that (ii) follows from (i) because

$$D_p \subset S_p \subset L_p .$$

Note also that $\|f\|_{S_p} \leqq \|f\|_{D_p}$ for every $f \in D_p$. Hence to prove that $D_p \neq S_p$ it is enough to find sequences $\{f^{(n)}\}$ in $D_p$ and $\{g^{(n)}\}$ in $T(G)$ such that

$$(1) \qquad \frac{\|f^{(n)} * g^{(n)}\|_p}{\|\widehat{g^{(n)}}\|_\infty \|f^{(n)}\|_{S_p}} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty .$$

We select these sequences as follows. Let $\sigma$ be the representation class on $\mathscr{S}_3$ of dimension 2 (see [4], 27.61). For $f$ and $g$ in $T_\sigma(\mathscr{S}_3)$ which will be specified later, form

$$f^{(n)}(\underline{\underline{x}}) = \prod_{k=1}^{n} f(x_k)$$

and

$$g^{(n)}(\underline{\underline{x}}) = \prod_{k=1}^{n} g(x_k) ,$$

where $\underline{\underline{x}} \in G$ is given by $\underline{\underline{x}} = (x_1, x_2, \cdots)$. Then $f^{(n)}$ and $g^{(n)}$ are elements of $T_{\sigma^{(n)}}(G)$ where $\sigma^{(n)}$ is the element of $\Sigma_G$ given by

$$U_{\underline{\underline{x}}}^{(\sigma^{(n)})} = U_{x_1}^{(\sigma)} \otimes \cdots \otimes U_{x_n}^{(\sigma)} \quad \text{for every } \underline{\underline{x}} \in G .$$

It is easily verified that

$$\|f^{(n)}\|_{S_p} = \|f^{(n)}\|_p = \|f\|_p^n ,$$

$$\|f^{(n)} * g^{(n)}\|_p = \|f * g\|_p^n ,$$

and

$$\|\widehat{g^{(n)}}\|_\infty = \|\hat{g}\|_\infty^n .$$

Hence, to show (1) it suffices to find $f$ and $g$ in $T_\sigma(\mathscr{S}_3)$ such that

$$\frac{\|f * g\|_p}{\|\hat{g}\|_\infty \|f\|_p} > 1 .$$

Let $g = 2u_{11}^{(\sigma)} + 2iu_{22}^{(\sigma)}$ and note that $\|\hat{g}\|_\infty = 1$. The rest of the argument divides into two cases.

*Case* 1. $1 \leqq p < 2$. In this case we let $f = 2\chi_\sigma$ so that $f * g = g$, and we compute

$$\| f \|_p = 2\left[\frac{2^p + 2}{6}\right]^{1/p} \qquad \text{(see [4], 27.61)} .$$

Also, we have

$$\| g \|_p = 2\left[\frac{(1 + 2^{1-p})\, 2\sqrt{2}^{\,p}}{6}\right]^{1/p} ,$$

and therefore we conclude

$$\frac{\| f * g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = 2^{1/p-1/2} > 1 .$$

*Case* 2. $2 < p < \infty$. In this case we let $f = 2iu_{12}^{(\sigma)} + 2u_{21}^{(\sigma)}$. Then $f * g = -2u_{12}^{(\sigma)} + 2u_{21}^{(\sigma)}$ and so we have

$$\| f \|_p = \sqrt{6}\left(\frac{2}{3}\right)^{1/p} \quad \text{and} \quad \| f * g \|_p = 2\sqrt{3}\left(\frac{1}{3}\right)^{1/p} .$$

Therefore, we conclude

$$\frac{\| f * g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = 2^{1/2-1/p} > 1 .$$

The question naturally arises as to whether $D_A^z$ is equal to $\mathscr{D}_A$. The next example shows that in some cases the answer is no.

THEOREM 4.2. *If* $G = \mathscr{S}_3^\infty$ *and* $1 \leqq p < 4$, *then* $D_p^z = \mathscr{D}_p$ *if and only if* $p = 2$.

*Proof.* By (1.5, iii) and (3.3, iv) we have

$$D_2^z = \mathscr{D}_2 = L_2^z(G) .$$

Suppose $p \neq 2$. Since $D_p^z \subset \mathscr{D}_p$ and $\| \ \|_{\mathscr{D}_p} \leqq \| \ \|_{D_p}$ on $D_p^z$, to show that $D_p^z \neq \mathscr{D}_p$, it suffices to find sequences $\{f^{(n)}\}$ in $D_p^z$ and $\{g^{(n)}\}$ in $T(G)$ such that

$$\frac{\| f^{(n)} * g^{(n)} \|_p}{\| \widehat{g^{(n)}} \|_\infty \| f^{(n)} \|_{\mathscr{D}}} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty .$$

As in the proof of (4.1) we construct the sequences by choosing $f$ and $g$ on $\mathscr{S}_3$ as follows. First, let $f = 2\chi_\sigma$. Then $f * g = g$ for any $g \in T_\sigma(\mathscr{S}_3)$, and $\| f \|_p = 2\,[(2^p + 2)/6]^{1/p}$. Also we have $f^{(n)} = 2^n \chi_{\sigma^{(n)}}$ and

$\| f^{(n)} \|_{\mathscr{D}_p} = \| f^{(n)} \|_p = \| f \|_p^n$. As before, it suffices to find $g \in T_\sigma(\mathscr{S}_3)$ with the property that

$$\frac{\| g \|_p}{\| \hat{g} \|_\infty \| f \|_p} > 1 \ .$$

Again we consider two cases.

*Case* 1. $1 \leqq p < 2$. Let $g = 2u_{11}^{(\sigma)} + 2iu_{22}^{(\sigma)}$. Then as in (4.1), Case 1, we have

$$\frac{\| g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = 2^{1/p-1/2} > 1 \ .$$

*Case* 2. $2 < p < 4$. Let $g = 2u_{12}^{(\sigma)} + 2u_{21}^{(\sigma)}$. Then $\| \hat{g} \|_\infty = 1$ and

$$\| g \|_p = 2\left[\frac{2\sqrt{3}^p}{6}\right]^{1/p} \ .$$

Therefore we see

$$\frac{\| g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = \left[\frac{2 \cdot 3^{p/2}}{2^p + 2}\right]^{1/p} > 1 \ .$$

Finally, we observe that for $G = \mathscr{S}_3^\infty$ we have the following.

THEOREM 4.3. $K(G) \subsetneqq S_{C(G)}$.

*Proof*. Since $\| f \|_u \leqq \| f \|_{K(G)}$ for $f$ is $K(G)$, it follows that

$$K(G) = S_{K(G)} \subset S_{C(G)} \ .$$

Also, since $\| f \|_{S_{C(G)}} \leqq \| f \|_{K(G)}$ for $f$ in $K(G)$, to show that $K(G) \neq S_{C(G)}$, we need only find $f \in T_\sigma(\mathscr{S}_3)$ such that

$$\frac{\| f \|_{K(\mathscr{S}_3)}}{\| f \|_\infty} > 1 \ .$$

If we let $f = u_{12}^{(\sigma)} + u_{21}^{(\sigma)}$, then we have $\| f \|_\infty = \sqrt{3}$ and $\| f \|_{K(\mathscr{S}_3)} = 2$. Hence, the proof is complete.

The techniques used to prove (4.1) – (4.3) can also be applied to show the following.

THEOREM 4.4. *If* $G = \mathscr{S}_3^\infty$ *and* $1 \leqq p < \infty$, *then*

$$\mathscr{D}_p(G) = L_p^z(G) \text{ if and only if } p = 2 \ .$$

## 5. Open questions.

(5.1) Is $T(G)$ dense in $D_A$? If so, then it can easily be shown that $D_{D_A}$ is isometrically isomorphic to $D_A$. One easily shows that the density of $T(G)$ is equivalent to the condition that $S_{D_A} = D_A$.

(5.2) Another question of interest is whether or not $D_A$ is self-adjoint (that is, closed under $f \to \tilde{f}$, where $\tilde{f}(x) = \overline{f(x^{-1})}$) whenever $A$ is. Equivalently, is $\hat{D}_A$ a left ideal in $\mathscr{E}_0(\Sigma)$ when $A$ is self-adjoint?

(5.3) Are there any conditions on a compact non-abelian group $G$ sufficient to imply that $D_p = S_p$ for $p \neq 2$?

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UNIVERSITY OF OREGON
AND
SIMMON COLLEGE, BOSTON, MASS.

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# UNIMBEDDABLE NETS OF SMALL DEFICIENCY

A. BRUEN

**We construct some new geometrical examples of unimbeddable nets $N$ of order $p^2$ with $p$ an odd prime. The deficiency of $N$ is $p-j$ where either $j=0$ or $j=1$. In particular, the examples show that a bound of Bruck is best possible for nets of order 9,25. Our proof also shows that deriving a translation plane of order $p^2$ is equivalent to reversing a regulus in the corresponding spread.**

2. **Background, summary.** Let $N$ be a net of order $n$, degree $k$ so that $N$ has deficiency $d = n + 1 - k$. Let the polynomial $f(x)$ be given by $f(x) = x/2[x^3 + 3 + 2x(x + 1)]$. The following result is shown in [1].

THEOREM 1 (Bruck). *Suppose $N$ is a finite net of order $n$, deficiency $d$. Then $N$ is embeddable in an affine plane of order $n$ provided $n > f(d-1)$.*

Thus a net of small deficiency is embeddable. However, as is pointed out in [1], little is known concerning the bound above. It is our purpose here to remedy this. In Theorem 2 we describe a construction used in [2] to obtain maximal partial spreads $W$ of $PG(3, q)$. $W$ yields a net $N$ of order $q^2$ and deficiency $q - j$ where either $j = 0$ or $j = 1$. Our main result is that $N$ is not embeddable if $q = p$ is an odd prime. This will show that Bruck's bound is best possible for nets of order 9,25 and is fairly good, if not best possible, for other nets of order $p^2$.

3. **The construction.** For definitions and proofs of Theorems 2, 3 we refer to [2].

THEOREM 2. *Let $S$ be a spread of $\Sigma = PG(3, q)$ with $q \geqq 3$, such that $S$ is not regular. Let $u$ be a line of $\Sigma$ with $u$ not in $S$, such that the $q+1$ lines $A$ of $S$ passing through the $q+1$ points of $u$ do not form a regulus. Let $W_1$ be the partial spread of $\Sigma$ which is got by removing $A$ from $S$ and adjoining $u$: in symbols $W_1 = H \cup \{u\}$ where $H = S - A$. Then there exists a maximal partial spread $W$ of $\Sigma$ which contains $W_1$. Furthermore, either*

(i) *$W = W_1$ so that $|W| = q^2 - q + 1$ or*

(ii) *$W = W_1 \cup \{v\}$ where $v$ is a line of $\Sigma$ which is skew to each line of $W_1$. In this case $|W| = q^2 - q + 2$.*

THEOREM 3. *For any (prime power) $q \geqq 3$ there exist examples of* *case* (i). *For any odd* $q$ *with* $q \geqq 5$ *there exist examples of case* (ii).

We can think of $\Sigma$ in terms of a 4-dimensional vector space $V = V_4(q)$ over $GF(q)$. The points and lines of $\Sigma$ are precisely the 1-dimensional and the 2-dimensional subspaces of $V$ respectively. The lines or *components* of $W$ in $\Sigma$ correspond to the components of a maximal partial spread $W$ of $V$, that is, a maximal collection $W$ of 2-dimensional subspaces of $V$ such that any 2 distinct members (components) of $W$ have only the origin of $V$ in common. For a proof of the next result see [7, p. 8], [4, p. 219].

THEOREM 4. *Let* $U$ *be a partial spread of* $V = V_4(q)$ *having exactly* $k$ *components. Then there is defined a net* $N = N(U)$ *of order* $q^2$ *and degree* $k$. *The points of* $N$ *are the* $q^4$ *vectors in* $V$. *The lines of* $N$ *are the components of* $U$ *and their translates (cosets) in* $V$. *Furthermore, if* $U$ *is a spread of* $V$, *then* $N(U)$ *is a translation plane.*

Our main result is that if $W$ is the maximal partial spread of Theorem 2 and $q$ is an odd prime, then $N(W)$ is not embeddable.

## 4. The main result.

In what follows, if $J$ is a set of vectors, then $\{J\}$ will denote the subspace spanned by the vectors in $J$.

LEMMA 5. *Let* $\Sigma = PG(3, q)$ *and let* $(V, +) = V_4(q)$ *be the corresponding vector space. Let* $a, b, c$ *be* 3 *distinct and pairwise skew lines of* $\Sigma$. *Then we may choose a basis* $e_1, e_2, e_3, e_4$ *of* $V$ *in such a manner that* $a$ *corresponds to* $\{e_1, e_2\}$, $b$ *corresponds to* $\{e_3, e_4\}$ *and* $c$ *corresponds to* $\{e_1 + e_3, e_2 + e_4\}$.

The following is crucial in our argument.

THEOREM 6. *Let* $n$ *be a square and let* $N$ *be a net of order* $n$ *and deficiency* $\sqrt{n} + 1$, *which is embedded in an affine translation plane* $\pi$. *Suppose further that* $N$ *is embedded in another affine plane* $\pi_1$. *Then* $\pi_1$ *is also an affine translation plane.*

*Proof.* $\pi_1$ is related to $\pi$ by Ostrom's technique of derivation (see [2, p. 383] and [6, p. 1382]). From this the result will follow, for it is easy to show that a plane $\pi_1$ obtained by deriving a translation plane $\pi$ is itself a translation plane [4, p. 224].

We revert to the notation of Theorem 2. Recall that $W$ is a maximal partial spread of $\Sigma = PG(3, q)$ with $q \geqq 3$. $W = H \cup \{u, v\}$ where (sometimes) $u = v$. $H$ is a partial spread contained in the nonregular spread $S$ of $\Sigma$. $H$ contains exactly $q^2 - q$ lines. Since

$q \geqq 3$ we have $|H| = q^2 - q > 3$. Thus $H$ contains 3 pairwise skew lines $a, b, c$ which we will refer to as the *fundamental components*. Corresponding to $\Sigma$ we have $V = V_4(q)$. As in Lemma 5 we have a basis $e_1, e_2, e_3, e_4$ of $V$ with $a = \{e_1, e_2\}$, $b = \{e_3, e_4\}$, $c = \{e_1 + e_3, e_2 + e_4\}$. Let $L = \{e_1, e_2\}$ and $M = \{e_3, e_4\}$. We can write $V = L \oplus M$ the direct sum of $L$ and $M$. Each vector in $V$ is uniquely expressible as an ordered pair $(x, y)$ with $x$ in $L$, $y$ in $M$. The fundamental components are then sets $y = 0$, $x = 0$, $y = x$ respectively. In the sequel it will be convenient to identify $M$ with $L$ and write $V = L \oplus L$. We also let 0 denote the null vector in $L$, so that $(0, 0)$ is the null vector of $V$.

THEOREM 7 (Main Result). *Let $W$ be the maximal partial spread of $PG(3, q)$ constructed in Theorem 2. Assume that $q = p \geqq 3$ is a prime. Then the net $N = N(W)$ obtained from $W$ as in Theorem 4 has order $p^2$ and deficiency $p - j$ where either $j = 0$ or $j = 1$. Moreover, $N$ is not embeddable in a plane.*

*Proof.* By way of contradiction assume that $N$ is embeddable in an affine plane $\pi_1$. Choose the origin of $\pi_1$ to be the origin of $V$. In the construction of $W$ recall that $H \subset S$. Denote the translation plane obtained from $S$ by $\pi$. Thus $N(H) \subset \pi$. Also $N(H) \subset N(W) \subset \pi_1$. Therefore, by Theorem 6, $\pi_1$ is a translation plane. We may use the fundamental components $a, b, c$ to set up Hall coordinates for $\pi_1$ using the set $L$ (see [5]). Actually it is easy to see that a vector $\lambda$ has in $\pi_1$ Hall coordinates $(s, t)$ if and only if $\lambda$ has vector space coordinates $(s, t)$ in $V = L \oplus L$. Also the Hall addition is precisely the vector space addition $+$ on $L$ (see [7, p. 4]). Thus the translation plane $\pi_1$ is then coordinatized by a quasifield $Q = (L, +, \cdot)$. Those lines of $\pi_1$ through the origin which are also lines of $N = N(W)$ correspond to the components of $W$. Let $l$ be a line of $\pi_1$ through the origin of $\pi_1$ such that $l$ is not a line of $N$. Then $l$ consists of all points with coordinates of the form $(x, x \cdot m)$ for some $m$ in $L$. Since $Q$ is a quasifield we have $(x + y) \cdot m = x \cdot m + y \cdot m$. Therefore $l$ is a set of $p^2$ vectors in $V$ which is closed under addition. Since $p$ is a prime, $l$ is a 2-dimensional subspace of $V$. And $l$ has only the origin of $V$ in common with any component of $W$. Thus $l$ yields a line of $PG(3, q)$ which is skew to each line of $W$. But this is a contradiction, since $W$ is maximal.

*Comments.* 1. Our argument in Theorem 7 above can be modified to show the following. Let $\pi_1$ be obtained from the translation plane $\pi$ of order $p^2$ by deriving with respect to a derivation set $D$ of $p + 1$ points on the line at infinity. Then the $p + 1$ lines of $\pi$ joining the origin to $D$ yield a regulus in the spread corresponding to $\pi$. Thus, in this case, derivation implies reversing a regulus. It can be shown (see [2]) that reversing a regulus implies derivation for translation planes of order $q^2$, whether or not $q$ is a prime. Thus the procedures of derivation and reversing a regulus are equivalent for the case of translation planes of order $p^2$. However, as is proved in [3], they are not in general equivalent if $q$ is not a prime. The reason is that $l$ above is not always a subspace in this general case. So it is not clear whether or not $N$ is embeddable if $q$ is not a prime.

2. For $q = p$ we have shown that $N = N(W)$ is unimbeddable. However except for $p = 3, 5$ we do not know whether $N(W)$ is contained in a larger net or even whether there exists a transversal $T$ of $N$ (that is, a set of $p^2$ points of $N$ no two of which are joined by a line of $N$). However, it follows from the work in [2], [6] that $T$ would have to be an affine subplane of $\pi$ having order $p$.

3. For $p = 3$, $N(W)$ has deficiency 3 or 2. By Theorem 3.3 in [2], $N(W)$ must have deficiency 3. We have shown that $N(W)$ is not embeddable. It follows that $N(W)$ is not contained in any larger net, and that the bound in Theorem 1 is best possible for nets of order 9.

4. For $p = 5$ we can obtain an unimbeddable net $N = N(W)$ of deficiency 4 using Theorem 3. By Theorem 1, $N$ is not contained in a larger net and so Bruck's bound is also the best possible for nets of order 25. Another way of putting it is to say that *we have produced a maximal set of* 20 *mutually orthogonal latin squares of order* 25.

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UNIVERSITY OF WESTERN ONTARIO

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# UNICOHERENT COMPACTIFICATIONS

M. H. CLAPP AND R. F. DICKMAN, JR.

**In this paper we give necessary and sufficient conditions for the Freudenthal compactification of a rimcompact, locally connected and connected Hausdorff space to be unicoherent. We give several necessary and sufficient conditions for a locally connected generalized continuum to have a unicoherent compactification and show that if such a space $X$ has a unicoherent compactification, then $\gamma X$ is the smallest unicoherent compactification of $X$ in the usual ordering of compactifications.**

A connected topological space $X$ is said to be *unicoherent* if, $H \cdot K$ is connected whenever $X = H + K$ where $H$ and $K$ are closed connected sets. A continuum is a compact connected metric space and a generalized continuum is a locally compact, connected, separable metric space. By a mapping we will always mean a continuous function. If $B$ is a subset of a space $X$, the closure of $B$ in $X$ will be denoted by $\mathrm{cl}_X B$ and the boundary of $B$ in $X$ will be denoted by $\mathrm{Fr}_X B$. An open set (respectively, a closed set) of a space $X$ will be called a $\gamma$-open (respectively, $\gamma$-closed) subset of $X$ provided it has a compact boundary in $X$. A space is rimcompact (or semicompact) provided every point has arbitrarily small neighborhoods with compact boundaries. All compactifications considered here are Hausdorff.

In [7] K. Morita showed that for any rimcompact Hausdorff space $X$ there exists a topologically unique compactification $\gamma X$ of $X$ satisfying:

(a) For every point $x$ of $\gamma X$ and every open set $R$ of $\gamma X$ containing $x$ there exists an open set $V$ of $\gamma X$ containing $x$ such that $V \subset R$ and $\mathrm{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint $\gamma$-closed subsets of $X$ have disjoint closures in $\gamma X$.

Furthermore if $C$ is any compactification of $X$ satisfying (a), there exists a mapping $h$ of $\gamma X$ onto $C$ such that $h|X$ is the identity map. The compactification $\gamma X$ of $X$ is called the Freudenthal compactification of $X$ after H. Freudenthal who first defined it [4].

DEFINITION. We say that a connected space $X$ is $\gamma$-unicoherent if whenever $X = H + K$, where $H$ and $K$ are $\gamma$-closed and connected sets, $H \cdot K$ is connected.

THEOREM 1. *If $X$ is a locally connected, connected, rimcompact Hausdorff space, then $\gamma X$, the Freudenthal compactification of $X$, is* *unicoherent iff $X$ is $\gamma$-unicoherent.*

*Proof.* Suppose that $X$ is $\gamma$-unicoherent and $\gamma X$ is not unicoherent. Then $\gamma X = H + K$ where $H$ and $K$ are closed and connected sets and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let $U$ and $V$ be open subsets of $\gamma X$ containing $A$ and $B$ respectively such that $\mathrm{cl}_{\gamma X} U \cdot \mathrm{cl}_{\gamma X} V = \Phi$ and $(\mathrm{Fr}_{\gamma X} V + \mathrm{Fr}_{\gamma X} U) \subset X$. By Propositions (2.8) and (4.1) of [1], $\gamma X$ is locally connected so if $C$ denotes the component of $U + V + H$ that contains $H$ and $D$ denotes the component of $U + V + K$ that contains $K$, $C$ and $D$ are open connected subsets of $\gamma X$ such that $(Fr_{\gamma X} C + \mathrm{Fr}_{\gamma X} D) \subset X$. By Lemma 5 of [6], $C \cdot X$ and $D \cdot X$ are connected so that $L = \mathrm{cl}_X (C \cdot X)$ and $M = \mathrm{cl}_X (D \cdot X)$ are $\gamma$-closed and connected subsets of $X$. Furthermore $X = L + M$ and $L \cdot M$ is not connected. This contradicts our hypothesis that $X$ is $\gamma$-unicoherent and thus $\gamma X$ must be unicoherent.

Now suppose that $\gamma X$ is unicoherent and $X$ is not $\gamma$-unicoherent. Then $X = H + K$ where $H$ and $K$ are $\gamma$-closed and connected subsets of $X$ and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let $H'$, $K'$, $A'$ and $B'$ denote the closures of $H$, $K$, $A$ and $B$, respectively, in $\gamma X$. Since the boundary of $H \cdot K$ in $X$ is a subset of the union of the boundaries of $H$ and $K$ in $X$, $H \cdot K$ and hence $A$ and $B$ are $\gamma$-closed subsets of $X$. Then by property (b) of Morita's characterization of $\gamma X$, $A'$ and $B'$ are disjoint closed subsets of $\gamma X$. We now argue that $H' \cdot K'$ is a subset of $A' + B'$. Suppose to the contrary that there exists a point $x$ in $H' \cdot K'$ that does not belong to $A' + B'$. Let $U$ be any open subsets of $\gamma X$ containing $x$ such that $U$ does not intersect $A' + B'$ and such that $\mathrm{Fr}_{\gamma X} \subset X$. Let $Q$ be the component of $U$ that contains $x$ and note that $\mathrm{Fr}_{\gamma X} Q$ is a subset of $X$ and $Q$ is an open subset of $\gamma X$. Then since $X$ is dense in $\gamma X$ and $x$ is a limit point of $H'$ and $K'$, $Q \cdot H$ and $Q \cdot K$ are nonempty sets. But by Lemma 5 of [6], $Q \cdot X$ is connected and since $Q$ misses $H \cdot K$, $Q \cdot X$ must lie entirely in $H$ or $K$. Of course this implies that either $Q \cdot H$ or $Q \cdot K$ is empty and this is a contradiction. Thus $H' \cdot K' = A' + B'$ and this contradicts the unicoherence of $\gamma X$. Therefore $X$ is $\gamma$-unicoherent.

We need the following notation and definitions. Let $S^1$ denote the unit circle in the complex plane, let $I_1 = \{z = e^{i\theta}: 0 \leqq \theta \leqq \Pi\}$ and let $I_2 = \{z = e^{i\theta}: \Pi \leqq \theta \leqq 2\Pi\}$. For any space $W$ let $\mathscr{A}(W)$ denote the set of mappings of $W$ into $S^1$ and let $\mathscr{A}_j(W)$ be the set of all mappings of $W$ into $I_j$, $j = 1, 2$. For each $f \in \mathscr{A}_j(W)$, $j = 1, 2$, let $B_j(f)$ denote the set of all points $t \in I_j$ such that $\mathrm{Fr}\, f^{-1}(t)$ contains a compact set $K$ that separates $W$ into two disjoint open sets $M$ and $N$ where $f$ maps $M$ into the arc from 1 to $t$ on $I_j$ and $f$ maps $N$ into the arc from $t$ to $-1$ on $I_j$. Finally let $E(W) = \{f \in \mathscr{A}(W): B_1(f \mid f^{-1}(I_1)) +$

$B_2(f \mid f^{-1}(I_2))$ is dense in $S^1\}$.

THEOREM 2. *Suppose that $X$ is a locally connected, rimcompact Hausdorff space. A necessary and sufficient condition that $\gamma X$ be unicoherent is that every element of $E(X)$ be nullhomotopic.*

*Proof of the necessity.* Suppose that $\gamma X$ is unicoherent and let $f$ be an element of $E(X)$. For $j = 1, 2$, there exists a point $t_j \in I_j$ such that $\mathrm{Fr}_X f^{-1}(t_j)$ contains a compact set $K_j$ that separates $f^{-1}(I_j)$ into two disjoint open sets $M_j$ and $N_j$ where $f$ maps $M_j$ into the arc from 1 to $t_j$ on $I_j$ and $f$ maps $N_j$ into the arc from $t_j$ to $-1$ on $I_j$. Then if we let $M$ denote $K_1 + K_2 + M_1 + M_2$ and let $N$ denote $K_1 + K_2$ $N_1 + N_2$, $X = M + N$ and the boundaries (relative to $X$) of $M$ and $N$ are subsets of $K = K_1 + K_2$. We assert that the boundaries of $M_0 = \mathrm{cl}_{\gamma X} M$ and $N_0 = \mathrm{cl}_{\gamma X} N$ relative to $\gamma X$ are also subsets of $K$. In order to see this suppose that $x$ is an element of the boundary of $M_0$ and $x \notin K$. Then since $\gamma X$ is locally connected, there exists an open connected set $R$ of $\gamma X$ containing $x$ such that $R \cdot K = \Phi$ and $\mathrm{Fr}_{\gamma X} R \subset X$. Then $R \cdot M \neq \Phi$ and $R \cdot (X \backslash M) \neq \Phi$ since $X$ is dense in $\gamma X$. Furthermore $R \cdot X$ is connected by Lemma 5 of [6] and so $R \cdot X$ is a connected subset of $X$ that meets $M$ and $X \backslash M$. This implies that $R$ meets $K$ and this contradicts our selection of $x$. Hence the boundaries of $M_0$ and $N_0$ in $\gamma X$ are subsets of $K$. Also by Theorem 3 of [7], $M_0$ and $N_0$ are topologically equivalent to $\gamma M$ and $\gamma N$ respectively. Then by Lemma 1 of [3], $f \mid M$ has a continuous extension $f_M$ to $M_0$ and $f \mid N$ has a continuous extension $f_N$ to $N_0$. Then since $N_0 \cdot M_0 \subset K$, the function $h$ of $\gamma X$ into $S^1$ defined by $h \mid M_0 = f_M$ and $h \mid N_0 = f_N$ is continuous. By Lemma (7.4) of [9, p. 228], $h$ is exponentially representable on $\gamma X$, i.e. there exists a real valued function $\theta$ on $\gamma X$ such that $h(x) = e^{i\theta(x)}$ for all $x \in X$. It is evident that this implies that $f = h \mid X$ is exponentially representable an $X$ and by Theorem (6.2) of [9, p. 226], $f$ is nullhomotopic.

*Proof of the sufficiency.* Suppose that every element of $E(X)$ is nullhomotopic and suppose that $\gamma X$ is not unicoherent. Then by the proof of Theorem 1 there exists closed and connected sets $H$ and $K$ of $\gamma X$ such that $H \cdot K$ is not connected, $\mathrm{Fr}\, H$ and $\mathrm{Fr}\, K$ are subsets of $X$ and $L = H \cdot X$ and $M = K \cdot X$ are connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$. We note that $L$ and $M$ are $\gamma$-closed subsets of and thus by Theorem 3 of [7], $\gamma L$ is homeomorphic to $H$ and $\gamma M$ is homeomorphic to $K$. It then follows from Lemma 2 of [3] that there exists a mapping $f$ of $H$ into $I_1$ such that $f(A) = 1$, $f(B) = -1$ and $B_1(f \mid H \cdot X)$ is dense in $I_1$. Similarly there exists a mapping $g$ of $K$ into $I_2$ such that $g(A) = 1$, $g(B) = -1$ and $B_2(g \mid K \cdot X)$ is dense in $I_2$. Then if we define $h: \gamma X \rightarrow S^1$ by $h|H = f$ and $h|K = g$ we have that $h$ is continuous and $k = h|X$ is an element of $E(X)$. Then by our hypothesis and Proposition 6.2 of [9, p. 226], $k$ is exponentially representable, i.e. there exists a real-valued mapping $\theta$ on $X$ such that for each $x \in X$, $k(x) = e^{i\theta(x)}$. But then $\theta(A) \subset \{0, \pm 2\Pi, \pm 4\Pi, \cdots\}$ and $\theta(B) \subset \{\pm \Pi, \pm 3\Pi, \cdots\}$ and so if $a \in \theta(A)$ and $b \in \theta(B)$, the interval $[a, b]$ lies in $\theta(A) \cdot \theta(B)$ since $L$ and $M$ are connected. This is a contradiction since then $k(L) \cdot k(M)$ would then contain a semicircle whereas it consists of the points $-1$ and $1$. Hence $\gamma X$ is unicoherent.

DEFINITION. A connected space $X$ is said to be weakly unicoherent if whenever $X = H + K$ where $H$ and $K$ are closed and connected sets and $K$ is compact, $H \cdot K$ is connected.

THEOREM 3. *Let $X$ be a locally connected generalized continuum. A necessary and sufficient condition for $\gamma X$ to be unicoherent is that $X$ be weakly-unicoherent.*

*Proof of the necessity.* Suppose that $\gamma X$ is unicoherent. Since $X$ is locally compact, $X$ is open in $\gamma X$ and $X^* = \gamma X \backslash X$ is closed. Then by Theorem (2.3) of [2], $X = \gamma X \backslash X^*$ is weakly-unicoherent.

*Proof of the sufficiency.* Suppose that $\gamma X$ is not unicoherent. Then as in the proof of Theorem 1, $\gamma X$ has a representation $\gamma X = P + Q$ where $P$ and $Q$ are open connected subsets of $\gamma X$, the boundaries of $P$ and $Q$ in $\gamma X$ are subsets of $X$, $\mathrm{cl}_{\gamma x} P \cdot \mathrm{cl}_{\gamma x} Q = A + B$ where $A$ and $B$ are disjoint nonempty closed sets and $P$ has a nonempty intersection with both the boundary of $A$ and the boundary of $B$. By Lemma 5 of [6], $P' = P \cdot X$ is a connected open subset of $X$ and thus is arcwise connected. Furthermore since the boundaries of $A$ and $B$ are subsets of $X$ there exists an arc $\alpha\beta$ in $P'$ such that $\alpha\beta \cdot A = \alpha$ and $\alpha\beta \cdot B = \beta$. Let $R$ be the component of $P' \backslash (A + B)$ that contains $\alpha\beta \backslash (\alpha + \beta)$ and let $W$ be an open subset of $\gamma X$ containing $A$ such that $B \cdot \mathrm{cl}\, W = \phi$ and the boundary of $W$ is a subset of $X$. Then $H = R \cdot \mathrm{Fr}_{\gamma x} W$ is a nonempty compact subset of $R$ and there exists a continuum $K_0$ of $X$ such that $H \subset K_0 \subset R$. Let $K$ be the union of $K_0$ together with all the components of $R \backslash K_0$ with boundary entirely in $K_0$, i.e. having no boundary points in $X \cdot (A + B)$. Then $K$ separates $R$ since $W \cdot R$ contains a subarc $\alpha b \backslash \alpha$ from some point $b \in \alpha\beta$ and $X \backslash \mathrm{cl}_x W$ contains a subarc $a\beta$ of $\alpha\beta$. But $X \backslash K$ is connected since $X \backslash K$ is the union of the closure of $Q$ in $X$ plus all of the components of $X \backslash (A \cdot B)$ except $R$ plus all of the components of $R - K_0$ having a boundary point in $X \cdot (A + B)$. This contradicts Whyburn's characterization of weak-unicoherence in [8, p. 185].

COROLLARY 3.1. *Let $X$ be a locally connected generalized continuum. Then $X$ is weakly-unicoherent iff $X$ is $\gamma$-unicoherent.*

This corollary follows immediately from Theorems 1 and 3.

REMARK. The authors have been unable to discover a direct proof of Corollary (3.1). In general the two types of unicoherency are not equivalent and in the absence of local compactness, Theorem 3 is not valid.

EXAMPLE. Let $Y = \{z \text{ complex } | 1/2 \leqq |z| \leqq 1\}$,

$$S = \{z | \, |z| = 1\},\ A \text{ a countable dense subset of } S\,,$$
$$L_z = Y\cdot\{\text{ray from origin thru } z\}$$
$$C_r = \{z | \, |z| = r\},\ r \in [1/2, 1];$$
$$Z = \{C_r \cdot L_a | r \text{ is rational},\ a \in A\}\,.$$

The set $Z$ is countable and dense in $Y$. Let $X = Y - Z$. The set $X$ is evidently $T_2$, connected and locally connected (in fact, path connected and locally path connected), rim compact but not locally compact. Moreover:

1. $X$ is weakly-unichoherent. To see this, note that any continuum $K \subset X$ has empty interior in $X$. If therefore $X = H + K$, $H$ closed and connected and $K$ compact and connected, then necessarily the open set $X - H$ is a subset of $K$, and thus empty. It follows that $H \cdot K = K$, which is connected.

2. $X$ is not $\gamma$-unicoherent. For let $p, q \in S - A$ be two distinct points. Then $L_p$ and $L_q$ are compact and disjoint subsets of $X$. Assume $0 \leqq ARGp < ARGq$. Then

$$H = \{z \in X | ARGp \leqq ARGz \leqq ARG_q\} \text{ and}$$
$$K = \{z \in X | ARGq \leqq ARG_z \leqq ARGp + 2\pi\}$$

are closed, connected subsets of $X$ such that $X = H + K$, $H \cdot K = L_p + L_q$ is compact but not connected.

3. $\gamma X$ is not unicoherent. To show this it is sufficient to show that $\gamma X$ is just the set $Y$. To this end we use the characterization of $\gamma X$ obtained by Morita [6]. We show that

(a) For any point $x \in \gamma X$ and open set $R$ of $\gamma X$ containing $x$, there is an open set $V$ of $rX$ containing $x$ such that $V \subset R$ and $\mathrm{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint $\gamma$-closed subsets of $X$ have disjoint closures in $\gamma X$.

That (a) holds is evident from the definition of $X$. To see that (b) holds, let $A$ and $B$ be disjoint $\gamma$-closed subsets of $X$ and suppose that $p \in \mathrm{cl}_{\gamma X} A \cdot \mathrm{cl}_{\gamma X} B$. First of all we note that $p$ cannot belong to $X$ for then it would lie in $A \cdot B$ which is empty. In particular $p$ does not lie in the compact set $(\mathrm{Fr}_X A + \mathrm{Fr}_X B)$. By our construction of $X$ there exists an open subset $V$ of $Y$ containing $p$ such that $V \cdot (\mathrm{Fr}_X A + \mathrm{Fr}_X B) = \Phi$ and $V \cdot X$ is connected. Since $p$ belongs to the closure of $A$ in $Y$, $V \cdot X \cdot A$ is not empty and since $V \cdot X$ misses $\mathrm{Fr}_X A$, $V \cdot X$ must lie entirely in $A$. But this is a contradiction since $V \cdot X$ must meet $B$. Therefore $A$ and $B$ have disjoint closures in $Y$.

DEFINITION. A mapping $f: X \in Y$ is *monotone* provided for every $y \in Y$, $f^{-1}(y)$ is compact and connected.

THEOREM 4. *If $X$ is a locally connected generalized continuum and $Y$ is any unicoherent compactification of $X$, then there exists a monotone mapping $g$ of $Y$ onto $\gamma X$ such that $g|X$ is the identity.*

*Proof.* Let $Z$ denote the quotient space of $Y$ obtained from the decomposition whose only nondegenerate elements are the components of $Y \backslash X$ and let $p$ denote the natural map of $Y$ onto $Z$. Then since $X$ is open in $Y$, $Z$ is a Hausdorff compactification of $X$. Furthermore since point inverses of $p$ are connected, it follows from Proposition (2.2.1) of [9], that $Z$ is unicoherent. Also $Z \backslash X$ is totally disconnected and by the maximality of $\gamma X$ there exists a mapping $h$ of $\gamma X$ onto $Z$ such that $h|X$ is the identity and $h(\gamma X \backslash X) = Z \backslash X$. We assert that $h$ is a homeomorphism. In order to prove this we need only show that $h$ is one-to-one on $\gamma X \backslash X$. To this end let $x, y \in \gamma X$, $x \neq y$ and suppose that $h(x) = h(y)$. There exists a connected and open set $R$ of $\gamma X$ containing $x$ such that $y \notin \mathrm{cl}_\gamma R = K$ and $\mathrm{Fr}_\gamma R \subset X$. Then $Z = h(K) + h(\gamma X \backslash R)$ and $h(K) \cdot h(\gamma X \backslash R) = h(x) + h(\mathrm{Fr}\, R)$ is not connected. This contradicts the unicoherence of $Z$ and hence $h$ must be a homeomorphism. Then $g = h^{-1} \circ p$ is the desired monotone mapping.

COROLLARY 4.1. *Suppose that $X$ is a locally connected generalized continuum. Then $X$ has a unicoherent compactification if and only if $\gamma X$ is unicoherent.*

*Proof.* This result follows immediately from Theorem 4 and the fact that monotone images of compact unicoherent continua are unicoherent.

THEOREM 5. *Suppose that $X$ is a locally connected generalized continuum. Then the following are equivalent*

( i ) *$X$ is weakly-unicoherent*
( ii) *$\gamma X$ is unicoherent*
(iii) *$X$ is $\gamma$-unicoherent*
(iv) *$X$ has a unicoherent compactification*
( v ) *every mapping of $X$ into $S^1$ with compact boundaries of point inverses is null-homotopic.*

*Proof.* The equivalence of (i)—(iv) has been established in Theorems (1) – (4). As an immediate consequence of Theorem (3.3) of [2], we have that (v) implies (i) and (ii) implies (v) follows from Theorem 1 of this paper.

DEFINITION. A connected space $X$ is said to have the *complementation property* provided whenever $K$ is a compact set in $X$, $X/K$ has at most one component with a non-compact closure. See [2] for some characterizations of this property.

THEOREM 6. *Let $X$ be a locally connected generalized continuum and let $Y$ be any unicoherent, locally connected continuum. There exists a unicoherent compactification $Z$ of $X$ with $Z\backslash X$ homeomorphic to $Y$ if and only if $X$ is weakly-unicoherent and has the complementation property.*

*Proof of the necessity.* Suppose that $Z$ is a unicoherent compactification of $X$ and $Z\backslash X$ is homeomorphic to $Y$. Then by Theorem (4.2) of [2], $X$ is weakly-unicoherent and has the complementation property.

*Proof of the sufficiency.* Suppose that $X$ is weaklyunicoherent and has the complementation property. Then by Theorem (2.2) of [5] there exists a compactification $Z$ of $X$ with $Z\backslash X$ homeomorphic to $Y$ and by Theorem (4.2) of [2], $Z$ is unicoherent. This completes the proof.

REMARK. It appears to be difficult to establish results concerning the unicoherence of a compactification of an arbitrary completely regular space. We can show that the Freudenthal compactification of a rim-compact, locally connected $\gamma$-unicoherent space is the smallest unicoherent compactification of $X$ with $\gamma X\backslash X$ zero-dimensional.

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CALIFORNIA STATE COLLEGE AT FULLERTON
AND
VIRGINIA POLYTECHNIC INSTITUTE AND
STATE UNIVERSITY

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# APPROXIMATE IDENTITIES AND THE STRICT TOPOLOGY

H. S. COLLINS AND R. A. FONTENOT

**This paper studies relationships between approximate identities on a $B^*$ algebra $A$ and other properties of the algebra. If $A$ is commutative, conditions on the approximate identity for $A$ are related to topological properties of the spectrum of $A$. The principal result of this paper is that for a locally compact Hausdorff space $S$, $C_0(S)$ has an approximate identity that is totally bounded in the strict topology (or compact open topology) if and only if $S$ is paracompact.**

**1. Introduction.** The problem of extending theorems about commutative $B^*$ algebras to the non-commutative case has received a great deal of attention in recent years. Because many proofs made in the commutative case make use of the spectrum (= maximal ideal space), an obvious question is: what is to replace this device in the case of a non-commutative $B^*$ algebra? Various possible replacements have been sought; e.g.; see Akemann [1] and Pedersen [15, 16]. Much progress has been made for certain types of problems by means of restrictions on approximate identities for the algebra in question by Taylor [20, 21], Akemann [2], and others. The class of problems solved or seemingly susceptible to this technique is rather large. This fact and the paucity of results for this class of problems obtained by studying Prim $A$ and the space of equivalence classes of irreducible representations suggest that the approximate identity is a useful tool for extending many commutative theorems to a non-abelian setting. A question that arises immediately in the case of a commutative $B^*$ algebra is: what do restrictions on the approximate identity imply about the spectrum of $A$ and vice versa? Along this line, Collins-Dorroh [6] characterize $\sigma$-compactness of the spectrum and ask for necesary and sufficient conditions on $S$ that $C_0(S)$ (in this paper, $S$ always denotes a *locally compact Hausdorff space*) have an approximate identity that is totally bounded in the strict topology (called $\beta$ by Buck). This paper answers this question and several related ones, including some in the non-commutative context.

**2. Preliminaries.**

DEFINITION 2.1. Let $A$ be a Banach algebra. An *approximate identity* for $A$ is a net $\{e_\lambda \mid \lambda \in \Lambda\}$ (we generally write simply $\{e_\lambda\}$) with $\lim_\lambda \|e_\lambda x - x\| = \lim_\lambda \|x e_\lambda - x\| = 0$ for $x \in A$ and $\|e_\lambda\| \leqq 1$ for all $\lambda$.

It is well known that all $B_*$algebras have approximate identities.

DEFINITION 2.2. The *double centralizer algebra* $M(A)$ of a $B^*$ algebra $A$ was studied by R. C. Busby [5] who defined the strict topology as that topology on $M(A)$ generated by the seminorms $x \to \max\{\|xy\|, \|yx\|\}$ for $x \in M(A)$ and $y \in A$. Two motivating examples for the double centralizer algebra concept are the algebra $C_0(S)$ of continuous complex functions on $S$ which vanish at infinity (this class is identical with the class of all commutative $C^*$ algebras by the theorem of Gelfand), whose double centralizer algebra was identified by Wang [22] as $C_b(S)$, the algebra of all bounded continuous complex functions on $S$; and the algebra of compact operators on a Hilbert space $H$, whose double centralizer algebra was shown to be the bounded linear operators on $H$ by Busby. For a definition of $M(A)$ and some of its properties, the reader is referred to Busby [5]. By $M(A)_\beta$ we shall mean $M(A)$ endowed with the strict topology $\beta$.

DEFINITION 2.3. If $f \in C_b(S)$, the *support* of $f$, spt $f$, is the closure in $S$ of $N(f) = \{x: f(x) \neq 0\}$.

DEFINITION 2.4. $S$ is *sham compact* if each $\sigma$-compact subset is relatively compact.

DEFINITION 2.5. Let $A$ be a $B^*$ algebra and $\{e_\lambda\}$ be an approximate identity for $A$. We shall be interested in the following conditions:
(a) $\{e_\lambda\}$ is *countable*, i.e., the range of $\{e_\lambda\}$ is a countable set;
(b) $\{e_\lambda\}$ is *sequential*, i.e., $\Lambda$ is the set of positive integers with the usual order;
(c) $\{e_\lambda\}$ is *canonical*, i.e., $e_\lambda \geqq 0$ and if $\lambda_1 < \lambda_2$ then $e_{\lambda_1}e_{\lambda_2} = e_{\lambda_1}$;
(d) $\{e_\lambda\}$ is *well-behaved* (after Taylor [21]), i.e., $\{e_\lambda\}$ is canonical and if $\lambda \in \Lambda$ and $\{\lambda_n\}$ is a strictly increasing sequence in $\Lambda$, there is a positive integer $N$ so that $e_\lambda e_{\lambda_n} = e_\lambda e_{\lambda_m}$ for $n, m > N$;
(e) $\{e_\lambda\}$ is *$\beta$ totally bunded*; i.e., totally bounded in the strict topology;
(f) $\{e_\lambda\}$ is abelian;
(g) $\{e_\lambda\}$ is *chain totally bounded*, i.e., if $\{\lambda_n\}$ is an increasing sequence in $\Lambda$, then $\{e_{\lambda_n}\}$ is $\beta$ totally bounded;
(h) $\{e_\lambda\}$ is $\sigma(M(A), M(A)^*_\beta)$ relatively compact, where $\sigma$ denotes the weak topology on $M(A)$ in the pairing with its $\beta$ dual;
(i) $\{e_\lambda\}$ is *sham compact*, i.e., $\{e_\lambda\}$ is canonical and if $\{\lambda_n\}$ is a sequence in $\Lambda$, then there is a $\lambda$ in $\Lambda$ so that $\lambda > \lambda_n$ for all integers $n$.

REMARK 2.6. A sequence $\{e_n\}$ in a $B^*$ algebra $A$ which satisfies

$\lim_n \|e_n x - x\| = \lim_n \|x e_n - x\| = 0$ is norm bounded by the uniform boundedness principle and the $B*$norm property. Thus it is not necessary to require norm boundedness in 2.1 for this case.

REMARK 2.7. Taylor [21] introduced the notion of a well-behaved approximate identity and used it to prove many interesting improvements of results of Phillips [9, p. 32], Akemann [2], Bade [3], Collins-Dorroh [6], and Conway [7 and 8].

**3. A characterization of paracompact spaces.** Our main result in this section, 3.10, answers two questions posed in [6, Remark 4.3]. Our interest centers exclusively on $B^*$ algebras without identity; for these, we need information about increasing sequences in the directed set of an appropriate identity and about supports. Lemmas 3.1 and 3.2 provide what we need.

LEMMA 3.1. *If $A$ is a Banach algebra without identity, $\{e_\lambda\}$ an approximate identity for $A$, and $\lambda_0 \in \Lambda$, then $\exists \lambda \in \Lambda \cdot \ni \cdot \lambda > \lambda_0$.*

*Proof.* If the conclusion does not hold, then $\forall \lambda \in \Lambda$, $\lambda \leqq \lambda_0$, from which it follows that $e_{\lambda_0}$ is an identity for $A$.

LEMMA 3.2. *Let $\{e_\lambda\}$ be an approximate identity for $C_0(S)$.*

(a) *If $\{e_\lambda\}$ is canonical, then $\lambda_1 < \lambda_2$ implies* $\operatorname{spt} e_{\lambda_1} \subset e_{\lambda_2}^{-1}\{1\} \subset N(e_{\lambda_2})$ *and $\lambda \in \Lambda$ implies that the* $\operatorname{spt} e_\lambda$ *is compact;*

(b) *If $K$ is a compact subset of $S$, then $\exists \lambda \in \Lambda$ so that $|e_\lambda| > 3/4$ on $K$.*

*Proof.* This is straightforward.

We are mainly interested (in §3) in two types of approximate identities, viz., well-behaved ones, shown to be important by Taylor [21], and $\beta$ totally bounded ones, the study of which motivated this paper.

LEMMA 3.3. *Let $\{e_\lambda\}$ be an approximate identity for $C_0(S)$ which is either $\beta$ totally bounded or well-behaved. Then there exists a cover of $S$ by clopen $\sigma$-compact sets.*

REMARK 3.4. All topologies between the compact open and the strict agree on norm bounded sets. Thus "$\beta$ totally bounded" may be replaced in 3.3 by "compact open totally bounded."

*Proof of* 3.3. We assume that $S$ is not compact in either case to avoid trivialities. Assume first that $\{e_\lambda\}$ is $\beta$ totally bounded. Replacing $\{e_\lambda\}$ by $\{|e_\lambda|^2\}$, performing a straight forward computation and using 3.4, we may assume that $\{e_\lambda\}$ is compact open totally bounded and $e_\lambda \geqq 0$ for each $\lambda$. Let $x \in X$ and choose by 3.2 (b) $\lambda_1 \in \Lambda$ so that $e_{\lambda_1}(x) > 3/4$. Let $K_1 = \{x \in S: e_{\lambda_1}(x) \geqq 1/4\}$. Suppose that $\{K_j\}$ $j = 1, \cdots, n$ and $\{\lambda_j\}$ $j = 1, \cdots, n$ have been chosen so that

$$(1) \qquad e_{\lambda_j} > \frac{3}{4} \quad \text{on} \quad K_{j-1}, \quad j = 1, \cdots, n$$

$$(2) \qquad K_j = \left\{x \in S: e_{\lambda_i}(x) \geqq \frac{1}{4j} \quad \text{for some} \quad i, 1 \leqq i \leqq j\right\}.$$

By 3.2 (b) again, choose $\lambda_{n+1} \in \Lambda$ so that $e_{\lambda_{n+1}} > 3/4$ on $K_n$ and let

$$K_{n+1} = \left\{x \in S: e_{\lambda_i}(x) \geqq \frac{1}{4^{n+1}} \quad \text{for some} \quad i, 1 \leqq i \leqq n+1\right\}.$$

By induction we obtain sequences $\{\lambda_n\}$ and $\{K_n\}$ satisfying (1) and (2) above. Let $X = \bigcup_n K_n$. $X$ is clearly $\sigma$-compact and contains $x$. It is open since $K_n \subset$ interior of $K_{n+1}$. To show that $X$ is closed, take a compact set $K$. It suffices to show $K \cap X$ is closed [13, p. 231]. The total boundedness condition of $\{e_\lambda\}$ gives the existence of an integer $i_0$ so that for all positive integers $j$,

$$(3) \qquad \min_{1 \leqq i \leqq i_0} \|e_{\lambda_j} - e_{\lambda_i}\|_K < \frac{1}{4}$$

($\|f\|_K = \sup_{x \in K} |f(x)|$ for $f \in C_b(S)$). Let $y \in K_m \cap K$ where $m > i_0$. By construction $e_{\lambda_{m+1}}(y) > 3/4$ so by (3) there is an integer $1 \leqq i \leqq i_0$ so that $e_{\lambda_i}(y) \geqq 1/2$ which shows that $y \in K_i$. Thus $X \cap K = K \cap \bigcup_{i=1}^{i_0} K_i$ so $X \cap K$ is closed.

For the other part of the lemma, let $x \in X$, assume that $\{e_\lambda\}$ is well-behaved, and choose by 3.1 and 3.2 an increasing sequence $\{\lambda_n\}$ so that $e_{\lambda_1}(x) > 0$. Let $K_n = \operatorname{spt} e_{\lambda_n}$ and note, by 3.2, that $K_n \subset$ interior of $K_{n+1}$. Let $X = \bigcup K_n$ and note that $X$ is open, $\sigma$-compact and $x \in X$. From 3.2 (a) and the definition of well-behaved approximate identity, it follows that $\{e_{\lambda_i}\}$ is totally bounded in the compact open topology and that $y \in X$ implies $e_{\lambda_j}(y) = 1$ for $j$ large enough. With these observations, the proof that $X$ is closed is the same as in the first part of the lemma.

REMARK 3.5. Note that in 3.3, $\bigcup_{n=1}^{\infty} \operatorname{spt} e_{\lambda_n} \subset X$.

COROLLARY 3.6. *If $S$ is connected and has an approximate identity that is either well-behaved or $\beta$ totally bounded, then $S$ is $\sigma$-compact.*

PROPOSITION 3.7. *Let $F$ be a closed subset of $S$. If $C_0(S)$ has either a well-behaved or a $\beta$ totally bounded approximate identity, then $F$ contains a $\sigma$-compact set that is relatively clopen in $F$.*

*Proof.* Let $\{e_\lambda\}$ be an approximate identity with either of the properties above. For $\lambda \in \Lambda$, let $d_\lambda$ be the restriction of $e_\lambda$ to $F$. Since $F$ is closed, $\{d_\lambda\} \subset C_0(F)$. We claim that $\{d_\lambda\}$ has the same property as $\{e_\lambda\}$ does; i.e., that $\{d_\lambda\}$ is a well-behaved (resp. $\beta$ totally bounded) approximate identity for $C_0(F)$. To show this, it suffices to show that if $f \in C_0(F)$, then there is an extension $g$ in $C_0(S)$ of $f$. Let $S^*$ denote the one-point compactification of $S$ and $\infty$ denote the point at infinity. Let $f'$ be an extension of $f$ to $F \cup \{\infty\}$ obtained by defining $f'(\infty) = 0$. Since $f \in C_0(F)$, $f'$ is continuous and extends to a continuous function $p$ on all of $S^*$ by Tietze's Theorem since $F \cup \{\infty\}$ is closed in $S^*$. The restriction $g$ of $p$ to $S$ is clearly an extension of $f$ in $C_0(S)$. This concludes the proof of 3.7.

COROLLARY 3.8. *If $S$ is locally connected and $C_0(S)$ has an approximate identity that is either well-behaved or $\beta$ totally bounded, then $S$ is paracompact.*

*Proof.* By [11, Theorem 7.3], it suffices to show that $S$ is a disjoint union of clopen $\sigma$-compact subspaces. In a locally connected space, the components are clopen and connected and so $\sigma$-compact by 3.7.

LEMMA 3.9. *Suppose that $C_0(S)$ has a $\beta$ totally bounded approximate identity and let $\mathscr{W}$ be the family of all clopen $\sigma$-compact subsets of $S$ constructed by the method of the first part of 3.3. If $\mathscr{U} \subset \mathscr{W}$, then $\bigcup_{W \in \mathscr{U}} W$ is clopen.*

*Proof.* We may assume $e_\lambda \geqq 0$ as in 3.3. Let $X = \bigcup_{W \in \mathscr{U}} W$ and $K$ be an arbitrary compact subset of $S$. Since $S$ is locally compact, it suffices to show that $X \cap K$ is closed. With each $W$ in $\mathscr{U}$ is associated a sequence $\{e_n^W\}$ from the approximate identity such that

$$\bigcup_{n=1}^{\infty} \operatorname{spt} e_n^W \subset W$$

(see 3.4) and if $\gamma \in W$, $e_n^W(\gamma) > 3/4$ for $n$ large enough. From $\beta$ total boundedness of $\{e_n^W : W \in \mathscr{U}, n = 1, 2, \cdots\}$, we get a set $\{W_i\}_{i=1,\ldots,n}$ from $\mathscr{U}$ and associated integers $\{n_i\}$ $i = 1, \cdots, n$ so that for any $V$ in $\mathscr{U}$ and positive integer $p$

$$\min_{1 \leqq i \leqq n} \| e_{n_i}^{W_i} - e_p^V \|_K < \frac{1}{4} . \tag{4}$$

If $\gamma \in X \cap K$, then $\gamma \in K \cap W$ for some $W \in \mathscr{U}$, so choosing $p$ large enough so that $e_p^V(\gamma) > 3/4$ we see that $e_{n_i}^{W_i}(\gamma) > 0$ for some $1 \leqq i \leqq n$ so that $\gamma \in W_i$. We have established that $X \cap K = K \cap \bigcup_{i=1}^n W_i$ so $X \cap K$ is closed. This concludes the proof of 3.9.

In [6] Collins and Dorroh show that if $S$ is paracompact then $C_0(S)$ has a $\beta$ totally bounded approximate identity and ask two questions: (1) Does the existence of a $\beta$ totally bounded approximate identity imply the existence of a canonical one that is $\beta$ totally bounded? and (2) Does the existence of a $\beta$ totally bounded approximate identity in $C_0(S)$ imply that $S$ is paracompact? We add to these a third question: Does the existence of a $\beta$ totally bounded approximate identity in $C_0(S)$ imply the existence of a well-behaved one? The answer to all these questions is given in 3.10.

THEOREM 3.10. *These are equivalent*: (1) $S$ *is paracompact*; (2) $C_0(S)$ *has a canonical approximate identity that is* $\beta$ *totally bounded*; (3) $C_0(S)$ *has a approximate identity that is* $\beta$ *totally bounded.*

*Proof*. For the first implication see [6]. Since the second implication is trivial, we prove only that if $\{e_\lambda\}$ is a $\beta$ totally bounded approximate identity for $C_0(S)$ then $S$ is paracompact. Take $\mathscr{W}$ to be the set in 3.9 and well order it. Let $W_0$ be the first element in $\mathscr{W}$ and $W_0' = W_0$. If $W \in \mathscr{W}$, and $W \neq W_0$, let $W' = W \backslash (\bigcup_{V<W, V \in \mathscr{W}} V)$.

Each set $W'$ is clopen and $\sigma$-compact by 3.3 and 3.9.

If $x \in S$ and $W$ is the least element in $\{W\colon W \in \mathscr{W}$ and $x \in W\}$, then $x$ clearly belongs to $W'$. The collection $\{W'\colon W \in \mathscr{W}\}$ then consists of disjoint sets and so forms a partition of $S$ by clopen $\sigma$-compact subsets. We apply [11, Theorem 7.3] to conclude the proof.

**4. Non-commutative results and examples.** Taylor [21] gives the following examples of $B^*$ algebras with well-behaved approximate identities: algebras with countable approximate identities, algebras with series approximate identities (for a definition, see Akemann [23]) such as the compact operators on a Hilbert space, and subdirect sums of algebras having well-behaved approximate identities, such as dual $B^*$ algebras which are subdirect sums of algebras of compact operators.

In this section, we give examples of algebras with $\beta$ totally bounded approximate identities using some techniques borrowed from Taylor and some of our own. We also give some partial results, e.g., 4.1, relating the existence of approximate identities of one type to existence of another type.

PROPOSITION 4.1. *Let $A$ be a Banach algebra with a sequential canonical approximate identity $\{e_n\}$. Then $\{e_n\}$ is $\beta$ totally bounded and well behaved.*

The proof requires the following observation whose proof is straightforward:

REMARK 4.2. If $\{f_\lambda\}$ is an approximate identity for $A$, then the locally convex topology on $M(A)$ (see 2.2) generated by the seminorms $x \to \max\{\|f_\lambda x\|, \|x f_\lambda\|\}$ agrees with the strict topology on norm bounded sets in $M(A)$.

*Proof of* 4.1. Let $m$ and $n_1 < n_2 < \cdots$ be positive integers. Choose a positive integer $i_0$ so that $n_i > m$ for $i \geqq i_0$. Then

$$e_m(e_{n_i} - e_{n_j}) = 0$$

for $i, j > i_0$ by the canonical property so $\{e_n\}$ is well-behaved. Total boundedness in the strict topology follows from 4.2 and the fact that $\{e_n\}$ is well-behaved. Part (a) of the next result was used by Taylor [21] in his study of well-behaved identities. We shall use it in 4.5 to show that algebras with countable approximate identities have ones with other nice properties.

LEMMA 4.3. *Let $A$ be a Banach algebra.* (a) *If $\{e_\lambda\}$ is an approximate identity for $A$ and $\{f_p\}$ is an approximate identity for the normed algebra generated by $\{e_\lambda\}$, then $\{f_p\}$ is an approximate identity for $A$;* (b) *If $\{e_\lambda\}$ is a norm bounded net in $A$ and $D$ a dense subset in the Hermitian part of the unit ball of $A$ so that $e_\lambda x \to x$ and $x e_\lambda \to x$ for each $x$ in $D$, then $\{e_\lambda\}$ is an approximate identity for $A$ (here we assume $A$ is $B^*$).*

*Proof.* This is a straightforward computation.

Separable $B^*$ algebras have many types of approximate identities as 4.4 shows.

LEMMA 4.4. *Let $A$ be a separable $B^*$ algebra. Then $A$ contains an approximate identity that is canonical, sequential, and abelian (and by* 4.1, *well-behaved and $\beta$ totally bounded).*

*Proof.* Let $\{x_n\}$ be a countable dense set in the Hermitian part of the unit sphere of $A$, and let $x = \sum (1/2^n)x_n^2$. Since $x$ is a positive element of $A$, the $B^*$ algebra $C$ generated by $x$ is isometrically $*$-isomorphic to the algebra $C_0(S)$, where $S$ is the maximal ideal space of

$C$. Since $C_0(S)$ is generated by a single function, $S$ is $\sigma$-compact. We may select from $C(=C_0(S))$ an approximate identity $\{e_k\}$ for $C$ possessing all the properties mentioned in the statement of 4.4. It remains only to show that $\{e_k\}$ is an approximate identity for $A$. Adjoin a unit $I$ to $A$ in the customary manner so that the adjoined algebra is $B^*$, hence we have that $\|(I - e_k)x(I - e_k)\| \xrightarrow[k]{} 0$. From [10, p. 14] we have that

$$\|(I - e_k)x_n x_n^*(I - e_k)\| \leqq 2^n \|(I - e_k)x(I - e_k)\|$$

so that $\|(I - e_k)x_n\| = \|x_n(I - e_k)\| \rightarrow_k 0$. Thus applying 4.3 (b) to $D = \{x_n\}$ and $\{e_k\}$ we see that $\{e_k\}$ is an approximate identity for $A$.

DEFINITION 4.5. Let $\{A_\gamma\}$ be a family of normed algebras. The subdirect sum, $(\sum A_\gamma)_0$, *of the family* $\{A_\gamma\}$ is that subset of $P_{\gamma \in \Gamma} A_\gamma$ consisting of all $a = (a_\gamma) \in PA_\gamma$ so that $\{\gamma \in \Gamma : \|a_\gamma\| \geqq \varepsilon\}$ is finite for each $\varepsilon > 0$. The algebraic operations are pointwise and $\|a\| = \sup\{\|a_\gamma\| : \gamma \in \Gamma\}$.

PROPOSITION 4.6. *If* $A = (\sum A_\gamma)_0$ *and each* $A_\gamma$ *has a* $\beta$ *totally bounded approximate identity, then so does* $A$.

*Proof*. The proof is the same as Proposition 3.2 in [21] where the same result is proved for well-behaved approximate identities.

REMARK 4.7. Proposition 4.6 is true when "totally bounded" is replaced by any of the types of approximate identities listed in §2, except countable and sequential. Dual $B^*$ algebras have $\beta$ totally bounded approximate identities by 4.6, and 4.5 and 4.6 give a proof, different from that in [6], that $C_0(S)$, for $S$ paracompact, has a $\beta$ totally bounded approximate identity.

CONJECTURE 4.8. We conjecture that $C_0(S)$ has a well-behaved approximate identity if and only if $S$ is paracompact. As we indicated earlier, our results on this question are incomplete, but we give an example in §6 that is perhaps illuminating.

## 5. Sham compact spaces and approximate identities.

The definition of sham compact space and sham compact approximate identity, given in 2.5, is motivated by the space $X$ of ordinals less than the first uncountable ordinal with the order topology, and the algebra $C_0(X)$. For example, let $\Lambda = X$ with the usual order and if $\lambda \in \Lambda$, let $f_\lambda$ be the characteristic function of the interval $[0, \lambda]$. It is clear that $\{f_\lambda\}$ is a sham compact approximate identity for $C_0(X)$. We note that $C_0(X)$ cannot have a $\beta$ totally bounded approximate identity since $X$ is not paracompact. Furthermore, it cannot have a well-behaved approximate identity either since it is pseudocompact.

PROPOSITION 5.1. *Let $S$ be pseudocompact. If $C_0(S)$ has a well-behaved approximate identity, then $S$ is compact.*

*Proof.* Let $\{e_\lambda\}$ be a well-behaved approximate identity for $C_0(S)$, suppose that $S$ is not compact, and choose, by 3.1, an increasing sequence $\{\lambda_n\}$ so that $e_{\lambda_i} \neq e_{\lambda_{i+1}}$ for any integer $i$. Note that $e_{\lambda_1} < e_{\lambda_2} < \cdots$, i.e., $\{e_{\lambda_i}\}$ is an increasing sequence. Since the sequence $\{e_{\lambda_i}\}$ is Cauchy in the compact open topology and $C_b(S)$ is complete in this topology, there is a function $f$ in $C_b(S)$ so that $e_{\lambda_i} \to f$ uniformly on compact subsets of $S$. By [12, Theorem 2], $e_{\lambda_i} \to f$ in norm so $f$ is in $C_0(S)$. By 3.2, $f \equiv 1$ on $\bigcup_{i=1}^{\infty} \operatorname{spt} e_{\lambda_i}$ which then is contained in the compact set $K = f^{-1}\{1\}$. Choosing $\lambda \in \Lambda$ so that $e_\lambda \equiv 1$ on $K$, we obtain a contradiction to the fact that $e_{\lambda_i} \neq e_{\lambda_{i+1}}$ for all $i$.

REMARK 5.2. Proposition 5.1 admits the following non-abelian generalization, stated here, without proof, for completeness: *Suppose a $B^*$ algebra $A$ has a well-behaved approximate identity and $M(A)$ satisfies the following condition: whenever $\{a_n\}$ is an increasing sequence in $A$ and $\{a_n\}$ converges in the strict topology to $x$ in $M(A)$, then $||a_n - x|| \to 0$. Then $A$ has an identity and $A = M(A)$.* (See [12, Proposition 2] to see that this result includes 5.1.)

The next proposition relates sham compactness of $S$, existence of sham compact approximate identities in $C_0(S)$ and the property $(DF)$ of Grothendieck.

DEFINITION 5.3. Let $E$ be a locally convex topological vector space with dual $E^*$. The space $E$ is $(DF)$ if there is a countable base for bounded sets in $E$ and if every countable intersection of closed convex circled zero neighborhoods which absorbs bounded sets is a zero neighborhood.

REMARK 5.4. The vector space $C_b(S)_\beta$ is complete and the $\beta$ bounded sets coincide with the norm bounded sets so $C_b(S)_\beta$ is $(DF)$ if each countable intersection of closed convex circled zero neighborhoods which absorbs points of $C_b(S)$ is a zero neighborhood [17, p. 67].

We shall use the following remark in the proof of Theorem 5.6.

REMARK 5.5. W. H. Summers [19] has recently shown that $C_b(S)_\beta$

is $(DF)$ if $C_b(N; C_0(S))$ is essential, where $C_b(N; C_0(S))$ is the Banach algebra of all norm bounded sequence from $C_0(S)$ with the sup norm topology $(\| \|_\infty)$ and "essential" means that $\|e_\lambda\{f_n\} - \{f_n\}\|_\infty \to_\lambda 0$ where $\{e_\lambda\}$ is any approximate identity for $C_0(S)$ and $\{f_n\}$ any element of $C_b(N; C_0(S))$.

THEOREM 5.6. *These are equivalent*: (a) $C_b(S)_\beta$ *is* $(DF)$; (b) $S$ *is a sham compact space* (c) $C_0(S)$ *has a sham compact approximate identity.*

*Proof.* Assume that $C_b(S)_\beta$ is $(DF)$ and $X$ is the union of compact sets $K_n$, i.e., $X = \bigcup_{n=1}^\infty K_n$. For each integer $n$, let $\varphi_n$ be a function in $C_0(S)$ so that $0 \leqq \varphi_n \leqq 1$ and $\varphi_n \equiv 1$ on $K_n$. Let

$$V = \{f \in C_b(S): \|f\varphi_n\| \leqq 1, \forall_n\} .$$

$V$ absorbs points of $C_b(S)$; therefore it is a zero neighborhood in the strict topology by (a). It is obvious that the sets $\{f \in C_b(S): \|f\varphi\| \leqq 1\}$ (for $\varphi \geqq 0$ in $C_0(S)$) is a base at zero for the strict topology. Thus $\exists \varphi \geqq 0$ in $C_0(S)$ so that $\{f \in C_b(S): \|f\varphi\| \leqq 1\} \subset V$. This shows that $\varphi(x) \geqq 1$ for $x$ in $X$. For if not, there is an integer $n$ and a point $x_0$ in $K_n$ so that $\varphi(x_0) < 1$. By a standard Urysohn's lemma argument $\exists f \in C_0(S)$ so that $f(x_0) > 1$ and $\|\varphi f\| < 1$. This contradiction establishes our claim, i.e., $X \subset \varphi^{-1}\{1\}$, so $X$ is compact.

Suppose that (b) holds. Let $\Lambda$ be the set of all pairs $(K, 0)$ where $K \subset D \subset S$, $K$ is compact and $0$ is open with compact closure. If $\lambda = (K, 0)$ and $\lambda_1 = (K_1, 0_1)$, we define $\lambda \geqq \lambda_1$ if $\lambda = \lambda_1$ or if $0_1 \subset K$. If $\lambda = (K, 0)$ let $f_\lambda$ be a function in $C_0(S)$ which satisfies: (1) $0 \leqq f_\lambda \leqq 1$; (2) $f_\lambda \equiv 1$ on $K$; and (3) $\operatorname{spt} f_\lambda \subset 0$. The net $\{f_\lambda\}$ is by (b) a sham compact approximate identity for $C_0(S)$.

Assume (c), with $\{e_\lambda\}$ a sham compact approximate identity, and let $\{f_n\}$ be a sequence contained in the unit ball of $C_0(S)$, and $\varepsilon > 0$. Choose a sequence $\{\lambda_n\}$ from $\Lambda$ so that $\|e_{\lambda_n} f_n - f_n\| < \varepsilon$ for each integer $n$. Let $\lambda_0 \in \Lambda$ be such that $\lambda_0 > \lambda_n$ for all integers $n$. Remark 5.5 and the following computation finish the proof;

$$\begin{aligned} \lambda > \lambda_0 \quad \text{implies} \quad \|e_\lambda f_n - f_n\| &= \|(1 - e_\lambda)f_n\| \\ &= \|(1 - e_{\lambda_n})(1 - e_\lambda)f_n\| \\ &\leqq \|(1 - e_{\lambda_n})f_n\| < \varepsilon \quad \text{for all } n . \end{aligned}$$

6. **Metacompact spaces—an example.** We have been unable to prove our conjecture that $S$ is paracompact if $C_0(S)$ has a well-behaved approximate identity except in special cases (see § 3), but we are able to give an example that shows that metacompactness is not sufficient for existence of a well-behaved approximate identity.

EXAMPLE 6.1. Let $I$ be the unit interval with the discrete topology and $I^*$, the one-point compactification of $I$, with $\infty$ denoting the point at infinity. Similarly, let $N$ denote the positive integers with discrete topology, $N^*$ the one-point compactification of $N$, and $w$ the point at infinity. Let $S = I^* \times N^* \backslash \{(\infty, w)\}$. Being an open set in a compact Hausdorff space, $S$ is locally compact Hausdorff.

To show that $X$ is metacompact, take an open cover $\mathscr{U}$ of $X$. For each point $(\infty, n)$, there is a finite set $F_n$ of $I$ so that a member of $\mathscr{U}$ contains the open set $U_n = \{(x, n): x \notin F_n\}$. Similarly, for each point $(x, w)$ there is a finite set $G_X$ of $N$ with a member of $\mathscr{U}$ containing the open set $W_x = \{(x, n): n \notin G_x\}$. If $(x, y) \in X$ and $x \neq \infty$ and $y \neq w$, $(x, y)$ is discrete. Let $W_{x,y} = \{(x, y)\}$. It is easily checked that the sets $\{W_x\}$, $\{U_n\}$, and $\{W_{x,y}\}$ from a point-finite open refinement of $\mathscr{U}$. Recalling that a space is metacompact if each open cover has a point-finite open refinement, we see that $X$ is metacompact.

Before we show that $C_0(X)$ has no well-behaved approximate identity, we point out that $X$ is not pseudocompact; thus we cannot simply apply 5.1. In our demonstration that $C_0(X)$ does not have a well-behaved approximate identity, we first exhibit a $\sigma(M(X)), C_b(X))$ convergent sequence $\{\mu_n\}$ which is not *tight*, where a subset $H$ of $M(X)$ is tight and if it is bounded and for each $\varepsilon > 0$ there is a compact set $K_\varepsilon$ in $X$ so that $|\mu|(X \backslash K_\varepsilon) < \varepsilon$ for all $\mu \in H$ ($|\mu|$ denotes the total variation of $\mu$). We may then apply corollary 3.4 in [21] to conclude that $C_0(X)$ does not have a well-behaved approximate identity.

For each positive integer $n$, let $\mu_n$ be the member of $M(X)$ defined by the equation $\mu_n(f) = f((\infty, n)) - f((\infty, n + 1))$ for $f$ in $C_b(X)$. Note that the total variation of $\mu_n$ satisfies the equation $|\mu_n|(f) = f((\infty, n)) + f((\infty, n + 1))$ for $f$ in $C_b(X)$ and so $\|\mu_n\| \leqq 2$ for each integer $n$. We now show that $\mu_n \to 0$ in the weak-* topology of $M(X)$. Let $f \in C_b(X)$ and $n \in N$. Since $f$ is continuous at $(\infty, n)$, for each $\varepsilon > 0$, there is a finite subset $I_{\varepsilon,n}$ of $I$ so that if $x \notin I_{\varepsilon,n}$, $|f(x, n) - f(\infty, n)| < \varepsilon$. Thus there is a countable subset $I_n$ of $I$ so that if $x \notin I_n, f(x, n) = f(\infty, n)$. If $I_f$ is the union of the sets $\{I_n\}$, we see that it is countable and if $x \notin I_f$ then $f(x, n) = f(\infty, n)$ for all integers $n$. Choose a point $x_f \notin I_f$. Then the sequence $\{(x_f, n)\}$ converges to the point $(x_f, w)$ so that $f((x_f, n)) \to f((x_f, w))$. Thus

$$\lim_n f((\infty, n)) = \lim_n f((x_f, n)) = f((x_f, w))$$

so that

$$\lim_n f((\infty, n)) - f((\infty, n + 1)) = 0 \text{ , i.e., } \mu_n(f) \to 0 \text{ .}$$

Since $f$ is arbitrary, we have shown that $\mu_n \to 0$ weak-*.

We next see that $\{\mu_n\}$ cannot be tight: let $\varepsilon = 1/2$ and note that a compact set in $X$ can contain only finitely many of the points $(\infty, n)$. If $K$ is a compact subset of $X$ and $(\infty, p) \notin K$, we can choose $f \in C_b(X)$ so that spt $f$ is compact, $f((\infty, p)) = 1$, $f \equiv 0$ on $K$, and $0 \leqq f \leqq 1$, i.e., so that $|\mu_p|(X\backslash K) \geqq |\mu_p(f)| \geqq |f((\infty, p))| = 1$. Applying [21, Cor. 3.4.], we see that $C_0(X)$ does not have a well-behaved approximate identity (note that $X$ is not paracompact by [20, 3.1 and 3.2]).

REMARK 6.2. The space $C_b(X)_\beta$ where $X$ is as in 6.1 is interesting for several other reasons. First $C_b(X)_\beta$ is not a strong Mackey space (see [7] for a definition). Conway in [7] has shown that $C_b(X)$ is strong Mackey if $X$ is paracompact. The problem of finding topological conditions on $X$ necessary and sufficient for $C_b(X)_\beta$ to be a strong Mackey (or Mackey) space is an intriguing problem. If we let $\mu_n$ be the element of $M(X)$ whose value at $f$ in $C_b(X)$ is $f((\infty, n))$, arguments similar to the above show that $\{\mu_n\}$ is weak* Cauchy but has no weak-* limit in $M(X)$, i.e., $M(X)$ is not weak-* sequentially complete (see [6, 5.1]). $C_b(X)_\beta$ is also not sequentially barrelled (see [23]).

## 7. Miscellaneous remarks.

REMARK 7.1. It is easy to show that if $\{e_\lambda\}$ is a sham compact approximate identity for a (possibly non-abelian) Banach algebra $A$, then $\{e_\lambda\}$ cannot be well-behaved unless $A$ has an identity. The question one really wants to answer is whether $A$ can have another approximate identity that is well-behaved unless $A$ has an identity element. If $A$ is commutative, the question is answered in the negative by 5.1 and 5.6 of this paper. We have the following generalization of Theorem 4.1 in [19]:

THEOREM 7.2. *These are equivalent*: (1) $M(A)_\beta$ *is* $(DF)$ (2) $M(A)_\beta$ *is* $(WDF)$ (3) $l^\infty(A)$ *is both a right and a left essential module* ($l^\infty(A)$ *is the set of all bounded sequences in* $A$; $l^\infty(A)$ *is a right essential module means that if* $\{f_\lambda\}$ *is any approximate identity for* $A$ *and* $x = \{x_n\} \in l^\infty(A)$ *then* $\lim_\lambda (\sup_n \|x_n f_\lambda - x_n\|) = 0$).

PROPOSITION 7.3. *Let* $A$ *have a well-behaved approximate identity and suppose that* $\{e_\lambda\}$ *is a sham compact approximate identity for* $A$. *Then* $A$ *has an identity.*

*Proof.* $x = (x_n) \in l^\infty(A)$; we can choose, by induction, a sequence $\{\lambda_k\}$ from $\Lambda$ so that

$$\lim_k \|e_{\lambda_k}x_n - x_n\| = \lim_k \|x_n e_{\lambda_k} - x_n\| = 0$$

for all positive integers $n$. By the sham compact property, choose $e_\lambda$ so that $\lambda > \lambda_k$ for all integers $k$. Thus $e_\lambda e_{\lambda_k} = e_{\lambda_k}$ so that $e_\lambda x_n = x_n$ for all $n$. Thus $\lim_\lambda (\sup \|x_n e_\lambda - x_n\|) = 0$ and $\lim_\lambda (\sup_n \|e_\lambda x_n - x_n) = 0$, i.e., $l^\infty(A)$ is both left and right essential. Suppose $\{f_\gamma\}$ is a well-behaved approximate identity for $A$ and $\gamma_1 < \gamma_2 < \cdots$ is a sequence in $\Gamma$ so that $0 \neq f_{\gamma_i} \neq f_{\gamma_{i+1}}$ for all integers $i$. Since $l^\infty(A)$ is essential, there is an element $\gamma_0$ is $\Gamma$ so that

$$\|f_{\gamma_0} f_{\gamma_i} - f_{\gamma_i}\| < \frac{1}{4}$$

for all positive integers $i$. Since $\{f_\gamma\}$ is well-behaved, there is a positive integer $N$ so that $n, m \geqq N$ implies that

$$f_{\gamma_0}(f_{\gamma_n} - f_{\gamma_m}) = 0$$

which further implies that $\|f_{\gamma_n} - f_{\gamma_m}\| < 1/2$ for $n, m \geqq N$. Let $C$ be the commutative $B^*$ algebra generated by $\{f_{\gamma_n}: n \geqq N\}$. We claim that $\operatorname{spt} f_{\gamma_N} \subsetneqq N(f_{\gamma_{N+2}})$. If this is not true, then $\operatorname{spt} f_{\gamma_N} = \operatorname{spt} f_{\gamma_{N+1}} = \operatorname{spt} f_{\gamma_{N+2}}$ and so $f_{\gamma_{N+1}} = f_{\gamma_{N+2}} =$ the characteristic function of $\operatorname{spt} f_{\gamma_N}$ by 3.2, contradicting the choice of $\{f_{\gamma_n}\}_{n=1}^\infty$. Thus $\exists x \in N(f_{\gamma_{N+2}}) \backslash \operatorname{spt} f_{\gamma_N}$ which implies that $\|f_{\gamma_{N+3}}(x) - f_{\gamma_N}(x)\| = 1$. This contradiction concludes the proof that a (*nonabelian*) $B^*$ algebra $A$ cannot have both a well-behaved and a sham compact approximate identity.

It is is easy to give an example of a $\beta$ totally bounded approximate identity in $C_0(S)$ that is not canonical (and a fortiori, not well-behaved). Our next result points out the rather interesting fact that in an abelian $B^*$ algebra a canonical chain totally bounded approximate identity is well-behaved.

PROPOSITION 7.4. *Let $\{e_\lambda\}$ be a canonical chain totally bounded approximate identity for $C_0(S)$. Then $\{e_\lambda\}$ is well-behaved.*

*Proof.* Let $\{\lambda_n\}$ be an increasing sequence in $\Lambda$ and $F = \bigcup_{n=1}^\infty \operatorname{spt} f_{\lambda_n}$. Then $F$ is clopen as in the proof of 3.3 and, for any compact subset $K$ of $F$, $K \subset N(e_{\lambda_N})$ for some integer $N$, so that $e_{\lambda_n} \equiv 1$ on $K$ for $n > N$. If $\lambda \in \Lambda$, let $K = \operatorname{spt} e_\lambda \cap F$: then $e_\lambda(e_{\lambda_n} - e_{\lambda_m}) = 0$ for $n$ and $m$ large enough by the preceding remarks. Therefore $\{e_\lambda\}$ is well-behaved.

Taylor [21] prove several interesting theorems about $M(A)$ assuming that the $B^*$ algebra $A$ has a well-behaved approximate identity. From 4.3 and 7.4 we see that (looking at the algebra generated by the approximate identity) an abelian, canonical, and chain totally bounded approximate identity for $A$ is a well-behaved approximate identity so Taylor's theorems hold in this case. We conjecture even more, viz., that if $A$ has a canonical chain totally bounded approximate identity, then the theorems in [21] hold. Our reason for believing this is the next proposition, which shows that a canonical chain totally bounded approximate identity is "almost" well-behaved.

PROPOSITION 7.5. *If $\{e_\lambda\}$ is a canonical chain totally bounded approximate identity in a Banach algebra $A$, then $\{e_\lambda\}$ satisfies the following condition: if $\varepsilon > 0$, $\{\lambda_n\}$ is an increasing sequence in $\Lambda$, and $\lambda \in \Lambda$ there exists a positive integer $N$ so that $n, m > N$ implies*

$$\|e_\lambda(e_{\lambda_n} - e_{\lambda_m})\| < \varepsilon .$$

*Proof.* By chain total boundedness of $\{e_\lambda\}$, there is an integer $P$ so that for all positive integers $n$

$$\min_{1 \leqq p \leqq P} \|e_\lambda(e_{\lambda_n} - e_{\lambda_p})\| < \frac{\varepsilon}{2} .$$

Choose $N \geqq P$ so that if $N < n < p$, $\exists q > p$ so that $\|e_\lambda(e_{\lambda_n} - e_{\lambda_q})\| < \varepsilon$. If $n, m > N$ and $n < m$, choose $q > m$ so that $\|e_\lambda(e_{\lambda_n} - e_{\lambda_q})\| < \varepsilon$. Then $\|e_\lambda(e_{\lambda_n} - e_{\lambda_m})\| = \|e_\lambda(e_{\lambda n} - e_{\lambda_q})e_{\lambda_m}\| \leqq \|e_\lambda(e_{\lambda_n} - e_{\lambda_q})\| < \varepsilon$.

EXAMPLE 7.6. We now give an example of an approximate identity that is *well-behaved* and *not $\beta$ totally bounded.* Let $R$ denote the real line and $\Lambda$ be the set of pairs $(i, j)$ where $i$ is any positive integer and $j = 0$ or $j = 1$. Order $\Lambda$ as follows:

(1) $(i, j) = (i', j')$ if $i = i'$ and $j = j'$;

(2) $(j, 0) > (i, 1)$ for all integers $i$ and $j$;

(3) $(i, 0) > (j, 0)$ if $i > j$.

If $\lambda = (i, 0)$ let $f_\lambda$ be in $C_0(R)$ so that $0 \leqq f_\lambda \leqq 1$ and $f_\lambda \equiv 1$ on $[-i, i]$ and $f_\lambda \equiv 0$ off $[-(i+1), (i+1)]$. If $\lambda = (i, 1)$, let $f_\lambda$ again be in $C_0(R)$ so that $0 \leqq f_\lambda \leqq 1, f_\lambda(x_i) = 1$ where $x_i = 1/2(1/(i+1) + 1/i)$ and $f_\lambda \equiv 0$ off $[1/(i+1), 1/i]$. The net $\{f_\lambda\}$ is easily seen to be well-behaved but the infinite sequence $\{f(i, 1)\}$ is clealy not $\beta$ totally bounded.

EXAMPLE 7.7. In 3.3, we showed that if $C_0(S)$ has an approximate identity that is well-behaved (or $\beta$ totally bounded) then $S$ contains a clopen set $X$ so that $C_0(S) = B_1 \oplus B_2$ where $B_1 = \{f \in C_0(S); f \equiv 0$ on $X\}$ and $B_2 = \{f \in C_0(S): f \equiv 0$ on $S \backslash X\}$ are 2-sided ideals of $C_0(S)$. Obvious non-commutative generalizations of the above fail as we now show. Let $A$ be the algebra of compact operators on a Hilbert space $H$, $\{e_\gamma: \gamma \in \Gamma\}$ an orthonormal basis for $H$, and $\Lambda$ the set of finite subsets of $\Gamma$ ordered by inclusion. If $\lambda \in \Lambda$, let $P_\lambda$ be the finite-dimensional projection defined by the equation

$$P_\lambda(h) = \sum_{\gamma \in \lambda} < h, e_\gamma > e_\gamma \quad \text{for} \quad h \in H .$$

It is easy to show that $\{P_\lambda\}$ is a well-behaved and totally bounded approximate identity for $A$, but $A$ has no non-trivial decomposition as a direct sum of two-sided ideals [14].

REMARK 7.8. It is perhaps worth pointing out that if $C_0(S)$ and $C_0(T)$ have approximate identities with certain properties, so does $C_0(SXT)$ and the converse is also true. Suppose for example that $C_0(S)$ has a well-behaved approximate identity $\{e_\lambda\}$ and $C_0(T)$ has a well-behaved approximate identity $\{f_\alpha\}$. If $f$ and $g \in C_0(S\}$ and $C_0(T)$ respectively let $f \otimes g$ be the function on $S \times T$ defined by $f \otimes g(s, t) = f(s)g(t)$. It is easy to see that $f \otimes g \in C_0(SXT)$. Because the algebra generated by $\left\{f \otimes g \middle| \begin{matrix} f \in C_0(S) \\ g \in C_0(T) \end{matrix}\right\}$ is dense in $C_0(SXT)$ by the Stone-Weierstrass Theorem, the net $\{e_\lambda \otimes f_\alpha\}$ with directed set all pairs $(\lambda, \alpha)$ where $(\lambda, \alpha) > (\lambda', \alpha')$ if $\lambda > \lambda'$ and $\alpha > \alpha'$ is an approximate identity for $C_0(SXT)$ which is easily seen to be well-behaved. Conversely, if $\{e_\lambda\}$ is a well-behaved approximate identity for $C_0(SXT)$ and $t_0 \in T$, the net of function $(f_\lambda)$ defined by $f_\lambda(s) = e_\lambda(s, t_0)$ is a well behaved approximate identity for $C_0(S)$.

EXAMPLE 7.9. Our investigations of $\sigma(M(A), M(A)_\beta^*)$ relatively compact approximate identities is in the first stages only. We wish to present the following example, however, as it seems interesting. Let $S =$ the ordinals less than first uncountable with the order topology. $C_0(S)$ has no $\sigma(C_b(S), M(S))$ relatively compact approximate identity. For, suppose that $C_0(S)$ has an approximate identity $\{e_\lambda\}$ which is $\sigma(C_b(S), M(S)$ relatively compact. Note that $(|e_\lambda|^2)$ is an approximate identity which is also $\sigma(C_b(S), M(S))$ relatively compact, so we may suppose $e_\lambda \geqq 0$. Let $\lambda_1 \in \Lambda$ and $x_1 = \min\{x \in S: y > x \Rightarrow e_{\lambda_1}(y) = 0\}$. Choose $\lambda_2 \in \Lambda$ so that $e_{\lambda_2} > 2/3$ on $[0, x_1 + 1]$ and let $x_2 = \min\{x \in S: y > x \Rightarrow e_{\lambda_2}(y) = 0$. Note $x_2 \geqq x_1 + 1$ so $x_2 > x_1$.

Suppose $\lambda_1, \cdots, \lambda_n$ and $x_1, \cdots, x_n$ have been chosen so that:

(1) $e_{\lambda_k} > k/(k+1)$ on $[0, x_{k-1} + 1]$ for $2 \leqq k \leqq n$

(2) $x_k = \min\{x \in S: y > x \Rightarrow e_{\lambda_k}(y) = 0\}$

(3) $x_n > x_{n-1} > \cdots > x_2 > x_1$.

By induction we select a sequence $(\lambda_n)$ in $\Lambda$ and a sequence $(x_n)$ from $X$ satisfying (1) and (2) and (3). Let $x = \text{lub}\{x_n\}$. By assumption, $\exists f \in C_b(S)$ so that $e_{\lambda_n}$ clusters $\sigma(C_b(S), M(S))$ to $f$. If $y > x$, $e_{\lambda_n}(y) = 0$ for all $n$ so that $f(y) = 0$. If $y < x$, then there is an integer $N$ so that $y < x_n$ for $n > N$ so that $e_{\lambda_n}(y)$ clusters to 1; therefore $f(y) = 1$. We now show that $f$ cannot be continuous at $x$. Since $\{x_n\}$ is strictly increasing, $x_n < x$ for all $n$ so that $e_{\lambda_n}(x) = 0$ for all

$n$ and so $f(x) = 0$; on the other hand, $x_n \to x$, so, if $f$ were continuous, $f(x)$ would be the limit of the constant sequence $f(x_n)$, i.e. 1. This contradiction concludes the proof that $C_0(S)$ has no $\sigma(C_b(S), M(S))$ relatively compact approximate identity.

Our last result (7.10 below) answers only one of a number of questions of the following form: given an algebra $A$ with an approximate identity having property $P$ and another approximate identity $\{e_\lambda\}$, can we select from $\Lambda$ a subset $\Lambda_0$ (cofinal, perhaps) so that $\{e_\lambda: \lambda \in \Lambda_0\}$ has property $P$. Easy examples show that the subset $\Lambda_0$ in 7.10 need not be cofinal in $\Lambda$.

PROPOSITION 7.10. *If a Banach algebra $A$ has a countable approximate identity $\{f_\gamma\}$ and $\{e_\lambda\}$ is another approximate identity, then there is a countable subset $\Lambda_0$ of $\Lambda$ so that $\{e_\lambda: \lambda \in \Lambda_0\}$ is an approximate identity for $A$.*

*Proof.* Choose a countable subset $\Lambda_0$ of $\Lambda$ so that $\lim_{\lambda \in \Lambda_0} e_\lambda f_\gamma = \lim_{\lambda \in \Lambda_0} f_\gamma e_\lambda = f_\gamma$ for each $\gamma \in \Gamma$.

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LOUISIANA STATE UNIVERSITY

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# CONVERGENCE IN SPACES OF SUBSETS

R. J. GAZIK

**Under certain conditions on a class $\mathscr{C}$ of subsets of either a uniform convergence space, uniform space, or bounded metric space, a natural convergence structure for $\mathscr{C}$ is defined which is, respectively, $u$-uniformizable, uniformizable, metrizable. Conditions which are sufficient for the convergence structure to be separated, topological, regular, are given. In the uniform space case some convergence properties of $\mathscr{C}$ are investigated and a fixed point theorem is proved for certain $\mathscr{C}$-multifunctions.**

**1. Introduction.** In order to establish notation and provide some motivation we will, in this section, review a few basic results which deal with uniform convergence structures. The reader is assumed to be familiar with the very basic theorems from the theory of convergence spaces [5].

In order to obtain concepts like Cauchy filter, uniform convergence, total boundedness, which were previously defined only in uniform spaces, Fischer and Cook began the study of uniform convergence spaces in [4]. A uniform convergence structure $\Sigma$ on a set $E$ is an intersection ideal in the collection of filters on $E \times E$ which satisfies the following axioms:

($U_1$) The filter of supersets of the diagonal in $E \times E$ is a member of $\Sigma$.

($U_2$) If $\mathscr{G} \in \Sigma$, so its inverse.

($U_3$) If $\mathscr{C}, \mathscr{F} \in \Sigma$ and the composition filter $\mathscr{G} \circ \mathscr{F}$ exists, then it belongs to $\Sigma$.

A uniform convergence space $(E, \Sigma)$ is a set $E$ along with a convergence structure $\Sigma$ on $E$. A convergence structure $\sigma(\Sigma)$ is induced on $E$ in a natural way: define $\mathscr{F} \in \sigma(\Sigma)$ $(x)$ if and only if $\mathscr{F} \times \dot{x} \in \Sigma$. If $P$ is a property which can be defined by convergence (for instance compactness, regularity, Hausdorffness, etc.) then, by definition, $(E, \Sigma)$ has property $P$ if and only if $\sigma(\Sigma)$ has it. Also, most definitions of uniform properties are available in uniform convergence spaces and are generalizations of the uniform topology case. For example, a filter $\mathscr{F}$ on $E$ is a Cauchy filter if $\mathscr{F} \times \mathscr{F} \in \Sigma$; $(E, \Sigma)$ is complete if each Cauchy filter converges with respect to $\sigma(\Sigma)$; $(E, \Sigma)$ is totally bounded if each filter on $E$ is coarser than a Cauchy filter; a map $f$ between uniform convergence spaces $(E, \Sigma)$, $(F, \psi)$ is uniformly continuous on $E$ if $(f \times f)\Sigma \subset \psi$.

With these definitions one obtains results which, for the most part, parallel the uniform space case. For example, it is true that each uniform convergence space has a completion [8], and that a uniform convergence space is compact if and only if it is complete and totally bounded [4]. The following result is due, independently, to Keller [6] and Cochran [3].

THEOREM 1.1. *Each Hausdorff convergence space* $(E, \delta)$ *is u-uniformizable. That is, there exists a uniform convergence structure* $\Sigma$ *on* $E$ *such that* $\delta = \sigma(\Sigma)$.

If $(E, \Sigma)$ is a uniform convergence space, a subset $\psi$ of $\Sigma$ is a base for $\Sigma$ if each member of $\Sigma$ is finer than a member of $\psi$. The following result (see [4]) shows the relationship between uniform convergence spaces and uniform spaces.

THEOREM 1.2. *If a uniform convergence structure* $\Sigma$ *for* $E$ *has a base consisting of exactly one filter* $\mathscr{U}$ *then* $\mathscr{U}$ *is a uniform structure for* $E$; *each uniform structure* $\mathscr{U}$ *for* $E$ *is a base of exactly one element for a uniform convergence structure* $[\mathscr{U}]$ *for* $E$; $\mathscr{U}$ *and* $[\mathscr{U}]$ *have exactly the same set of Cauchy filters and exactly the same set of convergent filters.*

Now consider the following well known construction: If $(E, \mathscr{U})$ is a uniform space and $\mathscr{C}$ is the class of nonempty, closed subsets of $E$, then a uniform structure for $\mathscr{C}$ is generated by sets of the form $\{(A, B): A, B \in \mathscr{C}, A \subset U(B), B \subset U(A)\}$, $U \in \mathscr{U}$. It follows that a filter $\boldsymbol{\mathscr{F}}$ on $\mathscr{C}$ converges to $A \in \mathscr{C}$, with respect to the completely regular topology on $\mathscr{C}$ induced by $\mathscr{U}$, if and only if for each $U \in \mathscr{U}$, there exists $\mathscr{F} \in \boldsymbol{\mathscr{F}}$ such that $F \subset V(A)$ and $A \subset V(F)$ for each $F \in \mathscr{F}$. The topology induced on $\mathscr{C}$ is called the uniform topology on $\mathscr{C}$ ]7].

The remarks above motivate the consideration of convergence of sets of a class $\mathscr{C}$ (of not necessarily closed sets) in any space where "closeness of sets" is meaningful. We will begin the discussion with uniform convergence spaces. According to Theorem 1.1, these include Hausdorff topological spaces and many others which are not topological spaces.

**2. Convergence classes.** For the remainder of this section a uniform convergence space $(E, \Sigma)$ will be a set $E$ along with a base $\Sigma$ for a uniform convergence structure on $E$.

DEFINITION 2.1. Let $(E, \Sigma)$ be a uniform convergence space. A nonempty class $\mathscr{C}$ of nonempty subsets of $E$ is called a convergence class for $(E, \Sigma)$ if and only if $A \subset V(A)$ for each $A \in \mathscr{C}$, $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$.

DEFINITION 2.2. Let $(E, \Sigma)$ be a uniform convergence space and let $\mathscr{C}$ be a nonempty collection of nonempty subsets of $E$. The function $\tau(\mathscr{C})$ from $\mathscr{C}$ into the power set of the filters on $\mathscr{C}$ is defined by $\boldsymbol{\mathscr{F}} \in \tau(\mathscr{C})(A)$ if and only if for each $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$, there exists $\mathscr{F} \in \boldsymbol{\mathscr{F}}$ such that $F \subset V(A)$ and $A \subset V(F)$ for each $F \in \mathscr{F}$.

THEOREM 2.1. *The function $\tau(\mathscr{C})$ of Definition 2.2 is a convergence structure on $\mathscr{C}$ if and only if $\mathscr{C}$ is a convergence class for $(E, \Sigma)$.*

*Proof.* It is clear that if $\boldsymbol{\mathscr{F}} \in \tau(\mathscr{C})(A)$ and $\boldsymbol{\mathscr{G}}$ is finer than $\boldsymbol{\mathscr{F}}$, then $\boldsymbol{\mathscr{G}} \in \tau(\mathscr{C})(A)$. If $\boldsymbol{\mathscr{F}}, \boldsymbol{\mathscr{G}} \in \tau(\mathscr{C})(A)$ then, since $\{\mathscr{F} \cup \mathscr{G} : \mathscr{F} \in \boldsymbol{\mathscr{F}}, \mathscr{G} \in \boldsymbol{\mathscr{G}}\}$ is a base for $\boldsymbol{\mathscr{G}} \wedge \boldsymbol{\mathscr{G}}$, $\boldsymbol{\mathscr{F}} \wedge \boldsymbol{\mathscr{G}} \in \tau(\mathscr{C})(A)$. Hence, $\tau(\mathscr{C})$ is a convergence structure for $\mathscr{C}$ if and only if the ultrafilter generated by $A$ is in $\tau(\mathscr{C})(A)$ for each $A \in \mathscr{C}$. But this is equivalent to the statement that $A \subset V(A)$ for each $A \in \mathscr{C}$, $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$.

Some additional properties which may sometimes be required of a convergence class $\mathscr{C}$ for a uniform convergence space $(E, \Sigma)$ are:

$(A_1)$ If $A, B \in \mathscr{C}$ and $A \subset V(B)$ for each $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$, then $A \subset B$.

$(A_2)$ If $A, B \in \mathscr{C}$ and $A \subset V(B)$, $B \subset V(A)$ for each $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$, then $A = B$.

$(A_3)$ For each $\mathscr{J} \in \Sigma$ and $V \in \mathscr{J}$, there exists $U \in \mathscr{J}$ such that $U^2(A) \subset V(A)$ for all $A \in \mathscr{C}$.

$(A_4)$ For each $\mathscr{J} \in \Sigma$, $V \in \mathscr{J}$, $A \in \mathscr{C}$, there exists $U \in \mathscr{J}$ such that $U^2(A) \subset V(A)$.

$(A_5)$ If $A, B \in \mathscr{C}$, then $A \cup B \in \mathscr{C}$.

THEOREM 2.2. *Let $\mathscr{C}$ be a convergence class for the uniform convergence space $(E, \Sigma)$. Then*

(1) *If either $(A_1)$ and $(A_4)$ or $(A_2)$ and $(A_4)$ hold, then $\tau(\mathscr{C})$ is separated and u-uniformizable.*

(2) *If $(A_5)$ holds and $\tau(\mathscr{C})$ is separated, then $(A_1)$ and $(A_2)$ hold.*

(3) *$(A_1)$ implies $(A_2)$ and, if $(A_5)$ holds, $(A_2)$ implies $(A_1)$.*

*Proof.* (1) Suppose $\boldsymbol{\mathscr{F}} \in \tau(\mathscr{C})(A) \cap \tau(\mathscr{C})(B)$ and let $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$. By $(A_4)$ there exist $U, W \in \mathscr{J}$ such that $U^2(B) \subset V(B)$, $W^2(A) \subset V(A)$. Then $S = U \cap W \in \mathscr{J}$ so $F \subset S(A)$, $A \subset S(F)$, $B \subset S(F)$, $F \subset S(B)$ for all $F \in \mathscr{F}$ and some $\mathscr{F} \in \boldsymbol{\mathscr{F}}$. From these relations, $A \subset U^2(B) \subset V(B)$ and $B \subset W^2(A) \subset V(A)$ so, if either $(A_1)$ or $(A_2)$ hold, $A = B$; that is, $\tau(\mathscr{C})$ is separated. It follows from Theorem 1.1 that $\tau(\mathscr{C})$ is $u$-uniformizable.

(2) Suppose $A \subset V(B)$ for all $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$. Then $A \cup B \subset V(B)$ and $B \subset V(A \cup B)$ so, since $(A_5)$ holds, the ultrafilter generated by $B$ converges to $A \cup B$. Since $\tau(\mathscr{C})$ is separated $B = A \cup B$ so $A \subset B$. A similar argument shows that $(A_2)$ holds.

(3) If $(A_1)$ holds, $(A_2)$ holds. If $A \subset V(B)$ for all $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$, then, since $A \cup B \in \mathscr{C}$, $A \cup B \subset V(B)$ and $B \subset V(A \cup B)$; it follows that $A \cup B = B$ and $A \subset B$.

THEOREM 2.3. *Let $\mathscr{C}$ be a convergence class for a uniform convergence space $(E, \Sigma)$. If $(A_3)$ holds then $\tau(\mathscr{C})$ is a topological space; that is, there is a topology $\sigma(\mathscr{C})$ on $\mathscr{C}$ such that a filter $\boldsymbol{\mathscr{F}}$ converges to $A \in \mathscr{C}$ with respect to $\sigma(\mathscr{C})$ if and only if $\boldsymbol{\mathscr{F}} \in \tau(\mathscr{C})(A)$.*

*Proof.* It suffices to show that if $\boldsymbol{\mathscr{F}} \notin \tau(\mathscr{C})(A)$, then there exists $\mathscr{X}$, a subset of $\mathscr{C}$, such that $A \in \mathscr{X}$, $\mathscr{X} \notin \boldsymbol{\mathscr{F}}$ and if $B \in \mathscr{X}$, $\boldsymbol{\mathscr{G}} \in \tau(\mathscr{C})(B)$, then $\mathscr{X} \in \boldsymbol{\mathscr{G}}$.

Now suppose $\boldsymbol{\mathscr{F}} \notin \tau(\mathscr{C})(A)$. Then for some $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$, no $\mathscr{F} \in \boldsymbol{\mathscr{F}}$ satisfies

(1) $F \in \mathscr{F}$ implies $F \subset V(A)$ and $A \subset V(F)$.

Define a subset $\mathscr{X}$ of $\mathscr{C}$ as follows: $\mathscr{X}$ consists of all $B \in \mathscr{C}$ such that

(2) $B \subset V(B)$, and $A \subset V(B)$, and

(3) there exists $U \in \mathscr{J}$ such that if $H \in \mathscr{C}$ and $H \subset U(B)$ and $B \subset U(H)$, then $H \subset V(A)$ and $A \subset V(H)$.

Now $A \in \mathscr{X}$ for $A \subset V(A)$ and we may take the $U$ required by (3) to be $V$. $\mathscr{X} \notin \boldsymbol{\mathscr{F}}$ by (1) and (2).

Suppose now that $\boldsymbol{\mathscr{G}} \in \tau(\mathscr{C})(B)$, $B \in \mathscr{X}$. We show $\mathscr{X} \in \boldsymbol{\mathscr{G}}$ by proving that $\mathscr{X}$ contains a member of $\boldsymbol{\mathscr{G}}$.

Since $B \in \mathscr{X}$, condition (2) holds for some $U \in \mathscr{J}$. By $(A_3)$ there exists $W \in \mathscr{J}$ such that $W^2(D) \subset U(D)$ for all $D \in \mathscr{C}$. Since $\boldsymbol{\mathscr{G}} \in \tau(\mathscr{C})(B)$, there exists $\mathscr{G} \in \boldsymbol{\mathscr{G}}$ such that $G \subset W(B)$ and $B \subset W(G)$ for all $G \in \mathscr{C}$.

Let $G \in \mathscr{G}$. Since $B \in \mathscr{X}$ and $G \subset W(B)$, $B \subset W(G)$, then $G \subset V(A)$ and $A \subset V(G)$ so $G$ satisfies (2).

Suppose $H \in \mathscr{C}$ and $H \subset W(G)$, $G \subset W(H)$. Then $H \subset W^2(B) \subset U(B)$ and $B \subset W(G) \subset W^2(H) \subset U(H)$ so, since $B \in \mathscr{X}$, $H \subset V(A)$ and $A \subset V(H)$. This shows that each $G \in \mathscr{G}$ satisfies (3).

In summary, $\mathscr{G} \subset \mathscr{X}$, $\mathscr{G} \in \boldsymbol{\mathscr{G}}$, so $\mathscr{X} \in \boldsymbol{\mathscr{G}}$.

THEOREM 2.4. *With the same assumptions as in Theorem 2.3, the topological space $(\mathscr{C}, \sigma(\mathscr{C}))$ is regular.*

*Proof.* Recall first that a net in a topological space converges to a point if and only if its filter of final sections converges to the same point. In the present context it follows from the previous theorem that a net $(A_n: n \in D)$ in $\mathscr{C}$ $\sigma(\mathscr{C})$-converges to $A \in \mathscr{C}$ (written $(A_n: n \in D) \to A$) if and only if for each $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$, $A_n \subset V(A)$ and $A \subset V(A_n)$ for $n$ sufficiently large.

Now let $(A_{ij}: i \in I, j \in J_i)$ be a simply convergent double net in $\mathscr{C}$ with $(A_{ij}: j \in J_i) \to P_i \in \mathscr{C}$ for each $i \in I$. Let $h$ be the diagonal net on $T = I \times \Pi(J_i: i \in I)$ defined by $h(i, g) = A_{i,g(i)}$ and suppose the diagonal net converges to $X \in \mathscr{C}$. We prove $(\mathscr{C}, \sigma(\mathscr{C}))$ is regular by showing that $(P_i: i \in I) \to X$.

Let $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$. By $(A_3)$ there exists $U \in \mathscr{J}$ such that $U^2(B) \subset V(B)$ for all $B \in \mathscr{C}$. Since the diagonal net converges to $X$,

(1) $A_{i,g(i)} \subset U(X)$, $X \subset U(A_{i,g(i)})$ for $(i, g) \geqq (i_0, g_0)$. Since each $(A_{i\,j}: j \in J_i) \to P_i$,

(2) $A_{ij} \subset U(P_i)$, $P_i \subset U(A_{ij})$ for each $i \geqq i_0$ and $j \geqq j(i, V)$. Define $w \in \Pi(J_i: i \in I)$ by requiring $w(i)$ to be greater than or equal to both $g_0(i)$, $j(i, V)$ if $i \geqq i_0$ and $w(i) = g_0(i)$ otherwise. Then, for $i \geqq i$, $(i, w) \geqq (i_0, g_0)$ so by (1), $A_{i,w(i)} \subset U(X)$ and $X \subset U(A_{i,w(i)})$. By (2) $A_{i\;w(i)} \subset U(P_i)$ and $P_i \subset U(A_{i,w(i)})$. Hence, for $i \geqq i_0$, $P_i \subset U^2(X) \subset V(X)$ and $X \subset U^2(P_i) \subset V(P_i)$. It follows that $(P_i: i \in I) \to X$ and $(\mathscr{C}, \sigma(\mathscr{C}))$ is regular.

It should be pointed out that a number of other natural convergences on a convergence class $\mathscr{C}$ might be studied. The following are a few such examples.

(1) $\boldsymbol{\mathscr{F}} \in \psi(\mathscr{C})(A)$ if and only if there exists $\mathscr{J} \in \Sigma$ such that: for each $V \in \mathscr{J}$ there is an $\mathscr{F} \in \boldsymbol{\mathscr{F}}$ such that $F \subset V(A)$ and $A \subset V(F)$ for each $F \in \mathscr{F}$.

(2) $\boldsymbol{\mathscr{F}} \in \lambda(\mathscr{C})(A)$ if and only if for each $V \in \mathscr{J}$, $\mathscr{J} \in \Sigma$, $a \in A$, there exists $\mathscr{F} \in \boldsymbol{\mathscr{F}}$ such that $F \subset V(A)$ and $F \cap V(a) \neq \phi$ for each $F \in \mathscr{F}$.

(3) $\boldsymbol{\mathscr{F}} \in \alpha(\mathscr{C})(A)$ if and only if there exists $\mathscr{J} \in \Sigma$ such that: for each $V \in \mathscr{J}$, $a \in A$, there exists $\mathscr{F} \in \boldsymbol{\mathscr{F}}$ such that $F \subset V(A)$ and $F \cap V(a) \neq \phi$ for each $F \in \mathscr{F}$.

EXAMPLE 1. Let $\Sigma$ consist of just one uniform structure $\mathscr{J}$ for $E$ and let $\mathscr{C}$ be any nonempty class of nonempty subsets of $E$. By Theorem 1.2 $\Sigma$ is a base for a uniform convergence structure on $E$. Clearly $A \subset V(A)$ for each $V \in \mathscr{J}$, $A \in \mathscr{C}$, so $\mathscr{C}$ is a convergence class for $(E, \Sigma)$. In particular, if $\mathscr{C}$ is the class of nonempty $\mathscr{J}$ closed subsets of $E$, then, by the discussion at the end of §1, $\tau(\mathscr{C})$ convergence is precisely the convergence of closed sets in the uniform topology on $\mathscr{C}$. (See [7].)

EXAMPLE 2. Let $E$ be a Hausdorff topological space and, for each finite subset $S$ of $E$, define $\mathscr{J}(S)$ to be the filter $\wedge\,(\mathscr{N}(x) \times \mathscr{N}(x): x \in S) \wedge \mathscr{D}$, where $\mathscr{D}$ is the filter of supersets of the diagonal in

$E \times E$ and $\mathscr{N}(x)$ is the neighborhood filter at $x$. The collection $\Sigma = \{\mathscr{J}(S): S$ is a finite subset of $E\}$ is a base for a uniform convergence structure on $E$. It is not hard to see that the convergence induced by $\Sigma$ is precisely convergence in the topological space $E$. Each member of each $\mathscr{J}(S)$ contains the diagonal so any $\mathscr{C}$ is a convergence class for $(E, \Sigma)$.

If $E$ is a closed interval of real numbers, $\mathscr{C}$ the class of nonempty closed subsets of $E$ and $\mathscr{J}$ the usual uniform structure on $E$, then, by Example 1 and results from [7], $(\mathscr{C}, \tau(\mathscr{C}))$ is compact with respect to the base $\{\mathscr{J}\}$. Now the base $\Sigma$ of Example 2 induces the same convergence on $E$ as does $\{\mathscr{J}\}$, but $(\mathscr{C}, \tau(\mathscr{C}))$ is not compact with respect to $\Sigma$.

*Question.* If $\mathscr{C}$ is the class of nonempty, closed subsets of a compact uniform convergence space $(E, \Sigma)$, is there a base $\Phi$ for a uniform convergence structure on $E$ such that $\Sigma$ and $\Phi$ induce the same convergence on $E$, and $(\mathscr{C}, \tau(\mathscr{C}))$ is compact with respect to $\Phi$?

EXAMPLE 3. Let $E$ be a Hausdorff topological space and $\mathscr{C}$ any collection of nonempty subsets of $E$. Define $\mathscr{J}$ to be the filter generated by sets of the form $\cup(G_i \times G_i: i \in I)$ where $I$ is finite, each $G_i$ is open and $\cup(G_i: i \in I) = E$. The collection $\Sigma = \{\mathscr{J}, \mathscr{J}^2, \mathscr{J}^3, \cdots\}$ is a base for a convergence structure on $E$ and $\mathscr{C}$ is a convergence class for $(E, \Sigma)$. The topological convergence on $E$ is generally not the same as that induced by $\Sigma$. In this case $\boldsymbol{\mathscr{F}} \in \tau(\mathscr{C})(A)$ if and only if for each $V \in \mathscr{J}$ and each natural number $n$, there exists $\mathscr{F} \in \boldsymbol{\mathscr{F}}$ such that $F \subset V^n(A)$ and $A \subset V^n(F)$ for each $F \in \mathscr{F}$.

EXAMPLE 4. Let $E$ be a regular, Hausdorff topological space, $\mathscr{C}$ the class of nonempty, closed subsets of $E$ and $\Sigma$ the base of the previous example. Then $\lambda(\mathscr{C})$ convergence is precisely the convergence of closed sets defined by Choquet on p. 90 of [2].

*Question.* If $E$ is a topological space, $\mathscr{C}$ its convergence class of closed sets, is there a base $\Sigma$ for a uniform convergence structure on $E$ such that one of the natural convergences $\tau(C), \lambda(\mathscr{C})$, etc. induces the convergence defined by Choquet on p. 87 of [2]?

Of course, the meaning of $\tau(\mathscr{C})$, $\psi(\mathscr{C})$, $\lambda(\mathscr{C})$ or $\alpha(\mathscr{C})$ convergence is known as soon as a base for a uniform convergence structure is given. In this regard, see [3] for an explicit construction of a uniform convergence structure for an arbitrary Hausdorff convergence space, and see [4] for construction of natural uniform convergence structures on function spaces.

**3. Convergence classes for uniform spaces.** Let $(E, \mathscr{J})$ be a uniform space and let $\mathscr{C}$ be any nonempty class of nonempty subsets of $E$. Since $A \subset V(A)$ for $V \in \mathscr{J}$ and $(A_3)$ of §2 holds, $\mathscr{C}$ is a convergence class for $(E, \mathscr{J})$ and $\tau(\mathscr{C})$ induces a regular topology $\sigma(\mathscr{C})$ on $\mathscr{C}$. A net $(A_n: n \in D)$ in $\mathscr{C}$ $\sigma(\mathscr{C})$-converges to $A \in \mathscr{C}$ if and only if for each $V \in \mathscr{J}$, $A_n \subset V(A)$ and $A \subset V(A_n)$ for $n$ sufficiently large. In fact, we have the following:

THEOREM 3.1. *If $(E, \mathscr{J})$ is a uniform space and $\mathscr{C}$ is a nonempty collection of nonempty subsets of $E$, then the topological space $(\mathscr{C}, \sigma(\mathscr{C}))$ is uniformizable.*

*Proof.* For each $V \in \mathscr{J}$, define $\mathscr{T}(V) = \{(A, B): A, B \in \mathscr{C}, A \subset V(B), B \subset V(A)\}$. Then each $\mathscr{T}(V)$ contains the diagonal in $\mathscr{C} \times \mathscr{C}$ and the inverse of $\mathscr{T}(V)$ is itself. Also $\mathscr{T}(V) \supset \mathscr{T}(U) \circ \mathscr{T}(U)$ if $U \circ U \subset V$. Thus $\mu(\mathscr{C})$, the filter generated by the $\mathscr{T}(V)$'s, is a uniform structure for $\mathscr{C}$. But, from the definitions and the remarks preceding the theorem, a net $\sigma(\mathscr{C})$-converges to $A \in \mathscr{C}$ if and only if it converges to $A$ with respect to the topology generated by $\mu(\mathscr{C})$.

Some additional axioms which may sometimes be required of a convergence class $\mathscr{C}$ for a uniform space $(E, \mathscr{J})$ are:

$(B_1)$ If $A, B \in \mathscr{C}$, then $A \cup B \in \mathscr{C}$.

$(B_2)$ If $A \subset \operatorname{clos} B$, then $A \subset B$.

$(B_3)$ If $A \subset \operatorname{clos} B$ and $B \subset \operatorname{clos} A$, then $A = B$.

$(B_4)$ If $S$ is linearly ordered and $(A_n: n \in S)$ is a decreasing net in $\mathscr{C}$ ($n \geqq m$ implies $A_n \subset A_m$) such that $\cap A_n \neq \phi$, then any net $(x_n: n \in R)$ with $R$ cofinal in $S$ and $x_n \in A_n$ for $n \in R$, which converges, converges to a point in $\operatorname{clos}(\cap A_n)$.

THEOREM 3.2. *If $\mathscr{C}$ is a convergence class for a uniform space $(E, \mathscr{J})$, then*

(1) *If $(B_2)$ or $(B_3)$ is satisfied, $(\mathscr{C}, \mu(\mathscr{C}))$ is Hausdorff.*

(2) *If $(\mathscr{C}, \mu(\mathscr{C}))$ is Hausdorff and $(B_1)$ holds, then $(B_2)$ and $(B_3)$ hold.*

(3) *$(B_2)$ implies $(B_3)$ and, if $(B_1)$ holds, $(B_3)$ implies $(B_2)$.*

*Proof.* This follows from Theorem 2.2.

EXAMPLE 5. A simple example of a convergence class $\mathscr{C}$ for a uniform space $(E, \mathscr{J})$ for which $(\mathscr{C}, \mu(\mathscr{C}))$ is Hausdorff and $\mathscr{C}$ does not consist of closed sets is obtained by taking $\mathscr{C}$ to be the class of all nonempty, regular open subsets of $E$. Recall that an open set $G$ is regular open if $G = \operatorname{Int}(\operatorname{clos} G)$. It is clear, then, that $\mathscr{C}$ satisfies $(B_3)$ so $(\mathscr{C}, \mu(\mathscr{C}))$ is Hausdorff.

DEFINITION 3.1. A net $(A_n: n \in D)$ in $\mathscr{C}$ is increasing (decreasing) if $D$ is totally ordered and $n \geqq m$ implies $A_n \supset A_m (A_m \supset A_n)$.

THEOREM 3.3 *Let* $(E, \mathscr{J})$ *be a compact uniform space and* $\mathscr{C}$ *a convergence class for* $(E, \mathscr{J})$. *Then*

(1) *An increasing net* $(A_n: n \in D)$ *in* $\mathscr{C}$ *converges if and only if there exists* $A \in \mathscr{C}$ *such that* $\cup A_n \subset \operatorname{clos} A$ *and* $A \subset \operatorname{clos}(\cup A_n)$.

(2) *If* $(A_n: n \in D)$ *is a decreasing net in* $\mathscr{C}$, $\cap A_n \neq \phi$ *and* $(B_4)$ *is satisfied, then* $(A_n: n \in D)$ *converges if and only if there exists* $A \in \mathscr{C}$ *such that* $A \subset \operatorname{clos}(\cap A_n)$ *and* $\cap A_n \subset \operatorname{clos} A$.

*Proof*. A proof of (1) is given. The proof of (2) is similar. If $(A_n: n \in D) \rightarrow A$ then, if $V \in \mathscr{J}$, $A_n \subset V(A)$ and $A \subset V(A_n)$ for $n$ sufficiently large. But since $(A_n: n \in D)$ is increasing, $\cup A_n \subset V(A)$ and $A \subset V(\cup A_n)$. Since $V$ was arbitrary, $\cup A_n \subset \operatorname{clos} A$ and $A \subset \operatorname{clos}(\cup A_n)$.

Now suppose $A \in \mathscr{C}$ exists which satisfies $\cup A_n \subset \operatorname{clos} A$ and $A \subset \operatorname{clos}(\cup A_n)$. Then, for $V \in \mathscr{J}$, $n \in D$, $A_n \subset V(A)$. Thus, to show $(A_n: n \in D) \rightarrow A$ it suffices to show that $A \subset V(A_n)$ for some $n \in D$.

Suppose this is not so. Then there are points $y_n \in A - V(A_n)$. The net $(y_n: n \in D)$ has a convergent subnet by the compactness of $(E, \mathscr{J})$. Clearly, the subnet converges to a point $x \in \operatorname{clos} A \subset \operatorname{clos}(\cup A_n)$. If $U^2 \subset V$, then $U(x) \cap A_n \neq \phi$ for $n$ sufficiently large. But $(y_n: n \in D)$ is frequently in $U(x)$ so there is an index $n \in D$ such that $y_n \in U(x)$, $t_n \in U(x)$, $t_n \in A_n$. Then $y_n \in U^2(t_n) \subset V(t_n) \subset V(A_n)$ which is a contradiction.

DEFINITION 3.2. If $\mathscr{C}$ is a convergence class for $(E, \mathscr{J})$ then $(\mathscr{C}, \mu(\mathscr{C}))$ is said to be monotone complete if and only if each increasing net in $(\mathscr{C}, \mu(\mathscr{C}))$ converges and each decreasing net $(A_n: n \in D)$ for which $\cap A_n \neq \phi$ converges.

THEOREM 3.4. *Let* $(E, \mathscr{J})$ *be a uniform space. Then*

(1) *If* $f: (E, \mathscr{J}) \rightarrow (E, \mathscr{J})$ *is uniformly continuous and* $\mathscr{C}$ *is any convergence class for* $(E, \mathscr{J})$ *such that* $A \in \mathscr{C}$ *implies* $f(A) \in \mathscr{C}$ *then* $g: (\mathscr{C}, \mu(\mathscr{C})) \rightarrow (\mathscr{C}, \mu(\mathscr{C}))$ *defined by* $g(A) = f(A)$ *is uniformly continuous.*

(2) *If* $(\mathscr{C}, \mu(\mathscr{C}))$ *is separated and monotone complete, then either,*

(a) $g(A) = A$ *for some* $A \in \mathscr{C}$, *or*

(b) *there exists* $A \in \mathscr{C}$ *such that* $g(A) \subset A$ *and* $\cap (g^n(A): n = 1, 2, \cdots) = \phi$ *or*

(c) $g(A), A$ *are not comparable for each* $A \in \mathscr{C}$.

*Proof*. (1) If $f$ is uniformly continuous then $(f \times f)\mathscr{J} \geqq \mathscr{J}$. Then, if $\mathscr{T}(V)$ is a generator of $\mu(\mathscr{C})$, there exists $U \in J$ such that

$V \supset (f \times f)(U)$. It is an easy computation to show that $(g \times g)\mathscr{T}(U) \subset \mathscr{T}(V)$ so $(g \times g)\mu(\mathscr{C}) \geqq \mu(\mathscr{C})$ and $g$ is uniformly continuous.

(2) If $A \subset g(A)$ for some $A \in \mathscr{C}$, then $A \subset g(A) \subset g^2(A) \cdots$ is a monotone net in $\mathscr{C}$ and hence converges to $B \in \mathscr{C}$. By (1) $g(A) \subset g^2(A) \subset \cdots$ converges to $g(B)$. Since $(\mathscr{C}, \mu(\mathscr{C}))$ is separated, $B = g(B)$.

If $A \supset g(A)$ for some $A \in \mathscr{C}$, $A \supset g(A) \supset g^2(A) \cdots$ is a decreasing net in $(\mathscr{C}, \mu(\mathscr{C}))$. If it is true that $(g^n(A): n = 1, 2, \cdots) \neq \phi$, then $g^n(A) \to B$, $g^n(A) \to g(B)$ and $B = g(B)$. Hence, if neither (a) nor (b) holds, it must be that $A \not\subset g(A)$ and $g(A) \not\subset A$ for all $A \in \mathscr{C}$. That is, (c) holds.

Recall that if $f: (E, \mathscr{J}) \to S$ is a bijection, then there is a finest uniform structure for $S$ which makes $f$ uniformly continuous, namely $(f \times f)(\mathscr{J})$.

DEFINITION 3.3. If $\mathscr{C}_1, \mathscr{C}_2$ are convergence classes for $(E_1, \mathscr{J}_1)$, $(E_2, \mathscr{J}_2)$ respectively, the natural uniformity $\mu[\mathscr{C}_1, \mathscr{C}_2]$ on $[\mathscr{C}_1, \mathscr{C}_2] = \{A \times B: A \in \mathscr{C}_1, B \in \mathscr{C}_2\}$ is the finest uniform structure on $[\mathscr{G}_1, \mathscr{C}_2]$ which makes the bijection $f: (\mathscr{C}_1 \times \mathscr{C}_2, \mu(\mathscr{C}_1) \times \mu(\mathscr{C}_2)) \to [\mathscr{C}_1, \mathscr{C}_2]$ defined by $f(A, B) = A \times B$ uniformly continuous.

THEOREM 3.5. (1) *Let $\mathscr{C}_1, \mathscr{C}_2$ be convergence classes for $(E, \mathscr{J})$. Then $(A_n \times B_n: n \in D)$ converges to $A \times B$ in $([\mathscr{C}_1, \mathscr{C}_2], \mu[\mathscr{C}_1, \mathscr{C}_2])$ if and only if $(A_n: n \in D)$, $(B_n: n \in D)$ converge to $A$, $B$ in $(\mathscr{C}_1, \mu(\mathscr{C}_1))$, $(\mathscr{C}_2, \mu(\mathscr{C}_2))$ respectively.*

(2) *If $(A_n: n \in D)$, $(B_n: n \in D)$ are nets in $(\mathscr{C}, \mu(\mathscr{C}))$ which converge to $A, B$ respectively and $A_n \subset B_n$ for $n$ sufficiently large, then $A \subset clos\ B$.*

*Proof.* (1) If $(A_n: n \in D) \to A$ and $(B_n: n \in D) \to B$, then $(A_n \times B_n: n \in D) \to A \times B$ by the continuity of the map $f: (A, B) \to A \times B$. If $(A_n \times B_n: n \in D) \to A \times B$ and $V \in \mathscr{J}$, then, when $\mathscr{T}(V) = \{(S, M): S \subset V(M), M \subset V(S)\}$, $\mathscr{H}(V) = \{((R, Y), (F, X)): (R, F) \in \mathscr{T}(V), (Y, X) \in \mathscr{T}(V)\}$ is in $\mu(\mathscr{C}_1) \times \mu(\mathscr{C}_2)$. Thus, by definition, $(f \times f)\mathscr{H}(V)(A \times B) = \{R \times Y: (R, A) \in \mathscr{T}(V), (Y, B) \in \mathscr{T}(V)\}$ is a neighborhood of $A \times B$. It follows that $(A_n, A) \in \mathscr{T}(V)$ and $(B_n, B) \in \mathscr{T}(V)$ for $n$ sufficiently large so $(A_n: n \in D) \to A$ and $(B_n: n \in D) \to B$.

(2) We have for $V \in \mathscr{J}$, an index $n \in D$ such that $A_n \subset B_n$, $A \subset V(A_n)$, $B_n \subset V(B)$ so $A \subset V^2(B)$ and the result follows from this fact.

The result above, as well as the theorem below will be used in the next section.

THEOREM 3.6. *Let $\mathscr{C}$ be a convergence class for the the uniform space $(E, \mathscr{J})$. Then*

(1) *If $(A_n: n \in D) \to A$ and $x \in A$, then there exists a directed set*

*$H$ and functions $p: H \to D$, $m: H \to E$, such that $p(H)$ is cofinal in $D$, the net $m$ converges to $x$ and $m(h) \in A_{p(h)}$ for all $h \in H$.*

(2) *If $(A_n: n \in D) \to A$ and a net $m: H \to E$ converges to $x$ with $m(h) \in A_{p(h)}$, $p(H)$ cofinal in $D$, $p: H \to D$, then $x \in clos\ A$.*

*Proof.* (1) Order $D \times \mathscr{J}$ by $(n, V) \geqq (m, U)$ if $n \geqq m$ and $V \subset U$. By convergence, if $(n, V) \in D \times \mathscr{J}$ there exists $p(n, V) \in D$ and $m(n, V) \in A_{p(n,V)}$ such that $p(n, V) \geqq (x, m(n, V)) \in V$. The result follows from this.

(2) If $V \in \mathscr{J}$, $m(h) \in V(x) \cap A_{p(h)}$ for $h$ sufficiently large. But, by convergence, there is an index $h$ such that $A_{p(h)} \subset V(A)$ also. It follows that for some $a \in A$, some $h \in H$, $(m(h), x) \in V$, $(m(h), a) \in V$. Thus $a \in V^2(x)$ and the result follows.

**4. Fixed point theorem for $\mathscr{C}$-multifunctions.** Let $\mathscr{C}$ be a convergence class for the uniform space $(E, \mathscr{J})$. If $F:(E, \mathscr{J}) \to (\mathscr{C}, \mu(\mathscr{C}))$ is a function, then $F^n$, $n = 2, 3, \cdots$ is defined inductively as follows. (Notice that $F^n(x)$ need not be in $\mathscr{C}$ if $n > 1$.) If $x \in E$, $F^2(x) = \cup F(y): y \in F(x))$ and $F^{n+1}(x) = \cup F(y): y \in F^n(x))$ for $n > 2$. If $F^n(x) \in \mathscr{C}$ for each $n$ and each $x \in E$, then $F$ is called a $\mathscr{C}$-multifunction.

DEFINITION 4.1. A $\mathscr{C}$-multifunction $F: (E, \mathscr{J}) \to (\mathscr{C}, \mu(\mathscr{C}))$ is condensing if $F$ is continuous and $V \in \mathscr{J}$, $x \neq y$, $x, y \in E$ implies there exists $n = n(x, y, V)$ such that $F^n(x) \times F^n(y) \subset V$.

EXAMPLE 6. With respect to the hypotheses of the next theorem, we remark that $(\mathscr{C}, \mu(\mathscr{C}))$ can be compact without $\mathscr{C}$ consisting only of closed sets. Let $E$ be the closed unit interval and let $\mathscr{C}$ consist of all subintervals (open, closed, or half open, half closed) of $E$ along with all singleton subsets of $E$. Then $(\mathscr{C}, \mu(\mathscr{C}))$ is compact.

THEOREM 4.1. *If $(E, \mathscr{J})$ is compact and Hausdorff, $(\mathscr{C}, \mu(\mathscr{C}))$ is compact and $F: (E, \mathscr{J}) \to (\mathscr{C}, \mu(\mathscr{C}))$ is condensing, then there exists $x_0 \in E$ such that $x_0 \in clos\ F(x_0)$.*

*Proof.* Suppose $x \notin F(x)$. Then for some $y \in F(x)$, $y \neq x$. If $V \in \mathscr{J}$, there exists $n(V)$ such that

(1) $F^{n(V)}(x) \times F^{n(V)}(y) \subset V$.

Since $(\mathscr{C}, \mu(\mathscr{C}))$ is compact so is $[\mathscr{C}, \mathscr{C}, \mathscr{C}]$ by Definition 3.3. Hence, with $\mathscr{J}$ directed by reverse inclusion, the net $p$ defined by $p = (F^{n(V)}(x) \times F^{n(V)}(y) \times F^{n(V)+1}(x): V \in \mathscr{J})$ has a convergent subnet $t: D \to [\mathscr{C}, \mathscr{C}, \mathscr{C}]$. If $t \to A \times B \times T$, then by (1) and Theorem 3.5, $A \times B \subset V$ for each $V \in \mathscr{J}$. Since $(E, \mathscr{J})$ is Hausdorff, $A \times B$ is contained in the diagonal of $E \times E$ so $A = B = \{x_0\}$ for some $x_0 \in E$.

Now $y \in F(x)$ so $F^{n(V)}(y) \subset F^{n(V)+1}(x)$. It follows from Theorem 3.5 that $x_0 \in \text{clos } T$.

Consider $z \in T$. By Theorem 3.6 and the fact that $t$ is a subnet of $p$, there is a net $m: H \rightarrow E$ and a function $f: H \rightarrow \mathscr{J}$, such that $f(H)$ is cofinal in $\mathscr{J}$, $m \rightarrow z$, $m(h) \in F(F^{n(f(h))}(x))$, $h \in H$. So,

(2) $m(h) \in F(u(h))$, $u(h) \in F^{n(f)h))}(x)$.

By compactness of $(E, \mathscr{J})$, $(u(h): h \in H)$ has a convergent subnet $w$. By (2) $w \rightarrow x_0$ and since $F$ is continuous, $F(w) \rightarrow F(x_0)$. By (2), the fact that $m \rightarrow z$, and Theorem 3.5, $z \in \text{clos } F(x_0)$.

In summary we have $T \subset \text{clos } F(x_0)$ and $x_0 \in \text{clos } T$ so $x_0 \in \text{clos } F(x_0)$.

COROLLARY 4.1. *Let $\mathscr{C}$ be thet set of all non-empty closed subsets of a compact, Hausdorff uniform space $(E, \mathscr{J})$ and let a continuous function $F: (E, \mathscr{J}) \rightarrow (\mathscr{C}, \mu(\mathscr{C}))$ satisfy the following condition: $V \in \mathscr{J}$, $x \neq y$ implies $F^n(x) \times F^n(y) \subset V$ for some $n = n(x, y, V)$. Then there is a unique $x_0 \in E$ such that $x_0 \in F(x_0)$.*

*Proof.* By results of [7], $F^n$ maps $E$ into $\mathscr{C}$ for each $n = 1, 2, 3, \cdots$ and $(\mathscr{C}, \mu(\mathscr{C}))$ is compact. Hence, by the previous theorem $x_0 \in \text{clos } F(x_0) = F(x_0)$ for some $x_0 \in E$. If also $x \in F(x)$, then given $V \in \mathscr{J}$, it is true that $(x, x_0) \in F^n(x) \times F^n(x_0) \subset V$ for some $n$. It follows that $(x, x_0) \in \cap \{V: V \in \mathscr{J}\}$, $x \neq x_0$, which contradicts the fact that $(E, \mathscr{J})$ is Hausdorff.

COROLLARY 4.2. (*Bailey* [1]). *Let $(E, d)$ be a compact metric space and $f: (E, d) \rightarrow (E, d)$ a continuous function such that if $x \neq y$, there exists $n = n(x, y)$ such that $d(f^n(x), f^n(y)) < d(x, y)$. Then $f$ has a unique fixed point.*

*Proof.* Under the hypothesis of the theorem it is easy to see that if $\delta > 0$ is given and $x \neq y$, there exists $n = n(x, y, \delta)$ such that $d(f^n(x), f^n(y)) < \delta$. Then, with $\mathscr{J}$ the natural uniform structure induced by $d$, the hypotheses of Corollary 4.1 are satisfied for $f$ and $(E, \mathscr{J})$ so the result follows.

Now let $(E, d)$ be a bounded metric space and $\mathscr{C}$ any class of nonempty subsets of $E$. The well-known Hausdorff function $h$ on $\mathscr{C}$ is defined by $h(a, b) = \max\{m(A, B), m(B, A)\}$ where $m(A, B) = \sup\{d(x, B): x \in A\}$ and $d(x, B) = \inf\{d(x, y): y \in B\}$.

THEOREM 4.2. *Let $(E, d)$ be a bounded metric space and let $\mathscr{C}$ be any nonempty class of nonempty subsets of $E$. Let $\mathscr{C}$ satisfy $(B_3)$ of* §3 *with respect to the natural uniform structure on E generated by the $V_\delta$'s, $V_\delta = \{(x, y): d(x, y) < \delta\}$. Then $(\mathscr{C}, \mu(\mathscr{C}))$ is uniformly metrizable* *and one metric for* $(\mathscr{C}, \mu(\mathscr{C}))$ *is the Hausdorff function on* $\mathscr{C}$.

*Proof.* If $h(A, B) = 0$, then $m(A, B) = m(B, A) = 0$. Given $\delta > 0$, it follows that $A \subset V_\delta(B)$ and $B \subset V_\delta(A)$. Since $(B_3)$ holds, $A = B$. Clearly $h(A, B) = h(B, A)$ and, if $A = B, h(A, B) = 0$.

To prove the triangle inequality it suffices to show that $m(A, B) \leqq m(A, X) + m(X, B)$ for each $A, B, X \in \mathscr{C}$. Let $\delta > 0$ be given.

(1) $m(A, B) < d(a_0, a) + d(a, x_0) + d(x, b) + \delta$ for some $a_0 \in A$ and all $a \in A, x \in X, b \in B$.

Also $m(A, X) \geqq d(a, X)$ for all $a \in A$ so given $a_0 \in A$,

(2) there exists $x_1 \in X$ such that $m(A, X) > d(a_0, x_1) - \delta$; similarly

(3) $m(X, B) > d(x_1, b_1) - \delta$ for some $b_1 \in B$. Combining (3), (2), and (1) we have $m(A, X) + m(X, B) > m(A, B) - 3\delta$ and it follows that $m(A, B) \leqq m(A, X) + m(X, B)$.

We have shown that $h$ is a metric on $\mathscr{C}$. Now let $U_\delta \in \mathscr{J}(h)$, $\mathscr{J}(h)$ the structure on $\mathscr{C}$ generated by $h$. A computation shows that if $(A, B) \in U_\delta$ then $A \subset V_{2\delta}(B)$ and $B \subset V_{2\delta}(A)$, $V_{2\delta} = \{(x, y): d(x, y) < 2\delta\}$, hence $U_\delta \subset \mathscr{T}(V_{2\delta})$ so $\mathscr{J}(h) \geqq \mu(C)$. Similarly $\mathscr{T}(V_\delta) \subset U_{2\delta}$ so then $\mathscr{J}(h) \leqq \mu(C)$.

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ARKANSAS STATE UNIVERSITY

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# AUTOMORPHISMS ON CYLINDRICAL SEMIGROUPS

JOAN MURPHY GERAMITA

**This paper characterizes the automorphisms of a cylindrical semigroup $S$ in terms of the automorphisms of the defining subgroups and subsemigroups. The following theorem is representative of the type of information given in this paper.**

**Let $F: \boldsymbol{R} \rightarrow A$ be a dense homomorphism of the additive real numbers to the compact abelian group $A$. Let $\lambda$ be a positive real number. Multiplication by $\lambda$ shall also denote the automorphism of $A$ whose restriction to $F(\boldsymbol{R})$ is given by $F\lambda F^{-1}$. The set of all such $\lambda$ for a given $F$ is called $\Lambda_F$.**

**Theorem. Let $f$ and $\lambda$ be as above. Let $G$ be a compact group. Let**

$$S = \{(p, f(p)\, g): p \in \boldsymbol{H} \text{ and } g \in G\} \cup \alpha \times A \times G .$$

**Then $\alpha: S \rightarrow S$ is an automorphism if and only if $\alpha(p, f(p), g) = (\lambda p, f(\lambda p), \tau(f(p))\xi(g))$; $\alpha(\infty, a, g) = (\infty, \lambda a, \tau(a)\xi(g))$, where $\tau: A \rightarrow G$ is a homomorphism into the centre of $G$ and, $\xi: G \rightarrow G$ is an automorphism. Theorem. Let $S$ be as in theorem above. Let $\mathscr{A}(G)$ be the automorphism group of $G$, and $Z(G)$, the center of $G$. The automorphism group of $S$ is isomorphic as an abstract group to $\mathscr{A}(G) \times (\Lambda_F \times \mathrm{Hom}(A, Z(G)))$ with the following multiplication**

$$(\xi, (\lambda, \tau))(\bar{\xi}, (\bar{\lambda}, \bar{\tau})) = (\xi \circ \bar{\xi}, (\lambda\bar{\lambda}, (\tau \circ \bar{\lambda})(\xi \circ \bar{\tau}))) .$$

Cylindrical semigroups play an important role Mislove's description of $Irr(X)$ and are the building blocks used in the construction of a hormos. Hofmann and Mostert [3] have shown that every compact irreducible semigroup is a hormos. The definition and description of a cylindrical semigroup, given in § I, is from their book.

**I. Definitions and notation.** All spaces are Hausdorff. All homomorphisms are continuous unless otherwise stated. A homomorphism will be called abstract if it is not assumed continuous. A group considered with the discrete topology will be called abstract. A topological semigroup is a topological space, $S$, together with a continuous associative multiplication $m: S \times S \rightarrow S$; $m(s, t) = st$. All semigroups are topological with identity 1. A topological group is a semigroup with the map $\phi: S \rightarrow S$, $\phi(s) = s^{-1}$, continuous also. An *ideal*, $I$, in a semigroup, $S$, is a subset of $S$ such that: if $x \in S$ then $(xI \cup Ix) \subset I$. If $S$ is compact and abelian then $S$ has an ideal $M(S)$ which is minimal with respect to set inclusion, is unique, and is a group. An *idempotent* $x \in S$ has the property $x^2 = x$. The maximal subgroup of $S$ containing an idempotent $e$ is called the *group of units of* $e$ and denoted $H(e)$. The group of units of 1 is also denoted $H(S)$ and called *the group of units of* $S$. If $\alpha: S \to S$ is an automorphism then $\alpha(H(S)) = H(S)$ and $\alpha(M(S)) = M(S)$.

NOTATION. The following notation is standard throughout the paper.

$[a, b]$—In a totally ordered set, the closed interval from $a$ to $b$.

$]a, b[$—The open interval from $a$ to $b$.

$\boldsymbol{H}$—The semigroup of nonnegative real numbers under addition with the usual topology.

$\boldsymbol{H}^*$—The one point compactification of $\boldsymbol{H}$, written $[0, \infty]$.

$\boldsymbol{H}_r^*$—$\boldsymbol{H}^*/[r, \infty]$.

$\Lambda$—The abstract group of positive real numbers under multiplication.

$\boldsymbol{R}$—The group of real numbers under addition with the usual topology.

$Z(G)$—The center of a group $G$.

$[p]$—The image of $p$ under the quotient map $\boldsymbol{H}^* \to \boldsymbol{H}_r^*$.

$*$—As in $B^*$, the closure of $B \subset X$, except as noted above for $\boldsymbol{H}$.

$X \backslash A$—For $A \subset X$, the complement of $A$ in $X$.

1. DEFINITION. Let $A$ and $G$ be compact groups. Let $A$ be an abelian and $f: \boldsymbol{H} \to A$ a homomorphism such that $f(\boldsymbol{H})^* = A$. Consider $\boldsymbol{H}^* \times A \times G$ with coordinate-wise multiplication, and let $S$ be that subsemigroup defined by:

$$S = \{(p, f(p), g): p \in \boldsymbol{H}, g \in G\} \cup \infty \times A \times G .$$

Any homomorphic image of $S$ is called a cylindrical semigroup.

The following theorem which describes cylindrical semigroups is from [3, p. 85, Prop. 2.2].

THEOREM A (Hofmann and Mostert). *Let $S$ be a cylindrical semigroup as defined above. Let $e$ be the identity of $G$ and*

$$S' = \{(p, f(p), e): p \in \boldsymbol{H}\} \cup \infty \times A \times e .$$

*Let $\phi: \to T$ be a surmorphism onto a compact semigroup $T$. Then there are*:

(i) *compact semigroups $T_1$, $T_1'$, $X$ and a compact group $B$,*

(ii) *surmorphisms $h_1$, $h_2$, $h_3$, $h_4$, $\phi_1$, $\phi_2$*

(iii) *monomorphisms $i_1$, $i_2$*

*such that the following diagram commutes*:

$$\begin{array}{ccccc}
\mathbf{H}^* & \xrightarrow{\quad h_1 \quad} & \mathbf{H}_r^* & \xrightarrow{id} & \mathbf{H}_r^* \\
\pi \uparrow & & \uparrow \pi' & & \uparrow h_4 \\
\mathbf{H}^* \times A \times G & \xrightarrow{h_1 \times h_2 \times id} & \mathbf{H}_r^* \times B \times G & \xrightarrow{h_3} & X \\
\text{IU} \uparrow & & \uparrow i_1 & & \uparrow i_2 \\
S & \xrightarrow{\quad \phi_1 \quad} & T_1 & \xrightarrow{\phi_2} & T \\
\text{IU} \uparrow & & \text{IU} \uparrow & & \uparrow \text{IU} \\
S' & \xrightarrow{\qquad} & T_1' & \longrightarrow & \phi(S')
\end{array}$$

($\pi$, $\pi'$ are projections; $\phi_2 \circ \phi_1 = \phi$).

*Moreover,* $h_3|_{\mathbf{H}^* \times B \times e}$ *is a monomorphism and* $h_4 \circ i_2$ *is a surmorphism.*

From this theorem it is possible to describe $T$ in terms of equivalence classes of elements in $\mathbf{H}_r^* \times B \times G$.

$f(0)$ is the identity of $A$. $r$, if it exists, is the least real number such that $\phi(r, f(r), e) = \phi(\infty, a, g)$ for some $a \in A$, $g \in G$.

$$B = \phi(\infty \times A \times e). \quad T_1' = \phi(S') \times e .$$

Let $\bar{f}\colon \mathbf{H} \to B$ be given by $\bar{f}(p) = \phi(\infty, f(p), e)$ then

$$i_1(T_1') = \{([p], f(p), e)\colon p \in \mathbf{H}\} \cup [r] \times B \times e .$$

If there is no such $r$, then $i_1(T_1') \subset \mathbf{H}^* \times B \times G$.

Let

$$G_{[p]} = \{g \in G\colon \phi(p, f(p), g) = \phi(p, f(p), e)\}$$

and

$$G_{[r]} = \{g \in G\colon \phi([r], f(0), g) = \phi([r], f(0), e)\}$$

where $r \leqq \infty$. $\{G_{[p]}\colon p \in \mathbf{H}^*\}$ has the following two properties:

(1) $$G_{[p]} \subseteqq G_{[q]} \quad \text{for} \quad p \leqq q ;$$

(2) $$G_{[p]} = \bigcap_{q > p} G_{[q]} .$$

Each $G_{[p]}$ is a normal subgroup of $G$. Denote $G/G_{[r]}$ by $\bar{G}$ and assume $G_{[0]} = \{e\}$.

$$i_2\phi(\{(p, f(p), g)\colon p \in \mathbf{H}, g \in G\}) = \{([p], f(p), gG_{[p]})\colon p \in \mathbf{H}, g \in G\}$$

where

$$(gG_{[p]})(\bar{g}G_{[\bar{p}]}) = g\bar{g}G_{[p+\bar{p}]} .$$

$i_2\phi(\infty \times A \times G) = ([r] \times B \times G)/K$ where $K$ is a normal subgroup of

$[r] \times B \times G$. $K$ has the property: if $([r], b, g) \in K$ and $([r], \bar{b}, \bar{g}) \in K$ then $b = \bar{b}$ if and only if $g = \bar{g}$.

We shall identify $T$ with its image $i_2(T)$ and refer to $i_1(T_1')$ as $T'$. Since $B$ is a compact abelian group and $\bar{f}: \boldsymbol{H} \to B$ is onto a dense subset of $B$, we may as well consider them as $f$ and $A$ to avoid extra notation. We say

$$T = \{([p], f(p), gG_{[p]}): p \in \boldsymbol{H}, g \in G\} \cup ([r] \times B \times G)/K .$$

**II. Automorphisms on semigroups of the form of $S$.** We first consider automorphisms of the cylindrical semigroup $S$ given in Definition 1. $M(S)$, the minimal ideal of $S$, is $\infty \times A \times G$. $H(S)$, the group of units, is $\{(0, f(0), g): g \in G\}$. From Theorem A we have that an automorphism $\alpha: S \to S$ can be thought of as an automorphism on $S' \times H(S)$.

Consider the situation where $G = \{e\}$. We have $S = S'$, $M(S') = \infty \times A \times e$ and $S' \backslash M(S')$ is isomorphic to $\boldsymbol{H}$ by $(p, f(p), e) \leftrightarrow p$. For an automorphism $\alpha: S' \to S'$, $\alpha(M(S')) = M(S')$; and, $\alpha$ restricted to $S' \backslash M(S')$ corresponds to an automorphism of $\boldsymbol{H}$. Since the only automorphisms of $\boldsymbol{H}$ are multiplication by a positive real number $\lambda$, we have $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$.

How shall $\alpha$ behave on $M(S')$? Let $\boldsymbol{R}$ be the additive group of real numbers, then $f: \boldsymbol{H} \to A$ can be extended to $F: \boldsymbol{R} \to A$ (for $x \notin \boldsymbol{H}$, $F(x) = f(-x)^{-1}$) and $F(\boldsymbol{R})$ will be dense in $A$. Let $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$. Then:

$$\begin{aligned} \alpha(\infty, f(p), e) &= \alpha((p, f(p), e)(\infty, f(0), e)) \\ &= \alpha(p, f(p), e)\alpha(\infty, f(0), e) \\ &= (\lambda p, f(\lambda p), e)(\infty, f(0), e) \\ &= (\infty, f(\lambda p), e) . \end{aligned}$$

Define $\bar{\lambda}: F(\boldsymbol{R}) \to F(\boldsymbol{R})$ by $\bar{\lambda}(F(x)) = F(\lambda x)$. $\alpha|_{M(S')}: M(S') \to M(S')$ must be an extension of $\bar{\lambda}$. This extension will be called $\lambda$.

Any homomorphism between dense subgroups of compact groups can be extended to a unique homomorphism between the groups. If original map is an automorphism then the extension is also. The existence and uniqueness of the extension, as a function, follow from the fact that the subgroups are uniform spaces and the groups are completions of them [1]. That the extension is a homomorphism is an easy consequence of the definition of the extension.

2. LEMMA. *Let $S' = \{(p, f(p), e): p \in \boldsymbol{H}\} \cup \infty \times A \times e$. If $f$ is neither one-to-one nor constant then the only automorphism of $S'$ is*

*the identity. Otherwise,* $\alpha\colon S'\to S'$ *is an automorphism iff* $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$, $\alpha(\infty, a, e) = (\infty, \lambda a, e)$ *where* $F\lambda F^{-1}$ *is open and continuous or* $F$ *is constant.*

*Proof.* If $\alpha\colon S'\to S'$ is an automorphism the discussion above shows that $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$ and $\alpha(\infty, a, e) = (\infty, \lambda a, e)$. If $f$ is constant then $A = \{e\}$; $S'$ is isomorphic to $\boldsymbol{H}^*$; and multiplication by any $\lambda$ is an automorphism.

Suppose $f$ is not constant. Consider the map $\bar\lambda\colon F(\boldsymbol{R})\to F(\boldsymbol{R})$ given by $\bar\lambda(F(x)) = F(\lambda x)$. If $F$ is not one-to-one then the kernel of $F$ in $\boldsymbol{R}$ is cyclic and $\lambda\colon \boldsymbol{R}\to\boldsymbol{R}$ must preserve this kernel. This implies $\lambda$ is an integer. Since $\lambda^{-1}$ must also be an integer, we have $\lambda = 1$.

If $F$ is one-to-one then $\bar\lambda$ is an automorphism of the abstract group $F(\boldsymbol{R})$. To be an automorphism of $F(\boldsymbol{R})$ with the induced topology from $A$, $\bar\lambda(=F\lambda F^{-1})$ must be open and continuous. The remark immediately preceding this lemma guarantees that $\bar\lambda$ can be extended to $A$ when it is open and continuous.

Let $\Lambda_F = \{\lambda\in\Lambda\colon F\lambda F^{-1}$ is open and continuous$\}$.

When $G\neq\{e\}$ we have $\alpha\colon S'\times H(S)\to S'\times H(S)$ where $H(S)$ is isomorphic to $G$ and $M(S) = \infty\times A\times G$. Since $\alpha(H(S)) = H(S)$, $\alpha(0, f(0), g) = (0, f(0), \xi(g))$ for some automorphism $\xi\colon G\to G$. Hence, the only possibility for $\alpha(\infty, f(0), g) = (\infty, a, h)$ is when $a = f(0)$. $\alpha$ restricted to $M(S)$ must therefore have the form $\alpha(\infty, a, g) = (\infty, \lambda a, \tau(a)\xi(g))$ with $\lambda\in\Lambda$, $\xi$ as above and $\tau\colon A\to Z(G)$ (center of $G$), a homomorphism. $\tau$ must be continuous since $\tau = \pi_G\circ\alpha\circ i$ where $\pi_G$ is the projection onto $G$, and $i$ is the map $A\to\infty\times A\times G$ given by $i(a) = (\infty, a, e)$. Similarly $\tau$ must be a homomorphism. Since elements in $\infty\times A\times e$ commute with elements of $\infty\times f(0)\times G$, $\tau$ maps $A$ into $Z(G)$.

3. THEOREM. *Let* $S$ *be as in Definition* 1. $\alpha\colon S\to S$ *is an automorphism iff* $\alpha(x, a, g) = (\lambda x, \lambda a, \tau(a)\xi(g))$ *where* $\lambda\in\Lambda_F$; $\tau\colon A\to Z(G)$ *is a homomorphism and* $\xi\colon G\to G$ *is an automorphism.*

*Proof.* The above discussion establishes the only if part. Let $\lambda, \tau, \xi$ be given as described in the theorem. $\hat\alpha\colon \boldsymbol{H}^*\times A\times G\to\boldsymbol{H}^*\times A\times G$ can be defined by $\hat\alpha(x, a, g) = (\lambda x, \lambda a, \tau(a)\xi(g))$. It is immediate that $\hat\alpha$ is an abstract automorphism. Since $\boldsymbol{H}^*\times A\times G$ is compact, we need only that $\hat\alpha$ is continuous. Let $U\times V\times W$ be a basis open set. $\hat\alpha^{-1}(U\times V\times W) = \lambda^{-1}U\times\lambda^{-1}V\times\xi^{-1}(\tau(\lambda^{-1}V)^{-1})\xi^{-1}(W)$. Since $\lambda$ and $\xi$ are continuous, $\lambda^{-1}U, \lambda^{-1}V$ and $\xi^{-1}(W)$ are open. Since $G$ is a topological group, for any set $X$, $X\xi^{-1}(W)$ is open. Hence $\hat\alpha^{-1}(U\times V\times W)$ is open. Let $\alpha = \hat\alpha|_S$.

**III. Automorphisms on semigroups of the form of $T$.** Recall $T = \{([p], f(p), gG_{[p]}) : p \in \boldsymbol{H}, g \in G\} \cup ([r] \times A \times \bar{G})/_K$. It is easier to keep track of the situation by considering cases determined by $r$, $G$, and $K$.

*Case* (a). Let $r < \infty$ and $G = \{e\}$. Then $K = \{([r], f(0), e)\}$.

4. LEMMA. *Let $T$ be given by Case* (a). *The only automorphism on $T$ is the identity.*

*Proof.* Let $\alpha$ be an automorphism of $T$.

Suppose $p < r$. $\alpha([p], f(p), e) = ([q], f(q), e)$ for some $q < r$ since $\alpha(M(T)) = M(T)$. First, let us take the case where $p = r/n$ for some integer $n$. If $p < q$ then there exists $p' < p$ such that $\alpha([p'], f(p'), e) = ([p], f(p), e)$ and $\alpha([np'], f(np'), e) = ([np], f(np), e) = ([r], f(r), e) \in M(T)$. But $np' < r$ since $np = r$ and $p' < p$. This means $\alpha([np'], f(np'), e) \notin M(T)$. We have a contradiction; so $p \geqq q$. If we assume $p > q$, a similar contradiction arises from $nq < r$. So, if $p < r$ and $p = r/n$ then $\alpha([p], f(p), e) = ([p], f(p), e)$.

For $p < r$, if $p \neq r/n$ then there exists a sequence, possibly finite, of integers $\{n_i\}$ such that $p = \sum r/n_i$. $\alpha$ is continuous so, again, $\alpha([p], f(p), e) = ([p], f(p), e)$.

$$\begin{aligned}\alpha([r], f(r), e) &= \lim_{\bar{p}<r} \alpha([\bar{p}], f(\bar{p}), e) \\ &= \lim_{\bar{p}<r} ([\bar{p}], f(\bar{p}), e) = ([r], f(r), e) .\end{aligned}$$

For $p > r$, $p = nr + p'$ where $p' < r$. We have:

$$\begin{aligned}\alpha([p], f(p), e) &= \alpha([nr], f(nr), e)\alpha([p'], f(p'), e) \\ &= (\alpha([r], f(r), e))^n([p'], f(p'), e) \\ &= ([r], f(r), e)^n([p'], f(p'), e) = ([p], f(p), e) .\end{aligned}$$

So $\alpha$ is the identity map.

*Case* (b). Let $r = \infty$, $G_p = G_\infty$ for all $p$ and $K = \{(\infty, f(0), e)\}$.

In this case, $T$ is of the form of $S$ where $\bar{G} = G/G_\infty$.

*Case* (c). Let $r < \infty$, $G_{[p]} = G_{[r]}$ for all $p$ and $K = \{([r], f(0), e)\}$. Let $G/G_{[r]} = \bar{G}$.

5. THEOREM. *Let $T$ be as in Case* (c). *$\alpha\colon T \to T$ is an automorphism iff $\alpha(x, a, g) = (x, a, \tau(a)\xi(g))$ where $\tau\colon A \to Z(\bar{G})$ is a homomor-*

*phism and* $\xi: \bar{G} \to \bar{G}$ *is an automorphism.*

*Proof.* From Lemma 4 we have $\lambda = 1$ and the precise arguments in the proof of Theorem 3 concerning $\tau$ and $\xi$ hold here.

*Case* (d). Let $r \leqq \infty$, $G_{[p]} \neq G_{[q]}$ for $[p] \neq [q]$ and $K = \{([r], f(0), e)\}$.

In this case, the description becomes more complicated but is in fact, no more difficult to prove. The previous cases allowed $\tau: A \to Z(\bar{G})$ to be defined in $M(T)$ and then used in $T\backslash M(T)$. Here, since $\bar{G} = G/G_{[r]} \neq G/G_{[p]}$ for $[p] \neq [r]$, it is not possible to start by taking $\tau$ defined in $M(T)$ to be any homomorphism in $\mathrm{Hom}\,(A, Z(\bar{G}))$. Rather, we start with a homomorphism $h: \boldsymbol{H} \to T\backslash M(T)$ which must also determine a homomorphism $f(\boldsymbol{H}) \to Z(\bar{G})$. The latter homomorphism can then be extended to define $\tau$. Without loss of generality, we may assume $G_{[0]} = \{e\}$.

6. THEOREM. *Let $T$ be as described for Case* (d). *Let $\xi: G \to G$ be an automorphism. If $r < \infty$, let $\xi(G_{[p]}) = G_{[p]}$ for all $p \in \boldsymbol{H}$. If $r = \infty$, let there exist a $\lambda \in \Lambda_F$ such that $\xi(G_{[p]}) = G_{[\lambda p]}$ for all $p \in \boldsymbol{H}$.*

*Let $h: \boldsymbol{H} \to T$ be a homomorphism such that $h(p) = ([p], f(p), gG_{[p]})$ and*

$$\{h(p)([r], f(0), G_{[r]})\} \subseteqq [r] \times A \times (G/G_{[r]})$$

*represents the graph of a homomorphism $f(\boldsymbol{H}) \to Z(G/G_{[r]})$.*

*$\alpha: T \to T$ is an automorphism iff $\alpha([p], f(p), gG_{[p]}) = h(\lambda p)(0, f(0), \xi(g))$, and $\alpha([r], a, gG_{[r]}) = ([r], \lambda a, \tau(a)\xi(g)G_{[r]})$ where $\tau: A \to Z(G/G_{[r]})$ is a homomorphism.*

*Proof.* Let us assume $r = \infty$. The proof for $r < \infty$ follows this one replacing $\lambda$ by 1 and $p$ by $[p]$. Let $\alpha$ be given.

Define $\xi: G \to G$ in the usual way by considering $\alpha|_{H(T)}$. It is still the case that $(p, f(p), G_p) \to (\lambda p, f(\lambda p), gG_p)$. This follows directly from the top level of the diagram in Theorem A. One can show that $\xi(G_p) = G_{\lambda p}$ by considering $(p, f(p), G_p)$ written as $(p, f(p), gG_p)$ for $g \in G_p$. $\lambda \in \Lambda_F$ since once again $\lambda$ must be extended to an automorphism of $A$ in $M(T)$ (see Theorem 3).

Define

$$h: \boldsymbol{H} \to T \text{ by } h(p) = \alpha(\lambda^{-1}p, f(\lambda^{-1}p), G_{\lambda^{-1}p}) .$$

$h$ is the composition of three homomorphisms

$$\boldsymbol{H} \xrightarrow{\lambda^{-1}} \boldsymbol{H} \xrightarrow{\hat{f}} T \xrightarrow{\alpha} T \quad \text{where} \quad \hat{f}(p) = (p, f(p), G_p) .$$

Define $\lambda h(p) = h(\lambda p)$. $\lambda h$ is also a homomorphism but not of the type specified by the theorem.

Define $\tau: A \to Z(G/G_\infty)$, as was done in Theorem 3, by considering $\alpha|_{\infty \times A \times e}$.

*Note*:

$$\begin{aligned} h(p)(\infty, f(0), G_\infty) &= \alpha(\infty, f(\lambda^{-1}p), G_\infty) \\ &= (\infty, f(p), gG_\infty) = (\infty, f(p), \tau(f(p))) . \end{aligned}$$

So $\{h(p)(\infty, f(0), G_\infty)\}$ represents the graph of a homomorphism from $f(\boldsymbol{H}) \to Z(G/G_\infty)$. We shall sometimes write $\tau(a)$ as $\tau(a)G_\infty$. We observe that $\{h(p)(\infty, f(0), G_\infty)\} = \{\lambda h(p)(\infty, f(0), G_\infty)\}$, so $h$ and $\lambda h$ can be made to determine the same $\tau$.

For the converse let $\xi$, and $h$ be given. $\xi$ determines $\lambda \in \Lambda_F$. $\lambda h$ determines the graph of a homomorphism since $h$ does. Define $\tau(f(p)) = \pi_\infty(h(\lambda p)(\infty, f(0), G_\infty))$ where $\pi_\infty$ is the projection. $\tau$ can be extended in the usual way to $A$.

Define $\alpha: T \to T$ by

$$\begin{aligned} \alpha(p, f(p), gG_p) &= \lambda h(p)(0, f(0), \xi(g)) \\ \alpha(\infty, a, gG_\infty) &= (\infty, \lambda a, \tau(a)\xi(g)) . \end{aligned}$$

Showing $\alpha$ is an abstract homomorphism is straightforward. One can prove $\alpha$ is continuous by writing $T$ as the image of $S$ and considering open sets. This proof is omitted because it is uninteresting and requires complicated notation.

*Case* (e). Let $r = \infty$, $G_p = G_q \neq G_\infty$ and $K = \{(\infty, f(0), e)\}$.

This situation is a simple version of Case (d). Since $G_p = G_{\lambda p}$ for all $\lambda$, we no longer have $\lambda$ determined by $\xi: G \to G$. Any choice of $\lambda \in \Lambda_F$ will give an automorphism.

*Case* (f). Let $K \neq \{([r], f(0), e)\}$ and $K \neq [r] \times A \times \bar{G}$. Let $\hat{T} = \{([p], f(p), gG_{[p]}): p \in \boldsymbol{H}, g \in G\} \cup [r] \times A \times \bar{G}$ and let $T = \{([p], f(p), gG_{[p]}\} \cup ([r] \times A \times \bar{G})/K$. Let $k: \hat{T} \to T$ be the map which is the identity on $\hat{T} \backslash M(\hat{T})$ and the quotient map on $M(\hat{T})$. Recall: if $([r], a, g) \in K$ and $([r], \bar{a}, \bar{g}) \in K$ then $a = \bar{a}$ iff $g = \bar{g}$. When $r < \infty$, if $k(t_\gamma)$ is a convergent net in $T$ such that $k(t_\gamma) \notin M(T)$ and $\lim k(t_\gamma) \in M(T)$, then $t\gamma$ is a convergent net in $\hat{T}$. Let $\pi_A(K) = \{a \in A: ([r], a, g) \in K$ for some $g \in \bar{G}\}$. Let $\beta$ be the abstract isomorphism $\beta: \pi_A(K) \to \bar{G}$ given by $g = \beta(a)$ if $([r], a, g) \in K$.

7. LEMMA. *Let $T$ and $\hat{T}$ be as above. Let $\hat{\alpha}: \hat{T} \to \hat{T}$ be charac-*

*terized by* $(\lambda, \tau, \xi)$ *or by* $(\lambda, h, \xi)$ *as given in* 3, 5, 6. *Let* $\pi_A(K)$ *and* $\beta$ *be as above. There exists an automorphism* $\alpha: T \to T$ *such that* $\alpha k = k\hat{\alpha}$ *iff* $\lambda|_{\pi_A(K)}$ *is an automorphism and* $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$ *for* $a \in \pi_A(K)$.

*Proof.* Suppose $\hat{\alpha}$ induces an automorphism $\alpha$ such that $\alpha k = k\hat{\alpha}$. Consider $\hat{\alpha}|_{M(\hat{T})}$ as an automorphism on the group $M(\hat{T})$. This induces $\alpha|_{M(T)}$ on $M(T)$ and for $\alpha|_{M(T)}$ to be well defined and one-to-one we must have $\hat{\alpha}(K) = K$. For $([r], a, \beta(a)) \in K$ we have $\hat{\alpha}([r], a, \beta(a)) = ([r], \lambda a, \tau(a)\xi(\beta(a))) \in K$. Hence, $\lambda a \in \pi_A(K)$ and $\beta(\lambda a) = \tau(a)\xi(\beta(a))$. Since $\hat{\alpha}^{-1}$ is also an automorphism $\lambda^{-1}a \in \pi_A(K)$ and $\lambda$ is onto. $\beta(\lambda a) = \tau(a)\xi(\beta(a))$ implies $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$.

The proof of the converse is straightforward. It is convenient to consider the continuity of $\alpha$ on $T\backslash M(T)$ and $M(T)$ separately and then consider a net converging to $M(T)$.

8. THEOREM. *Let* $\hat{T}$, $T$ *and* $k$ *be as in Lemma* 7. $\alpha: T \to T$ *is an automorphism iff there exists an automorphism* $\hat{\alpha}: \hat{T} \to \hat{T}$ *such that* $\alpha k = k\hat{\alpha}$.

*Proof.* Let $\alpha : T \to T$ be an automorphism. We consider two cases: $r < \infty$ and $r = \infty$. Let $r < \infty$. We know from Theorems 5 and 6 that $\hat{\alpha}$ is determined by $(\xi, h)$ or $(\xi, \tau)$. Constructing $h$ is the more general situation. An argument similar to that of Theorem 4 establishes that

$$\alpha k([p], f(p), G_{[p]}) = k([p], f(p), \bar{g}G_{[p]}) .$$

Let $G_{[0]} = \{e\}$ and $\bar{G} = G/G_{[r]}$.

Define $\xi: G \to G$ by $\xi(g) = \pi_G \alpha k([0], 1, g)$. Clearly $\xi$ is an automorphism.

Define $h: \boldsymbol{H} \to \hat{T}$ by:

$$h(p) = k^{-1}\alpha k([p], f(p), G_{[p]}) \quad \text{when } p < r ;$$
$$h(r) = \lim_{p<r} h(p) ;$$
$$h(p) = (h(r))^n h(q) \quad \text{when} \quad p = nr + q, q < r .$$

It is immediate that $h$ is a homomorphism. Since $\alpha k([p], f(p), G_{[p]}) = k([p], f(p), gG_{[p]})$, we have also $\alpha k([r], a, G_{[r]}) = k([r], a, gG_{[r]})$.

Define $\tau: A \to Z(\bar{G})$ by $\tau(a) = gG_{[r]}$ such that $\alpha k([r], a, G_{[r]}) = k([r], a, gG_{[r]})$. $\tau$ is well-defined since if $([r], a, y) \in ([r], a, g)K$ then $([r], f(0), yg^{-1}) \in K$ and $y = g$. It is also immediate that $\tau$ is an abstract homomorphism. $\tau(f(p)) = \pi_{\bar{G}}(h(p)([r], f(0), e))$ so $\tau$ is continuous on $f(\boldsymbol{H})$ and hence on $A$. Even if $\hat{\alpha}$ is more efficiently given by $(\xi, \tau)$, $h$

can be defined and the above will show $\tau$ continuous.

Define $\hat{\alpha}$: $\hat{T} \to \hat{T}$ by $(\xi, h)$ or $(\xi, \tau)$.

$$\alpha k([p], a, gG_{[p]}) = k([p], a, \tau(a)\xi(g)G_{[p]}) = k\hat{\alpha}([p], a, gG_{[p]}) .$$

So $\alpha k = k\hat{\alpha}$.

Now, let $r = \infty$ and $\bar{G} = G/G_\infty$. Define $\xi$ as before. Either $\xi$ determines $\lambda$ (as in 6); or, define $\lambda$ by checking $\alpha k(p, f(p), G_p)$. If $f$ is not one-to-one then, $\lambda = 1$ or $A = \{1\}$. If $f$ is one-to-one then $\lambda$ is one-to-one on $f(\boldsymbol{H}) \subset A$ and can be extended to $\lambda$ continuous on $A$. Since $\alpha^{-1}$ is also an automorphism the above process can be done for $\lambda^{-1}$ which means $\lambda$ is open on $A$ and hence $\lambda \in \Lambda_F$.

Define $h$: $\boldsymbol{H} \to \hat{T}$ by $h(p) = k^{-1}\alpha k(\lambda^{-1}p, f(\lambda^{-1}p), G_{\lambda^{-1}p})$. $h$ is a homomorphism since $k$ is an isomorphism.

Define $\tau(f(p)) = \pi_{\bar{G}}(h(p)(\infty, f(0), G_\infty))$. $\tau$ is continuous since $h$ and $\pi_{\bar{G}}$ are, and can be extended to $A$.

We define $\hat{\alpha}$: $\hat{T} \to \hat{T}$ by $(\lambda, \xi, h)$ or $(\lambda, \xi, \tau)$. Again, $\alpha k = k\hat{\alpha}$.

So, for each case, $\hat{\alpha}$, an automorphism of $\hat{T}$ inducing $\hat{\alpha}$, can be constructed.

**IV. Automorphism groups.** This section describes the group structure of the groups of automorphisms given in II and III. All groups discussed here are discrete. Bowman [2] has described the topology of these groups. Since in each case the group is described as a semidirect product of groups of homomorphisms; we give the definition of semidirect product below.

Let $A$ and $B$ be two groups. Let $g$: $A \to \mathscr{A}(B)$, the group of automorphisms of $B$, be a function such that:

(i) $g(a_2)(g(a_1)b) = g(a_2a_1)(b)$;

or

(ii) $g(a_2)(g(a_1)b) = g(a_1a_2)(b)$.

$A \times B$ is a group with the following multiplication: $(a, b)(\bar{a}, \bar{b}) = (a\bar{a}, b(g(a)\bar{b}))$ when $g$ is of type i; $(a, b)(\bar{a}, \bar{b}) = (a\bar{a}, (g(\bar{a})b)\bar{b})$ when $g$ is of type ii. The semidirect product will be denoted $A \times_g B$.

Recall, the operation in $\mathscr{A}(G)$ is composition of functions; in Hom $(A, Z(G))$, multiplication of functions; in $\Lambda_F$, multiplication of real numbers.

We begin with $\mathscr{A}(S)$ where $S$ is as in Definition 1. We have from Theorem 3 the correspondence $\alpha \leftrightarrow (\lambda, \tau, \bar{\xi})$ for $\alpha \in \mathscr{A}(S)$. It is immediate that this correspondence is one-to-one.

9. THEOREM. *Let $S$ be as in Definition* 1. *The automorphism group of $S$ is isomorphic to*

$$\mathscr{A}(G) \times_{g_2} (\Lambda_F \times_{g_1} \text{Hom}\,(A, Z(G)))$$

*where*

$$g_1(\lambda)(\tau) = \tau \circ \lambda \qquad (\textit{of type ii})$$
$$g_2(\bar{\xi})(\lambda, \tau) = (\lambda, \bar{\xi} \circ \tau) \qquad (\textit{of type i}) .$$

*Proof.* Showing that the correspondence given by Theorem 3 is a homomorphism is only a matter of computing $\alpha \circ \bar{\alpha}$ where $\alpha, \bar{\alpha}$ are in $\mathscr{A}(S)$. The multiplication given by $g_1$ and $g_2$ is as follows:

$$(\bar{\xi}, (\lambda, \tau))(\bar{\bar{\xi}}, \bar{\lambda}, \bar{\tau})) = (\bar{\xi} \circ \bar{\bar{\xi}}, (\lambda\bar{\lambda}, (\tau \circ \bar{\lambda})(\bar{\xi} \circ \bar{\tau}))) .$$

Proceeding to the various forms of $T$ discussed in §III, we have, in Case (a), $\mathscr{A}(T) = \{1_T\}$. In Case (b), $T$ is really of the form of $S$ so Theorem 9 applies. For Case (c) we have the following.

10. THEOREM. *Let $T$ be as in Theorem 5. $\mathscr{A}(T)$ is isomorphic to $\mathscr{A}(G) \times_g \mathrm{Hom}\,(A, Z(G))$ where $g(\bar{\xi})(\tau) = \bar{\xi} \circ \tau$ (of type i).*

*Proof.* In this case $T$ is almost like $S$. $\lambda$ is forced to be 1. $g$ here corresponds to $g_2$ in Theorem 9. $(\xi, \tau)(\bar{\xi}, \bar{\tau}) = (\xi \circ \bar{\xi}, \tau(\xi \circ \bar{\tau}))$.

For $T$ described by Case (d), we construct a group isomorphic to the desired subgroup of $\mathrm{Hom}\,(\boldsymbol{H}, T)$. Let $H = \{h \in \mathrm{Hom}\,(\boldsymbol{H}, T)\colon h$ is as in Theorem 6$\}$. $H$ is a group under the following operation$*$. Let $h_i(p) = ([p], f(p), g_i G_{[p]})$. Define $h_1 * h_2$ by $h_1 * h_2(p) = ([p], f(p), g_1 g_2 G_{[p]})$. This group can be mapped isomorphically into $\prod_{p \in \boldsymbol{H}} (G/G_{[p]})$ and $\hat{h}$ is given by $h(p) = ([p], f(p), \hat{h}(p))$. Let $\mathscr{H}$ be the image of $H$ in $\prod_{p \in \boldsymbol{H}} (G/G_{[p]})$. $\mathscr{H}$ is an abelian group under coordinate multiplication.

11. THEOREM. *Let $T$ and $\mathscr{H}$ be as above. Let $\Xi_F$ be the subgroup of $\mathscr{A}(G)$ satisfying Theorem 6, $(\xi(G_{[p]}) = G_{[\lambda p]})$. Consider $\xi \in \Xi_F$ inducing a map called $\bar{\xi}\colon G/G_{[p]} \to G/G_{[\lambda p]}$ $\mathscr{A}(T)$ is isomorphic to $\Xi_F \times_g \mathscr{H}$ where $g(\xi)\hat{h} = \xi \circ \hat{h} \circ \lambda^{-1}$ (of type i).*

*Proof.* There are several things to check in this theorem. Again we will consider $r = \infty$ as in the proof of Theorem 6. $\bar{\xi}\hat{h}\lambda^{-1}\colon \boldsymbol{H} \to G/G_p$ since $\bar{\xi}$ is the induced map $G/G_{\lambda^{-1}p} \to G/G_p$.

From Theorem 6, we note if $\alpha$ is given

$$h(p) = \alpha(\lambda^{-1}p, f(\lambda^{-1}p), G_p)$$

and

$$\begin{aligned} \tau(f(p)) &= \pi_\infty(\alpha(p, f(p), G_p)(\infty, f(0), G_\infty)) \\ &= \pi_\infty((h(\lambda p))(\infty, f(0), G_\infty)) . \end{aligned}$$

If $h$ is given $\alpha(p, f(p), G_p) = \lambda h(p) = h(\lambda p)$ and $\tau(f(p)) = \pi_\infty(h(\lambda p)(\infty, f(0), G_\infty))$. From this we see the correspondence between $\alpha$ and $(\xi, h)$ is one-to-one and that the construction of $\tau$ does not depend on which representation is used.

The multiplication in $\Xi_F \times {}_g\mathscr{H}$ is

$$(\xi_1, \hat{h}_1)(\xi_2, \hat{h}_2) = (\xi_1 \circ \xi_2, (\hat{h}_1)(\bar{\xi}_1 \circ \hat{h}_2 \circ \lambda_1^{-1})) .$$

We note that $\hat{h}_1(\bar{\xi}_1 \hat{h}_2 \lambda_1^{-1})$ determines $\tau$ where $\tau = (\tau_1 \circ \lambda_2)(\xi_1 \circ \tau_2)$ which is exactly the product we expect to see in $\alpha_1 \circ \alpha_2$. From here it is immediate that the correspondence is an isomorphism.

In Case (e) we replace $\Xi_F$ in Theorem 11 by $\Xi_0 \times \Lambda_F$ where $\xi \in \Xi_0$ if $\xi(G_\infty) = G_\infty$. The automorphism group of $T$ is isomorphic to $(\Xi_0 \times \Lambda_F) \times_g \mathscr{H}$ where $g((\xi, \lambda))\hat{h} = \bar{\xi}\hat{h}\lambda^{-1}$ and $g$ is of type i.

In Case (f) the isomorphism group of $T$ is a subgroup of $\mathscr{A}(\hat{T})$.

**V. Examples.** The following semigroups can be found in Chapter D of [3].

**12. Example.** Let $\boldsymbol{Z}$ be the integers under addition. Let $A = G = \hat{\alpha}/\boldsymbol{Z}$. Let $f: \boldsymbol{H} \to A$ be given by $f(p) = p + \boldsymbol{Z}$. Then

$$S = \{(p, p + \boldsymbol{Z}, q + \boldsymbol{Z}) : p \in \boldsymbol{H}, q \in \boldsymbol{R}\} \cup \infty \times \boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z} .$$

$\mathscr{A}(S)$ is given by 9. Since $f$ is not one-to-one $\Lambda_F = \{1\}$. $\mathscr{A}(\boldsymbol{R}/\boldsymbol{Z}) = \{-1, 1\}$ and $\operatorname{Hom}(R/\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}) = \boldsymbol{Z}$.

$\mathscr{A}(S) = \{-1, 1\} \times_{g_2} \boldsymbol{Z}$ and the multiplication is given by $(x, k)(y, n) = (xy, k + xn)$.

**13. Example.** Let $S$ be as in 12. Let $T$ be the homomorphic image of $S$ obtained by letting $r = 1$ and not changing $A$ or $G$. $\mathscr{A}(T)$ is given by 10 and $\mathscr{A}(T) = \mathscr{A}(S)$.

**14. Example.** Let $S$ be as in 12. Let $T$ be the homomorphic image of $S$ obtained by letting $G_p = \boldsymbol{Z}$ for $p < \infty$ and $G_\infty = \boldsymbol{R}/\boldsymbol{Z}$. $T$ is described in §II, Case (e). $\mathscr{A}(T)$ is given by Theorem 11 and the comment following it. This is a particularly simple example where $\Lambda_F = \{1\}$ and $\Xi_0 = \Xi = \mathscr{A}(G)$. $\mathscr{H} = \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{R}/\boldsymbol{Z}) = \boldsymbol{R}$. $\mathscr{H}$ must represent homomorphisms $h: \boldsymbol{H} \to T$. It does in this way: $h_r(p) = (p, p + \boldsymbol{Z}, rp + \boldsymbol{Z})$.

$\mathscr{A}(T) = \{-1, 1\} \times_g R$ where multiplication is given by $(x, r)(y, s) = (xy, r + xs)$.

**15. Example.** Let $S$ be as in 12. Let $T$ be the homomorphic image obtained from $S$ by letting $K = \{(\infty, p + \boldsymbol{Z}, p + \boldsymbol{Z}) : p \in \boldsymbol{R}\}$. The automorphisms of $T$ are given by 7 and 8. They are a subgroup of $\mathscr{A}(S)$.

We examine $\mathscr{A}(S) = \{-1, 1\} \times_{g_2} \boldsymbol{Z}$ to see which automorphisms satisfy 7. Let $(x, k) \in \mathscr{A}(S)$. $\pi_A(K) = \boldsymbol{R}/\boldsymbol{Z}$ and $\beta(p + \boldsymbol{Z}) = p + \boldsymbol{Z}$. $k$ is the homomorphism called $\tau$ in 7 and $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$. We have $k(p + \boldsymbol{Z}) = kp + \boldsymbol{Z} = p + \boldsymbol{Z} - xp + \boldsymbol{Z}$. If $x = 1, kp + \boldsymbol{Z} = Z$; if $x = -1, kp + \boldsymbol{Z} = 2p + \boldsymbol{Z}$. $\mathscr{A}(T) = \{(1, 0), (-1, 2)\}$ considered as a subgroup of $\mathscr{A}(S)$.

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QUEEN'S UNIVERSITY,
KINGSTON, ONTARIO

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# DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

K. R. GOODEARL

**This paper is an investigation of conditions on a module $A$ under which the natural map**

$$A \otimes (\Pi C_\alpha) \longrightarrow \Pi(A \otimes C_\alpha)$$

**is an injection. The investigation leads to a theorem that a commutative von Neumann regular ring is self-injective if and only if the natural map**

$$(\Pi F_\alpha) \otimes (\Pi G_\beta) \longrightarrow \Pi(F_\alpha \otimes G_\beta)$$

**is an injection for all collections $\{F_\alpha\}$ and $\{G_\beta\}$ of free modules. An example is constructed of a commutative ring $R$ for which the natural map**

$$R[[s]] \otimes R[[t]] \longrightarrow R[[s, t]]$$

**is not an injection.**

$R$ denotes a ring with unit, and all $R$-modules are unital. All tensor products are taken over $R$.

We state for reference the following theorem of H. Lenzing [2, Satz 1 and Satz 2]:

THEOREM L. (a) *A right $R$-module $A$ is finitely generated if and only if for any collection $\{C_\alpha\}$ of left $R$-modules, the natural map $A \otimes \Pi C_\alpha \rightarrow \Pi(A \otimes C_\alpha)$ is surjective.*

(b) *A right $R$-module $A$ is finitely presented if and only if for any collection $\{C_\alpha\}$ of left $R$-modules, the natural map $A \otimes \Pi C_\alpha \rightarrow \Pi(A \otimes C_\alpha)$ is an isomorphism.*

THEOREM 1. *For any right $R$-module $A$, the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of flat left $R$-modules, then the natural map $A \otimes \Pi C_\alpha \rightarrow \Pi(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set $X$ of cardinality at least card $(R)$ such that the natural map $A \otimes R^X \rightarrow A^X$ is an injection.*

(c) *If $B$ is any finitely generated submodule of $A$, then the inclusion $B \rightarrow A$ factors through a finitely presented module.*

Note that condition (c) always holds when $R$ is right noetherian, for then all finitely generated submodules of $A$ are finitely presented.

*Proof.* (b) $\Rightarrow$ (c): If $R$ is finite, then it is right noetherian and

(c) holds. Thus we may assume that $R$ is infinite.

Let $f: F \to A$ be an epimorphism with $F_R$ free, and set $K = \ker f$. There is a finitely generated submodule $G$ of $F$ such that $fG = B$.

We have a commutative diagram with exact rows as follows (Diagram I):

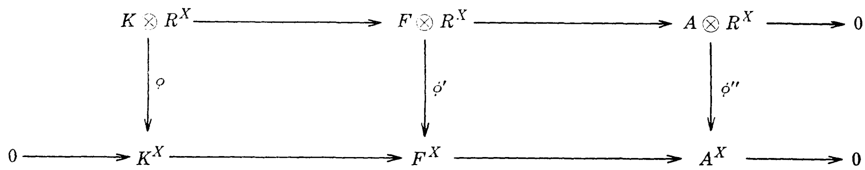


DIAGRAM I

Since $G$ is finitely generated, $G^X \leqq \phi'(F \otimes R^X)$. A short diagram chase (using the injectivity of $\phi''$) shows that $(G \cap K)^X \leqq \phi(K \otimes R^X)$.

card $(G) \leqq$ card $(R)$ because $R$ is infinite, hence card $(G \cap K) \leqq$ card $(X)$. Thus there is a surjection $\alpha \mapsto g_\alpha$ of $X$ onto $G \cap K$. The element $g = \{g_\alpha\}$ in $(G \cap K)^X$ must be the image under $\phi$ of some element $h_1 \otimes r_1 + \cdots + h_n \otimes r_n$ in $K \otimes R^X$. It follows easily that $G \cap K$ is contained in the submodule $H$ of $K$ generated by $h_1, \cdots, h_n$. Note that $G \cap H = G \cap K$.

$G + H$ is contained in some finitely generated free submodule $F_0$ of $F$. The map $f$ induces a monomorphism of $G/(G \cap H)$ into $A$, and this monomorphism factors through the finitely presented module $F_0/H$. Since $fG = B$, the inclusion $B \to A$ also factors through $F_0/H$.

(c) $\Rightarrow$ (a): Consider any $x$ belonging to the kernel of the natural map $\phi: A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$. There is a finitely generated submodule

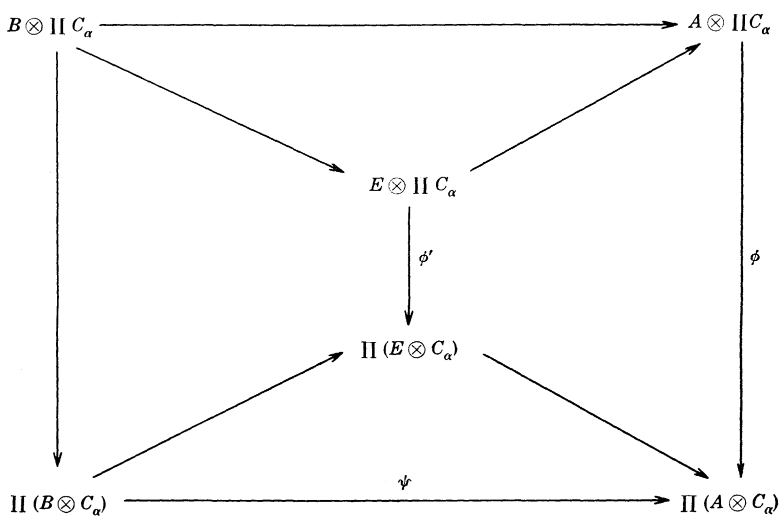


DIAGRAM II

$B$ of $A$ such that $x$ is in the image of the map $B \otimes \Pi C_\alpha \to A \otimes \Pi C_\alpha$. By (c), the inclusion $B \to A$ factors through some finitely presented module $E$.

We have a commutative diagram as follows (Diagram II):

$\phi'$ is an isomorphism by Theorem L, and $\psi$ is a monomorphism because all the $C_\alpha$'s are flat. Another diagram chase now shows that $x = 0$.

COROLLARY. *Suppose that $R$ is (von Neumann) regular. For any right $R$-module $A$, the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of left $R$-modules, then the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set $X$ of cardinality at least card$(R)$ such that the natural map $A \otimes R^X \to A^X$ is injective.*

(c) *All finitely generated submodules of $A$ are projective.*

*Proof.* (b) $\Rightarrow$ (c): If $B$ is a finitely generated submodule of $A$, then Theorem 1 says that the inclusion $B \to A$ factors through a finitely presented module $E$. $E$ is flat (because $R$ is regular) and hence is projective. Thus $B$ can be embedded in a projective module. Since $R$ is semihereditary, $B$ must be projective.

(c) $\Rightarrow$ (a): All the $C_\alpha$'s are flat (since $R$ is regular), and all finitely generated submodules of $A$ are finitely presented, so this follows directly from Theorem 1.

THEOREM 2. *Assume that $R$ is a commutative regular ring. Then the following conditions are equivalent:*

(a) *If $\{F_\alpha\}$ and $\{G_\beta\}$ are any collections of free $R$-modules, then the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi(F_\alpha \otimes G_\beta)$ is an injection.*

(b) *There is a set $X$ of cardinality at least card$(R)$ such that the natural map $R^X \otimes R^X \to R^{X \times X}$ is an injection.*

(c) *$R$ is injective as a module over itself.*

*Proof.* (b) $\Rightarrow$ (c): By [1, Theorem 2.1], it suffices to show that any finitely generated nonsingular $R$-module $B$ is projective.

[1, Lemma 2.2] says that we can embed $B$ in a finite direct sum $Q_1 \oplus \cdots \oplus Q_n$, where each $Q_i$ is a copy of the maximal quotient ring $Q$ of $R$. Then $B$ can be embedded in a direct sum $B_1 \oplus \cdots \oplus B_n$, where $B_i$ is a finitely generated $R$-submodule of $Q_i$. Since $R$ is semihereditary, $B$ will be projective provided each $B_i$ is projective. Thus without loss of generality we may assume that $B$ is an $R$-submodule of $Q$.

Let $b_1, \cdots, b_n$ generate $B$. Since $R$ is an essential submodule of $Q$, there is an essential ideal $I$ of $R$ such that $b_i I \leqq R$ for all $i$.

Since $R$ is commutative, the multiplications by the elements of $I$ induce homomorphisms of $B$ into $R$. Together, these homomorphisms induce a homomorphism $f: B \to R^I$. $Q$ is a nonsingular $R$-module because it has the nonsingular $R$-module $R$ as an essential submodule. Thus no nonzero element of $B$ is annihilated by $I$; i.e., $f: B \to R^I$ is an injection. Since $\text{card}(I) \leqq \text{card}(R) \leqq \text{card}(X)$, there must also be an embedding of $B$ into $R^X$.

Since the natural map $R^X \otimes R^X \to (R^X)^X$ is injective by (b), the corollary to Theorem 1 says that all finitely generated submodules of $R^X$ are projective. Thus $B$ must be projective.

(c) $\Rightarrow$ (a): By [1, Theorem 2.1], all finitely generated nonsingular $R$-modules are projective. Since $R_R$ is nonsingular, $\Pi F_\alpha$ is nonsingular; thus all finitely generated submodules of $\Pi F_\alpha$ are projective. By the corollary to Theorem 1, the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi_\beta[(\Pi F_\alpha) \otimes G_\beta]$ is an injection. Likewise, each of the maps $(\Pi F_\alpha) \otimes G_\beta \to \Pi_\alpha(F_\alpha \otimes G_\beta)$ is injective. Thus the map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi(F_\alpha \otimes G_\beta)$ must be injective.

In particular, Theorem 2 asserts that if $R$ is a countable commutative regular ring which is not self-injective, then the natural map $R^X \otimes R^X \to R^{X \times X}$ is not an injection for any infinite set $X$. For example, let $F_1, F_2, \cdots$ be a countable sequence of copies of some countable field $F$; let $R$ be the subalgebra of $\Pi F_n$ generated by 1 and $\bigoplus F_n$. $R$ is obviously a countable commutative regular ring. Since $\Pi F_n$ is a proper essential extension of $R_R$, $R_R$ is not injective.

If $N$ is the set of natural numbers, then the natural map $R^N \otimes R^N \to R^{N \times N}$ is not an injection. Thus the tensor product of two one-variable power series rings, $R[[s]] \otimes R[[t]]$, is not embedded in $R[[s, t]]$ by the natural map.

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UNIVERSITY OF WASHINGTON

*Author's current address*: UNIVERSITY OF UTAH

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# THE NON-CONJUGACY OF CERTAIN ALGEBRAS OF OPERATORS

JULIEN HENNEFELD

**Let $E$ be a Banach space and $B(E)$ be the space of all bounded linear operators on $E$. It was shown by Schatten, that if $E$ is a conjugate space then $B(E)$ is isometrically isomorphic to a conjugate space. The fact that for an arbitrary Banach space, the unit ball of $B(E)$ has extreme points suggests that $B(E)$ might always be a conjugate space. In this paper it is proved that if $E$ has an unconditional basis and is not isomorphic to a conjugate space, then $B(E)$ is not isomorphic to a conjugate space. An even stronger result is proved.**

**Furthermore, it is shown that if $E$ has an unconditional basis or a complemented subspace with an unconditional basis, then the space of all compact linear operators on $E$ is not isomorphic to a conjugate space.**

The result of Schatten is proved in [3; p. 4]. It is a theorem of Kakutani, that the identity of a Banach algebra is an extreme point of the unit ball. It follows that the invertible elements of norm one, whose inverses also have norm one, are extreme points of the unit ball. Hence, one cannot readily invoke the Krein Millman Theorem to prove non-conjugacy of $B(E)$. For $X$ and $E$ Banach spaces let $B(X, E)$ denote the space of all bounded linear operators from $X$ into $E$.

THEOREM 2.1. (Bessaga-Pełczynski). *A conjugate space contains no complemented subspace isomorphic to* $c_0$.

*Proof.* See [1; p. 250].

THEOREM 2.2. *Let $X, E$ be Banach spaces.*

(1) *If $E$ has an unconditional basis $\{e_i\}$ and $E$ is not isomorphic to a conjugate space, then $B(X, E)$ is not isomorphic to a conjugate space.*

(2) *If $E$ has a complemented subspace which is not isomorphic to a conjugate space and which has an unconditional basis, then $B(X, E)$ is not isomorphic to a conjugate space.*

*Proof.* (1) Since $E$ is not isomorphic to a conjugate space, the basis $\{e_i\}$ is not boundedly complete [2; Cor. 12, p. 37]. Since $\{e_i\}$ is also unconditional, $E$ cannot be weakly sequentially complete and hence has a subspace isomorphic to $c_0$ by [2; Thm. 5, p. 39 and Thm.

6, p. 71]. Then since $E$ is separable this subspace isomorphic to $c_0$ must be complemented [2; p. 92].

Let $Q$ be a projection from $E$ onto $M_0$, the subspace of $E$ isomorphic to $c_0$. Fix $x_0 \in X$. Let $R$ be a projection from $X$ to $[x_0]$. Define $\mathscr{P}: B(X, E) \to B(X, E)$ by $\mathscr{P}T = QTR$ for each $T \in B(X, E)$. Then $\mathscr{P}(\mathscr{P}T) = QQTRR = QTR$ and hence $\mathscr{P}$ is a bounded projection. The map which sends $\mathscr{P}T$ onto $\mathscr{P}Tx_0$ for each $T \in B(X, E)$ is a one-to-one, bounded map from the image of $\mathscr{P}$ onto $M_0$. Hence $B(X, E)$ has a complemented subspace isomorphic to $c_0$, and by Theorem 2.1 $B(X, E)$ cannot be isomorphic to a conjugate space.

(2) $E$ still has a complemented subspace isomorphic to $c_0$.

THEOREM 2.3. *Let $E$ have an unconditional basis $\{e_i\}$. Then $\mathscr{C}(E)$, the space of compact linear operators from $E$ to $E$, is not isomorphic to a conjugate space.*

*Proof.* The map which sends a compact operator $A$ onto the operator whose matrix with respect to $\{e_i\}$ consists of the diagonal of the matrix of $A$, is a bounded projection from $\mathscr{C}(E)$ onto a subspace isomorphic to $c_0$ [4; p. 493]. Then apply Theorem 2.1.

COROLLARY 2.3. *Let $E$ have a complemented subspace $M$ with an unconditional basis. Then $\mathscr{C}(E)$ is not isomorphic to a conjugate space.*

*Proof.* Let $Q: E \to M$ be a bounded projection. Define $\mathscr{P}: \mathscr{C}(E) \to \mathscr{C}(E)$ by $\mathscr{P}A = QAQ$ for each $A \in \mathscr{C}(E)$. Then $\mathscr{P}$ is a projection onto a subspace isomorphic to $\mathscr{C}(M)$. Since $\mathscr{C}(M)$ has a complemented subspace isomorphic to $c_0$ so does $\mathscr{C}(E)$.

REMARK. It is an open question whether a separable Banach space has a complemented subspace with an unconditional basis. It is a reasonable conjecture that for any separable Banach space $E$, $\mathscr{C}(E)$ is not isomorphic to a conjugate space.

The author wishes to thank the referee of a previous paper for calling his attention to the Bessaga-Pełczynski Theorem.

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BOSTON COLLEGE

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# THE NONSTANDARD HULLS OF A UNIFORM SPACE

C. WARD HENSON

**Let $(X, \mathscr{U})$ be a uniform space in some set theoretical structure $\mathscr{M}$ and let $^*X$ be the set corresponding to $X$ in an enlargement $^*\mathscr{M}$ of $\mathscr{M}$. In this paper a set of $\mathscr{U}$-finite elements of $^*X$ is defined and this set is used to define a nonstandard hull of $(X, \mathscr{U})$. The main result is that, with some specific exceptions depending on the existence of measurable cardinal numbers, this nonstandard hull is the same as the smallest of the nonstandard hulls defined by Luxemburg. This result is used in giving a characterization of subsets of $X$ on which every uniformly continuous, real valued function is bounded. Also, two examples are given to illustrate the possible structure of the nonstandard hulls.**

The nonstandard hulls defined by Luxemburg [4] are obtained from sets $F$ of "*finite*" elements of $^*X$ which may be written in the form

$$F = \{p \mid p \in {}^*X \text{ and } {}^*f(p) \text{ is finite for all } f \text{ in } \mathscr{F}\}$$

where $\mathscr{F}$ is a set of uniformly continuous, real valued functions on $(X, \mathscr{U})$. The concept of finiteness introduced in this paper is entirely different. An element $p$ of $^*X$ is *$\mathscr{U}$-finite* if, for each $A$ in $\mathscr{U}$ there is a sequence $q_0, \cdots, q_n$ in $^*X$ which satisfies the conditions (i) $q_0 = p$, (ii) $q_n = {}^*x$ for some $x$ in $X$, and (iii) for each $j = 0, \cdots, n-1$ the pairs $(q_j, q_{j+1})$ and $(q_{j+1}, q_j)$ are both in $^*A$.

Our main result is that the set of $\mathscr{U}$-finite elements of $^*X$ is equal to the set

$$\{p \mid p \in {}^*X \text{ and } {}^*f(p) \text{ is finite for every uniformly continuous, real valued function } f \text{ on } (X, \mathscr{U})\}$$

if and only if it is impossible to partition $X$ into a measurable cardinal number of subsets $\{X_a \mid a \in I\}$ which are "uniformly open" in the sense that there is an $A$ in $\mathscr{U}$ such that

$$x \in X_a \quad \text{implies} \quad \{y \mid (y, x) \in A\} \subset X_a$$

for every $a$ in $I$. In particular, these two sets of finite elements of $^*X$ are equal whenever the number of topologically connected components of $(X, \mathscr{U})$ is smaller than every measurable cardinal number.

This result is used in giving a characterization of those subsets $Y$ of $X$ such that every uniformly continuous, real valued function on $(X, \mathscr{U})$ is bounded on $Y$, generalizing a Theorem of Atsuji [2].

Also, two examples are presented which illustrate the possible structure of the nonstandard hulls defined using the set of $\mathscr{U}$-finite elements of $^*X$. These examples are based on ideas due to L. C. Moore, to whom the author is grateful for many helpful conversations on the subject of this paper.

1. Throughout this paper $\mathscr{M}$ denotes a set theoretical structure and $^*\mathscr{M}$ denotes an enlargement of $\mathscr{M}$. (The image of an element $x$ of $\mathscr{M}$ under the embedding into $^*\mathscr{M}$ is denoted by $^*x$.) Whether $\mathscr{M}$ and $^*\mathscr{M}$ are taken to be structures for type theory (as in [4] and [6]) or to be structures for the $\varepsilon$-language of ordinary set theory (as in [5] and [7]) is a matter of taste. Most references in this paper will be to [4], although the concepts and results in [4] can easily be set in the frameworks of nonstandard analysis described in [5] and [7].

As is usual, it is assumed here that the set $N$ of positive integers and the set $R$ of real number are elements of $\mathscr{M}$, and that the embedding $x \longmapsto {}^*x$ is the identity on $R$ (and thus also on $N$.) The extensions to $^*R$ of the operations $+$ and $\cdot$ on $R$, as well as of the ordering $<$ on $R$, will be denoted by the same symbols. In general the embedding $x \longmapsto {}^*x$ is not the identity on sets in $\mathscr{M}$. Given an element $A$ of $\mathscr{M}$ it is convenient to introduce the notation $^*[A]$ for the set of standard elements of $^*A$; that is,

$$^*[A] = \{^*a \mid a \in A\} .$$

In dealing with uniform spaces there are certain useful operations on subsets of a cartesian product $C \times C$. If $A$ and $B$ are subsets of $C \times C$, recall that $A \circ B$ and $A^{-1}$ are defined by

$$A \circ B = \{(x, z) \mid \text{ for some } y, (x, y) \in A \text{ and } (y, z) \in B\}$$

$$A^{-1} = \{(x, y) \mid (y, x) \in A\} .$$

The set $A^n$ is defined recursively for $n \geqq 1$ by:

$$A^1 = A, \quad A^{n+1} = A^n \circ A .$$

Also, given an element $x$ of $C$, the set $A(x)$ is defined by

$$A(x) = \{y \mid (y, x) \in A\} .$$

Note that if $A, B$ and $C$ are elements of $\mathscr{M}$ then $^*A$ and $^*B$ are subsets of $^*C \times {}^*C$ $(= {}^*(C \times C).)$ Moreover, the following equalities hold:

$$^*(A \circ B) = (^*A) \circ (^*B)$$
$$^*(A^{-1}) = (^*A)^{-1}$$

$$^*(A^n) = (^*A)^n$$
$$^*(A(x)) = (^*A)(^*x)$$

(where $x \in C$ and $n \geqq 1$.)

Throughout this paper $(X, \mathscr{U})$ denotes a uniform space which is an element of $\mathscr{M}$. The set of all uniformly continuous, real valued functions on $(X, \mathscr{U})$ is denoted by $C(X, \mathscr{U})$. It is assumed that the reader is familiar with certain parts of the nonstandard theory of uniform spaces, as presented in [4] or [5]. In particular, recall that the monad of the filter $\mathscr{U}$ (that is, the intersection of the family $^*[\mathscr{U}]$ of subsets of $^*X \times {}^*X$) is an equivalence relation on $^*X$. The equivalence class of $p$ is denoted by $\mu(p)$, for each $p$ in $^*X$.

The collection $^*[\mathscr{U}]$ generates a filter on $^*X \times {}^*X$ which will be denoted by $\widetilde{\mathscr{U}}$. A simple, direct argument can be used to show that $\widetilde{\mathscr{U}}$ is a uniform structure on $^*X$ and that the mapping $x \longmapsto {}^*x$ is a uniform space embedding of $(X, \mathscr{U})$ into $(^*X, \widetilde{\mathscr{U}})$. Alternately, let $\mathscr{R}$ be any set of bounded semimetrics on $X$ which defines $\mathscr{U}$. ($\rho(x, y)$ is a semimetric on $X$ if $\rho$ is nonnegative, symmetric, satisfies the triangle inequality and $\rho(x, x) = 0$ for any $x$ in $X$.) For each $\rho$ in $\mathscr{R}$ a function $\tilde{\rho}$ may be defined on $^*X \times {}^*X$ by

$$\tilde{\rho}(p, q) = \text{st}\,(^*\rho(p, q)) .$$

(Here "st" is the standard part operation on finite elements of $^*R$.) Then $\tilde{\rho}$ is a semimetric on $^*X$. For each $\rho \in \mathscr{R}$ and $\delta > 0$ in $R$, let

$$A(\rho, \delta) = \{(x, y) \mid \rho(x, y) < \delta\} .$$

Then the collection $\{A(\rho, \delta) \mid \rho \in \mathscr{R}, \delta > 0\}$ generates $\mathscr{U}$ so that the collection $\{^*A(\rho, \delta) \mid \rho \in \mathscr{R}, \delta > 0\}$ generates $\widetilde{\mathscr{U}}$. But

$$^*A(\rho, \delta) \subset \{(p, q) \mid \tilde{\rho}(p, q) \leqq \delta\}$$

and

$$\{(p, q) \mid \tilde{\rho}(p, q) < \delta\} \subset {}^*A(\rho, \delta) .$$

Therefore $\widetilde{\mathscr{U}}$ is the uniformity on $^*X$ defined by the set $\{\tilde{\rho} \mid \rho \in \mathscr{R}\}$ of semimetrics on $^*X$.

Let $X_0 = \{\mu(p) \mid p \in {}^*X\}$ and let $\mathscr{U}_0$ be the quotient uniformity on $X_0$ induced by $\widetilde{\mathscr{U}}$. Denote the quotient mapping from $(^*X, \widetilde{\mathscr{U}})$ onto $(X_0, \mathscr{U}_0)$ by $\pi$. The previous remarks show that $(X_0, \mathscr{U}_0)$ is the nonstandard hull for $(X, \mathscr{U})$ constructed in [4] using any set $\mathscr{R}$ of bounded semimetrics which defines $\mathscr{U}$. (See also p. 56 of [5], where $(X_0, \mathscr{U}_0)$ is constructed and called $T_{\mathscr{U}}$.)

The definition of $\widetilde{\mathscr{U}}$ makes it clear that $\mu(p) = \mu(q)$ if and only if $p$ and $q$ have exactly the same neighborhoods in the $\widetilde{\mathscr{U}}$-topology on $^*X$. Thus $\pi$ is not only uniformly continuous, but also $\pi(^*A)$ (which equals $\{(\mu(p), \mu(q)) \mid (p, q) \in {}^*A\}$ by definition) is in $\mathscr{U}_0$ whenever $A$ is in $\mathscr{U}$. Therefore $\pi$ is an open mapping. Moreover, any net in $^*X$ which is mapped by $\pi$ onto a Cauchy net (convergent net) in $(X_0, \mathscr{U}_0)$ is a Cauchy net (convergent net) in $(^*X, \widetilde{\mathscr{U}})$. If the $\mathscr{U}$-topology on $X$ is Hausdorff, then the map taking $x$ to $\mu(x)$ is a uniform space embedding of $(X, \mathscr{U})$ into $(X_0, \mathscr{U}_0)$. (Otherwise it simply identifies those pairs of points which have exactly the same neighborhoods in the $\mathscr{U}$-topology.)

Constructing "nonstandard hulls" of $(X, \mathscr{U})$ in general involves two distinct steps: (i) the identification of a set $F$ of "finite" elements of $^*X$, and (ii) the construction of a uniformity on $F$ (and then on the set $\{\mu(p) \mid p \in F\}$ by a quotient operation.) There are many different useful concepts of "finiteness" for elements of $^*X$, each one motivated by considerations depending on the kind of mathematical structure which $X$ is assumed to carry. However there seems to be only one natural way to carry out step (ii)—by putting on $F$ the uniformity obtained by restriction from $\widetilde{\mathscr{U}}$. In that case, the nonstandard hull constructed using $F$ is just the subspace $\pi(F)$ of $(X_0, \mathscr{U}_0)$.

For example, let $\mathscr{S}$ be any set of semimetrics which defines $\mathscr{U}$. In defining a nonstandard hull using $\mathscr{S}$, Luxemburg [4] takes $F$ to be the set

$$\{p \mid {}^*\rho(p, {}^*x) \text{ is finite if } x \in X \text{ and } \rho \in \mathscr{S}\} .$$

The uniformity put on $F$ is the one defined by a set $\{\rho' \mid \rho \in \mathscr{S}\}$ of semimetrics on $F$, where

$$\rho'(p, q) = \operatorname{st}({}^*\rho(p, q))$$

for each $\rho$ in $\mathscr{S}$ and $p, q$ in $F$. If $\mathscr{R}$ is the set $\{\min(\rho, 1) \mid \rho \in \mathscr{S}\}$, then $\mathscr{R}$ also defines $\mathscr{U}$. Moreover, the uniformity defined on $F$ by the set $\{\tilde{\rho} \mid \rho \in \mathscr{R}\}$ is easily seen to be the same as the one defined on $F$ by $\{\rho' \mid \rho \in \mathscr{S}\}$. That is, this uniformity is just the restriction of $\widetilde{\mathscr{U}}$ to $F$.

In this paper an entirely different concept of "finiteness" for elements of $^*X$ is introduced. It is based on the intuitive idea that a point is "finitely far away" from a set if there is a finite chain of small steps from the point to (some element of) the set, no matter how small the steps are required to be. Thus an element of $^*X$ is taken to be finite if it is "finitely far away" from $^*[X]$, relative to the uniform space $(^*X, \widetilde{\mathscr{U}})$. (See Definition 1.2)

DEFINITION 1.1. Let $(Y, \mathscr{V})$ be any uniform space.

(i) If $A \in \mathscr{V}$ and $x, y \in X$, then an *A-chain from $x$ to $y$* is a finite sequence $x_0, \cdots, x_n$ in $Y$ which satisfies: $x_0 = x$, $x_n = y$ and, for each $i = 0, \cdots, n-1$, $(x_i, x_{i+1}) \in A \cap A^{-1}$. (The *number of steps* for such an $A$-chain is $n$.)

(ii) If $x, y \in Y$, then $x \equiv_A y$ if and only if there is an $A$-chain from $x$ to $y$.

(iii) If $x, y \in Y$, then $x \equiv_{\mathscr{V}} y$ if and only if $x \equiv_A y$ for every $A$ in $\mathscr{V}$.

If $A$ is in $\mathscr{V}$, then $A \cap A^{-1}$ is symmetric and contains the diagonal of $Y \times Y$, so that $\equiv_A$ is an equivalence relation on $Y$. Therefore $\equiv_{\mathscr{V}}$ is also an equivalence relation on $Y$. The latter relation can be calculated from any collection $\mathscr{B}$ which generates $\mathscr{V}$ as a filter on $Y \times Y$, in the sense that

$$x \equiv_{\mathscr{V}} y \longleftrightarrow x \equiv_A y \text{ for every } A \in \mathscr{B} .$$

Also, observe that if $A$ is in $\mathscr{V}$, then the equivalence classes under $\equiv_A$ are both open and closed in the $\mathscr{V}$-topology on $Y$.

Definition 1.1 will be applied to both of the uniform spaces $(X, \mathscr{U})$ and $({}^*X, \widetilde{\mathscr{U}})$. Since ${}^*[\mathscr{U}]$ generates $\widetilde{\mathscr{U}}$ as a filter on ${}^*X \times {}^*X$, it follows that for each $p, q \in {}^*X$

$$p \equiv_{\widetilde{\mathscr{U}}} q \longleftrightarrow p \equiv_{{}^*A} q \text{ for every } A \in \mathscr{U} .$$

Note that for each $A \in \mathscr{U}$, ${}^*(\equiv_A)$ is also an equivalence relation on ${}^*X$, and in general it will not be the same relation as $\equiv_{{}^*A}$. Indeed, $p$ and $q$ are in the same ${}^*(\equiv_A)$ equivalence class if there is a ${}^*$-finite sequence (hence an internal element of ${}^*\mathscr{M}$) $q_0, \cdots, q_\omega$ in ${}^*X$ which satisfies: $q_0 = p$, $q_\omega = q$ and $(q_i, q_{i+1}) \in {}^*(A \cap A^{-1})$ for every $i = 0, \cdots, \omega - 1$. Such a ${}^*$-finite sequence may exist without any such finite sequence existing: in that case $p \equiv_{{}^*A} q$ would be false.

DEFINITION 1.2. An element $p$ of ${}^*X$ is *$\mathscr{U}$-finite* if, for each $A \in \mathscr{U}$, there exists an $x$ in $X$ which satisfies $p \equiv_{{}^*A} {}^*x$.

The set of $\mathscr{U}$-finite elements of ${}^*X$ will be denoted by $\mathrm{fin}_{\mathscr{U}}({}^*X)$.

It is clear that if $p$ is $\mathscr{U}$-finite, then every element of $\mu(p)$ is also $\mathscr{U}$-finite. In the language of [4], this says that $\mathrm{fin}_{\mathscr{U}}({}^*X)$ is $\mu$-saturated. Also the condition $p \in \mathrm{fin}_{\mathscr{U}}({}^*X)$ is equivalent to a condition on the ultrafilter $\{Y \mid Y \subset X \text{ and } p \in {}^*Y\}$ determined by $p$. Namely, $p$ is $\mathscr{U}$-finite if and only if for each $A \in \mathscr{U}$ there exist $x \in X$ and $n \geqq 1$ such that $p \in ({}^*A)^n({}^*x) = {}^*(A^n(x))$. Therefore, if $p$ is $\mathscr{U}$-finite then each element of the monad of the ultrafilter $\{Y \mid Y \subset X$ and $p \in {}^*Y\}$ is also $\mathscr{U}$-finite. In the language of [4] this says that $\mathrm{fin}_{\mathscr{U}}({}^*X)$ is $\mu_d$-saturated.

If $\rho$ is any semimetric on $X$ which defines a weaker uniformity than $\mathscr{U}$, and $a \in X$, then the function $f(x) = \rho(x, a)$ is $\mathscr{U}$-uniformly continuous (since $|\rho(x, a) - \rho(y, a)| \leqq \rho(x, y)$.) Thus the sets $F$ of finite elements of $^*X$ considered in [4] are all of the form

$$F = \{p \mid {}^*f(p) \text{ is finite for every } f \in \mathscr{F}\}$$

where $\mathscr{F}$ is a set of $\mathscr{U}$-uniformly continuous, real valued functions on $X$. The next result shows that each of these sets has $\text{fin}_{\mathscr{U}}({}^*X)$ as a subset.

THEOREM 1.3. *If* $f \in C(X, \mathscr{U})$ *and* $p \in \text{fin}_{\mathscr{U}}({}^*X)$ *then* ${}^*f(p)$ *is finite.*

*Proof.* Since $f$ is uniformly continuous, there exists $A$ in $\mathscr{U}$ which satisfies

$$(x, y) \in A \longrightarrow |f(x) - f(y)| \leqq 1 .$$

Since $p$ is $\mathscr{U}$-finite, there is a ${}^*A$-chain $q_0, \cdots, q_n$ from $p$ to ${}^*x$, for some $x$ in $X$. Therefore

$$\begin{aligned} |{}^*f(p) - {}^*f(x)| &\leqq \Sigma |{}^*f(q_i) - {}^*f(q_{i+1})| \\ &\leqq n . \end{aligned}$$

It follows that ${}^*f(p)$ must be finite.

THEOREM 1.4. $\text{fin}_{\mathscr{U}}({}^*X)$ *is closed in the* $\widetilde{\mathscr{U}}$*-topology on* ${}^*X$, *and* $\text{pns}_{\mathscr{U}}({}^*X) \subset \text{fin}_{\mathscr{U}}({}^*X)$.

*Proof.* For each $A$ in $\mathscr{U}$ the set

$$\{p \mid p \equiv_{*A} {}^*x \text{ for some } x \in X\}$$

is a disjoint union of $\equiv_{*A}$ equivalence classes, each of which is open and closed in the $\widetilde{\mathscr{U}}$-topology on ${}^*X$. It follows that this set is, itself, open and closed in that topology. Finally, $\text{fin}_{\mathscr{U}}({}^*X)$ is an intersection of such sets, so that it must be a closed set.

That $\text{pns}_{\mathscr{U}}({}^*X)$ is a subset of $\text{fin}_{\mathscr{U}}({}^*X)$ follows immediately, using the obvious fact that ${}^*[X]$ is a subset of $\text{fin}_{\mathscr{U}}({}^*X)$ and using Theorem 3.15.2 of [4]. (This Theorem implies that $\text{pns}_{\mathscr{U}}({}^*X)$ is the closure of ${}^*[X]$ in the $\widetilde{\mathscr{U}}$-topology on ${}^*X$. The extra assumptions on ${}^*\mathscr{M}$ made in [4] are not needed for this result. See also Theorem 7.5.3 of [5].)

Let $\kappa$ be an uncountable cardinal number which is strictly larger than the cardinality of some filter basis for $\mathscr{U}$. It is well known that there must be a set $\mathscr{R}$ of bounded semimetrics which defines $\mathscr{U}$ and which has cardinality less than $\kappa$. Theorem 3.15.1 of [4] implies that if $^*\mathscr{M}$ is $\kappa$-saturated, then $(X_0, \mathscr{U}_0)$ is a complete uniform space. (Theorem 3.15.1 has the added assumption that $^*\mathscr{M}$ is an ultrapower of $\mathscr{M}$, but this is not necessary. It may be removed by noting that the completeness of $(X_0, \mathscr{U}_0)$ can be proved by considering only Cauchy nets over index sets of cardinality less than $\kappa$, and then using Theorem 1.8.3.)

Therefore when $^*\mathscr{M}$ is $\kappa$-saturated the uniform space $(^*X, \widetilde{\mathscr{U}})$ is also complete. By Theorem 1.4 this implies that the restriction of $\widetilde{\mathscr{U}}$ to $\mathrm{fin}_{\mathscr{U}}(^*X)$ defines a complete uniform space. It should also be noted that each set of the form $\{p \mid {}^*f(p)$ is finite if $f \in \mathscr{F}\}$ is closed in the $\widetilde{\mathscr{U}}$-topology when $\mathscr{F}$ is a subset of $C(X, \mathscr{U})$. Therefore each of the nonstandard hulls of [4] is a complete uniform space when $^*\mathscr{M}$ is $\kappa$-saturated, even when $\mathscr{F}$ may have cardinality $\geqq \kappa$.

2. This section is concerned with the relationship between $\mathrm{fin}_{\mathscr{U}}(^*X)$ and the set

$$F_0 = \{p \mid {}^*f(p) \text{ is finite for all } f \in C(X, \mathscr{U})\} .$$

As argued in §1, $\pi(F_0)$ is the smallest of the nonstandard hulls of $(X, \mathscr{U})$ constructed in [4]. By Theorem 1.3, $\mathrm{fin}_{\mathscr{U}}(^*X)$ is a subset of $F_0$. In fact, the two sets are equal, except in certain circumstances depending on the existence of measurable cardinal numbers. (Corollary 2.5) The principal tool in proving this is the following result.

LEMMA 2.1. *If $A$ is in $\mathscr{U}$ and $x \equiv_A y$ for all $x, y \in X$, then there is a semimetric $\rho$ on $X$ which satisfies*

( i ) *the uniformity defined by $\rho$ contains $A$ and is weaker than $\mathscr{U}$, and*

(ii) *for each $p, q \in {}^*X$,*

$$p \equiv_{^*A} q \longleftrightarrow {}^*\rho(p, q) \text{ is finite} .$$

*Proof.* The proof uses a modification of a construction given in [3]. Let $A$ be in $\mathscr{U}$ and suppose $x \equiv_A y$ holds for all $x, y \in X$. It may be assumed that $A$ is symmetric (replacing $A$ by $A \cap A^{-1}$ if necessary.) Let $Z$ be the set of all the integers. Select a sequence $\{A_n \mid n \in Z\}$ of symmetric sets in $\mathscr{U}$ as follows: (i) $A_0 = A$, (ii) for $n > 0$ define $A_n$ inductively by

$$A_n = (A_{n-1})^3 ,$$

(iii) for $n < 0$ select $A_n$ inductively so that

$$(A_n)^3 \subset A_{n+1} .$$

Then $\{A_n \mid n \in Z\}$ is a chain of sets in $\mathscr{U}$, and it satisfies

$$(A_n)^3 \subset A_{n+1} \quad \text{for all } n \in Z . \tag{2.1}$$

Moreover, since $n \geqq 0$ implies $A_n = (A^3)^n$, it follows that

$$\cup \{A_n \mid n \in Z\} = \cup \{A^{3^n} \mid n \geqq 1\} . \tag{2.2}$$

The assumption that $x \equiv_A y$ holds for every $x, y \in X$ means that the right side of (2.2) is equal to $X \times X$. Therefore a function $g$ on $X \times X$ may be defined by

$$g(x, y) = \begin{cases} 2^n & \text{if } (x, y) \in A_n \sim A_{n-1} \\ 0 & \text{if } (x, y) \in A_n \quad \text{for all } n \in Z . \end{cases}$$

In particular, for $n \geqq 0$

$$g(x, y) \leqq 2^n \longleftrightarrow (x, y) \in A^{3^n} (= A_n) .$$

Passing this to $^*\mathscr{M}$, it follows that for any $p, q \in {}^*X$ and $n \in N$

$$^*g(p, q) \leqq 2^n \longleftrightarrow (p, q) \in ({}^*A)^{3^n} .$$

Therefore, if $p, q \in {}^*X$, then

$$^*g(p, q) \text{ is finite} \longleftrightarrow p \equiv_{^*A} q .$$

The desired semimetric $\rho$ is then defined from $g$ by

$$\rho(x, y) = \inf \left\{ \sum_{i=0}^{n-1} g(x_i, x_{i+1}) \mid x_0, \cdots, x_n \text{ is a sequence in } X, x_0 = x \text{ and } x_n = y \right\} .$$

(That $\rho$ is nonnegative, symmetric and satisfies $\rho(x, x) = 0$ for all $x$ in $X$ follows from the fact that the function $g$ has the same properties. That $\rho$ satisfies the triangle inequality is equally obvious.) The fundamental fact about $\rho$ is the inequality

$$\rho(x, y) \leqq g(x, y) \leqq 2\rho(x, y) \tag{2.3}$$

which holds for all $x, y \in X$. The first inequality follows immediately from the definition. The second is proved by showing that if $x_0, \cdots, x_n$ is a sequence in $X$,

$$g(x_0, x_n) \leqq 2 \cdot \sum_{i=0}^{n-1} g(x_i, x_{i+1}) . \tag{2.4}$$

The proof of (2.4) is by induction on $n$, using (2.1). The details are like those in the proof of Theorem 6.7 in [3], and they will be omitted.

Passing the inequality (2.3) to $^*\mathscr{M}$, it follows that $^*\rho(p, q)$ is finite exactly when $^*g(x, y)$ is finite. Therefore, for any $p, q \in {}^*X$

$$p \equiv_{*A} q \leftrightarrow {}^*\rho(p, q) \text{ is finite} .$$

It thus remains only to show that $\rho$ satisfies (i). The definition of $g$ implies that $A = \{(x, y) \mid g(x, y) \leqq 1\}$, and by equation (2.3) it follows that $A$ contains the set $\{(x, y) \mid \rho(x, y) \leqq 1/2\}$. This shows that $A$ is in the uniformity defined by $\rho$. Finally, for each $n \in Z$

$$A_n = \{(x, y) \mid g(x, y) \leqq 2^n\} \subset \{(x, y) \mid \rho(x, y) \leqq 2^n\} .$$

This shows that the uniformity defined by $\rho$ is weaker than $\mathscr{U}$, and completes the proof.

Throughout the rest of this section let $\mathscr{K}$ denote the set of all cardinal numbers $\kappa$ which support $\omega$-complete, free ultrafilters. It is well known that if $\mathscr{K}$ is nonempty, then the smallest member $\kappa_0$ of $\mathscr{K}$ is actually measurable. (In fact, every $\omega$-complete ultrafilter on $\kappa_0$ is $< \kappa_0$-complete.) Moreover, in that case the class $\mathscr{K}$ consists exactly of the cardinal numbers $\geqq \kappa_0$. (There does not seem to be any accepted term designating the members of $\mathscr{K}$. Some authors call them "measurable" but this does not agree with current terminology in set theory.)

Given a set $I$ in $\mathscr{M}$ and an element $p$ of $^*I$, let Fil$(p)$ denote the ultrafilter $\{J \mid J \subset I \text{ and } p \in {}^*J\}$ on $I$ determined by $p$. (Fil$_I(p)$ will be used for Fil$(p)$ if necessary to avoid confusion.) Recall that Fil$(p)$ is a free ultrafilter if and only if $p$ is not standard.

LEMMA 2.2. *For each* $p \in {}^*I$, Fil$(p)$ *is* $\omega$-*complete if and only if* $^*f(p)$ *is finite for every real valued function* $f$ *on* $I$.

*Proof*. Given any real valued function $f$ on $I$ and $n \geqq 1$, define

$$A_n(f) = \{x \mid x \in I \text{ and } |f(x)| \geqq n\} .$$

Then $\{A_n(f) \mid n \geqq 1\}$ is a decreasing chain of subsets of $I$ and the intersection of the chain is empty.

If $p \in {}^*I$ and there exists a real valued function $f$ on $I$ such that $^*f(p)$ is infinite, then $p \in {}^*A_n(f)$ for every $n \geqq 1$. That is,

$$\{A_n(f) \mid n \geqq 1\}$$

is contained in Fil$(p)$. This shows that Fil$(p)$ is not $\omega$-complete.

Conversely, suppose Fil$(p)$ is not $\omega$-complete. Then there exists a decreasing chain $\{A_n \mid n \geqq 1\}$ in Fil$(p)$ whose intersection is empty. It may be assumed that $A_1 = I$. Thus a real valued function $f$ may be defined on $I$ by

$$f(x) = \max\{n \mid x \in A_n\} .$$

Evidently $A_n(f) = A_n$ for each $n \geqq 1$. The assumption that $p \in {}^*(A_n)$ for all $n \geqq 1$ implies that $|{}^*f(p)| \geqq n$ for all $n \geqq 1$. That is, ${}^*f(p)$ is infinite.

Let $\mathscr{D}$ be the discrete uniformity on $X$ (that is, $\mathscr{D}$ is the principal filter on $X \times X$ generated by the diagonal set.) Clearly $C(X, \mathscr{D})$ is the set of all real valued functions on $X$ and $\text{fin}_{\mathscr{D}}({}^*X) = {}^*[X]$. Thus Lemma 2.2 says that

$$\text{fin}_{\mathscr{D}}({}^*X) = \{p \mid {}^*f(p) \text{ is finite if } f \in C(X, \mathscr{D})\}$$

if and only if the cardinality of $X$ is not in $\mathscr{K}$.

The next results describe completely the conditions under which an element of $F_0$ is not $\mathscr{U}$-finite.

THEOREM 2.3. *If $p \in {}^*X$ is not $\mathscr{U}$-finite but ${}^*f(p)$ is finite for every $f \in C(X, \mathscr{U})$, then there exists an element $A$ of $\mathscr{U}$ which satisfies*

$$Y \in Fil_X(p) \longrightarrow \textit{the number of } \equiv_A \textit{ equivalence classes which intersect } Y \textit{ is in } \mathscr{K} .$$

*Proof.* Assume that $p \in {}^*X$ is not $\mathscr{U}$-finite, and that ${}^*f(p)$ is finite whenever $f \in C(X, \mathscr{U})$. There exists a symmetric element $A$ of $\mathscr{U}$ such that $p \equiv_{{}^*A} {}^*x$ is false for every $x \in X$. Let $\{X_a \mid a \in I\}$ be a one-to-one enumeration of the $\equiv_A$ equivalence classes, and let a function $c$ from $X$ to $I$ be defined by

$$c(x) = a \longleftrightarrow x \in X_a .$$

It will be shown first that ${}^*c(p)$ is not a standard element of ${}^*I$. If otherwise, there exists $a \in I$ which satisfies ${}^*a = {}^*c(p)$, and hence $p \in {}^*(X_a)$. Let $A_a$ equal $A \cap (X_a \times X_a)$ and let $\mathscr{U}_a$ be the uniformity obtained by restricting $\mathscr{U}$ to $X_a$. Since $X_a$ is an $\equiv_A$ equivalence class, $x \equiv_{A_a} y$ holds for every $x, y \in X_a$. By Lemma 2.1 there exists a semimetric $\rho$ on $X_a$ which satisfies (i) the uniformity defined by $\rho$ on $X_a$ contains $A_a$ and is weaker than $\mathscr{U}_a$, and (ii) for any $r, s \in {}^*(X_a)$,

$$r \equiv_{{}^*(A_a)} s \longleftrightarrow {}^*\rho(r, s) \text{ is finite} .$$

Since $X_a$ is an $\equiv_A$ equivalence class, $r \equiv_{{}^*A} s$ is equivalent to $r \equiv_{{}^*(A_a)} s$, for elements $r, s$ of $^*(X_a)$. Thus (ii) implies

(ii′) for any $r, s \in {}^*(X_a)$,

$$r \equiv_{*A} s \longleftrightarrow {}^*\rho(r, s) \text{ is finite .}$$

Let $x_0$ be a fixed element of $X_a$ and define a function $h$ on $X$ by

$$h(x) = \begin{cases} 0 & \text{if } x \notin Y_a \\ \rho(x_0, x) & \text{if } x \in X_a \,. \end{cases}$$

Given $\delta > 0$, there exists an element $B_a$ of $\mathscr{U}_a$ which satisfies

$$(x, y) \in B_a \longrightarrow \rho(x, y) < \delta$$

by (i) above. This implies that $B_a$ contains a set of the form $B \cap (X_a \times X_a)$, where $B$ is in $\mathscr{U}$, and it may be assumed that $B \subset A$. If $(x, y) \in B$, then either $x$ and $y$ are both outside $X_a$, and $h(x) = h(y) = 0$, or $(x, y) \in B_a$. In the latter case

$$|h(x) - h(y)| = |\rho(x_0, x) - \rho(x, y)| \leqq \rho(x, y) < \delta \,.$$

Therefore, $h$ is an element of $C(X, \mathscr{U})$. This implies that $^*h(p)$ is finite. However, since $p \in {}^*(X_a)$, $^*h(p) = {}^*\rho({}^*x_0, p)$. Thus, by (ii′) above, $p \equiv_{*A} {}^*x_0$ which is a contradiction. This shows that $^*c(p)$ is not a standard element of $^*I$.

Now let $Y$ be any subset of $X$ which satisfies $p \in {}^*Y$, and let $J = c(Y)$. It must be shown that there exists an $\omega$-complete, free ultrafilter on $J$. If not, then the ultrafilter Fil($^*c(p)$) is not $\omega$-complete. (It is free since $^*c(p)$ is not standard.) In that case, by Lemma 2.2 there exists a real valued function $f$ on $J$ such that $^*f(^*c(p))$ is infinite. Define a function $g$ on $X$ by

$$g(x) = \begin{cases} 0 & \text{if } c(x) \notin J \\ f(c(x)) & \text{if } c(x) \in J \,. \end{cases}$$

If $(x, y) \in A$, then $x \equiv_A y$ and hence $c(x) = c(y)$. This implies that $g$ is in $C(X, \mathscr{U})$. But $^*g(p) = {}^*f({}^*c(p))$, so that $^*g(p)$ is infinite. This contradiction shows that $\text{Fil}_J({}^*c(p))$ is an $\omega$-complete, free ultrafilter on $J$, and completes the proof.

THEOREM 2.4. *If $Y \subset X$ and the number of $\equiv_A$ equivalence classes which intersect $Y$ is in $\mathscr{K}$, for some $A$ in $\mathscr{U}$, then there exists an element $p$ of $^*Y$ which is not $\mathscr{U}$-finite but which satisfies: $^*f(p)$ is finite for every $f \in C(X, \mathscr{U})$.*

*Proof.* Given $A \in \mathscr{U}$ and $Y \subset X$ as stated, there is a subset $W$ of $Y$ which has one element in common with each $\equiv_A$ equivalence class which intersects $Y$. Moreover, there exists an $\omega$-complete, free ultrafilter on $W$. Since $^*\mathscr{M}$ is an enlargement of $\mathscr{M}$, this means that there is an element $p$ of $^*W$ which is not standard and such that $\mathrm{Fil}_W(p)$ is $\omega$-complete. By Lemma 2.2, $^*f(p)$ is finite for every real valued function $f$ on $W$, hence for every $f$ in $C(X, \mathscr{U})$. It thus suffices to show that $p \equiv_{^*A} {}^*x$ is false for every $x$ in $X$. Otherwise, there exist $x \in X$ and $n \geqq 1$ which satisfy $(p, {}^*x) \in {}^*B^n$, where $B$ is $A \cap A^{-1}$. Since $p \in {}^*W$ it follows that for some $w \in W$, $(w, x) \in B^n$. Therefore $(p, {}^*w) \in ({}^*B)^{2n}$. But since $p$ is not standard, this implies that there exists $w' \in W$ such that $w'$ is distinct from $w$ and $(w', w) \in B^{2n}$. That is, $w' \equiv_A w$ and hence $W$ has two elements from the same $\equiv_A$ equivalence class. This contradiction proves that $p$ has the desired properties.

COROLLARY 2.5. *The equality*

$$\mathrm{fin}_{\mathscr{U}}({}^*X) = \{p \mid {}^*f(p) \text{ is finite for all } f \in C(X, \mathscr{U})\}$$

*holds if and only if the number of $\equiv_A$ equivalence classes is not in $\mathscr{K}$ for every $A \in \mathscr{U}$.*

In cases where the cardinality assumption of Corollary 2.5 holds (in particular, if there is no $\omega$-complete, free ultrafilter on $X$) then the smallest nonstandard hull constructed in [4] is also the subspace $\pi(\mathrm{fin}_{\mathscr{U}}({}^*X))$ of $(X_0, \mathscr{U}_0)$. This fact is helpful in determining the elements of this nonstandard hull, since it is usually easier to show that $\mu(p)$ is an element by showing that $p$ is $\mathscr{U}$-finite, and to show that $\mu(p)$ is not an element by exhibiting a function $f$ in $C(X, \mathscr{U})$ such that $^*f(p)$ is infinite. (See the examples in §4.)

3. Atsuji [2] has given a condition on $(X, \mathscr{U})$ which is equivalent to the statement that every function in $C(X, \mathscr{U})$ is bounded, and which is closely related to the concepts discussed above. In this section a nonstandard proof is given of a natural generalization of Atsuji's Theorem. (The ideas used in proving this Theorem are also used in §4.)

DEFINITION 3.1. A subset $Y$ of $X$ is *finitely chainable in* $(X, \mathscr{U})$ if, for each $A \in \mathscr{U}$, there exist $y_1, \cdots, y_k$ in $Y$ and $n \geqq 1$ which satisfy

$$Y \subset A^n(y_1) \cup \cdots \cup A^n(y_k) .$$

The uniform space $(X, \mathscr{U})$ is *finitely chainable* [2] if $X$ is finitely chainable in $(X, \mathscr{U})$.

THEOREM 3.2. *For any subset $Y$ of $X$, $Y$ is finitely chainable in*

*$(X, \mathscr{U})$ if and only if* $^*Y \subset \text{fin}_{\mathscr{U}}(^*X)$.

*Proof.* Suppose $Y$ is finitely chainable in $(X, \mathscr{U})$. Given $A \in \mathscr{U}$, there exist $y_1, \cdots, y_k$ in $Y$ and $n \geqq 1$ which satisfy

$$Y \subset A^n(y_1) \cup \cdots \cup A^n(y_k) .$$

If follows that $^*Y \subset (^*A)^n(^*y_1) \cup \cdots \cup (^*A)^n(^*y_k)$. If $A$ is symmetric, this implies that each element of $^*Y$ is in the same $\equiv_{*A}$ equivalence class with one of the elements $^*y_1, \cdots, ^*y_k$. Therefore $^*Y$ is contained in $\text{fin}_{\mathscr{U}}(^*X)$.

Conversely, suppose $Y$ is not finitely chainable in $(X, \mathscr{U})$. Thus there exists a symmetric set $A$ in $\mathscr{U}$ such that for any $n \geqq 1$ and $y_1, \cdots, y_k \in Y$, the union $A^n(y_1) \cup \cdots \cup A^n(y_k)$ does not contain $Y$. For each $y \in Y$ and $n \geqq 1$ define

$$S(n, y) = \{x \mid x \in Y \quad \text{and} \quad x \notin A^n(y)\} .$$

The assumptions on $Y$ imply that the collection $\{S(n, y)\}$ has the finite intersection property. Since $^*\mathscr{M}$ is an enlargement, there exists $p \in {}^*Y$ which satisfies $p \in {}^*S(n, y)$ for every $y \in Y$ and $n \geqq 1$.

It will be shown that $p$ is not $\mathscr{U}$-finite, thus showing that $^*Y$ is not contained in $\text{fin}_{\mathscr{U}}(^*X)$. Otherwise there exist $x \in X$ and $n \geqq 1$ which satisfy $(p, {}^*x) \in (^*A)^n$. This implies that there exists $y$ in $Y \cap A^n(x)$, and therefore $p \in A^{2n}(y)$. That is, $p \notin {}^*S(2n, y)$, which is a contradiction.

The following result generalizes the theorem due to Atsuji [2] which states that $(X, \mathscr{U})$ is finitely chainable if and only if every function in $C(X, \mathscr{U})$ is bounded.

THEOREM 3.3. *For any subset $Y$ of $X$, $Y$ is finitely chainable in $(X, \mathscr{U})$ if and only if every function in $C(X, \mathscr{U})$ is bounded on $Y$.*

*Proof.* If $Y$ is finitely chainable in $(X, \mathscr{U})$, then by Theorem 3.2 $^*Y \subset \text{fin}_{\mathscr{U}}(^*X)$. For any function $f$ in $C(X, \mathscr{U})$, this implies that $^*Y \subset \{p \mid {}^*f(p) \text{ is finite}\}$ by Theorem 1.3. Therefore the set

$$\{|{}^*f(p)| \mid p \in {}^*Y\} ,$$

which is internal, has a finite upper bound $M$ in $R$. But this implies that $f$ is bounded by $M$ on $Y$. That is, each member of $C(X, \mathscr{U})$ is bounded on $Y$.

Conversely, suppose each function in $C(X, \mathscr{U})$ is bounded on $Y$. To show that $Y$ is finitely chainable in $(X, \mathscr{U})$ it suffices to prove $^*Y \subset \text{fin}_{\mathscr{U}}(^*X)$, by Theorem 3.2. If not, then by Theorem 2.3 there must exist an element $A$ of $\mathscr{U}$ such that the number of $\equiv_A$ equivalence classes which intersect $Y$ is in $\mathscr{K}$. In particular there are countably many (distinct) $\equiv_A$ equivalence classes $X_1, \cdots, X_n, \cdots$, each of which intersects $Y$. The function $f$ defined on $X$ by

$$f(x) = \begin{cases} n & \text{if } x \in X_n \\ 0 & \text{if } x \notin X_n\,, \quad \text{all } n \geqq 1 \end{cases}$$

is therefore unbounded on $Y$. However, $f$ is constant on $\equiv_A$ equivalence classes, and thus $f$ is in $C(X, \mathscr{U})$. This is a contradiction, and completes the proof.

REMARK. Theorem 3.2 allows us to say exactly when there is a single function $f$ in $C(X, \mathscr{U})$ which satisfies

$$\mathrm{fin}_{\mathscr{U}}({}^*X) = \{p \mid {}^*f(p) \text{ is finite}\}\,.$$

Namely, this equality holds if and only if the sets $\{x \mid |f(x)| \leqq n\}$ (for $n \geqq 1$) are all finitely chainable in $(X, \mathscr{U})$. (The equality holds if and only if $\{p \mid |{}^*f(p)| \leqq n\} \subset \mathrm{fin}_{\mathscr{U}}({}^*X)$ for all $n \geqq 1$ (by Theorem 1.3) if and only if $\{x \mid |f(x)| \leqq n\}$ is finitely chainable in $(X, \mathscr{U})$ for all $n \geqq 1$ (by Theorem 3.2).)

In particular, if $\mathscr{U}$ is the uniformity defined by some metric $\rho$ on $X$, then the equality

$$\mathrm{fin}_{\mathscr{U}}({}^*X) = \{p \mid {}^*\rho(p, {}^*x) \text{ is finite}\}$$

holds for some (or, equivalently, every) $x$ in $X$, if and only if

$$\{y \mid \rho(y, x) \leqq n\}$$

is finitely chainable in $(X, \mathscr{U})$ for every $n \geqq 1$.

4. Given a metric $\rho$ on $X$, Robinson [6] says that $p$ and $q$ are in the same *galaxy* of ${}^*X$ if ${}^*\rho(p, q)$ is finite. Generalizing this idea Luxemburg [4] defines $p$ and $q$ to be in the same galaxy relative to a set $\mathscr{S}$ of semimetrics on $X$ if ${}^*\rho(p, q)$ is finite for every $\rho$ in $\mathscr{S}$. The following definition of the *$\mathscr{U}$-galaxies* of ${}^*X$ arises naturally from the considerations which led to Definition 1.2.

DEFINITION 4.1. If $p, q \in {}^*X$, then $p$ and $q$ are in the same *$\mathscr{U}$-galaxy* if $p \equiv_{\mathscr{U}} q$.

THEOREM 4.2. *If $p$ and $q$ are in the same $\mathscr{U}$-galaxy and $\rho$ is any semimetric on $X$ which defines a uniformity weaker than $\mathscr{U}$, then ${}^*\rho(p, q)$ is finite.*

*Proof.* Since $\rho$ defines a uniformity weaker than $\mathscr{U}$ there exists

$A \in \mathscr{U}$ which satisfies

$$(x, y) \in A \longrightarrow \rho(x, y) \leqq 1 .$$

Since $p$ and $q$ are in the same $\mathscr{U}$-galaxy, there is a ${}^*A$-chain $q_0, \cdots, q_n$ from $p$ to $q$. Using the triangle inequality for ${}^*\rho$ yields

$${}^*\rho(p, q) \leqq \sum {}^*\rho(q_i, q_{i+1}) \leqq n .$$

Therefore ${}^*\rho(p, q)$ is finite.

DEFINITION 4.3. A subset $Y$ of $X$ is *chain connected in* $(X, \mathscr{U})$ if $x \equiv_{\mathscr{U}} y$ for every $x, y \in Y$. The uniform space $(X, \mathscr{U})$ is *chain connected* if $X$ is chain connected in $(X, \mathscr{U})$.

THEOREM 4.4. *Let $\mathscr{S}$ be the set of all semimetrics which define weaker uniformities than $\mathscr{U}$ and suppose that $Y$ is chain connected in $(X, \mathscr{U})$. Then for every $p, q \in {}^*Y$: $p$ and $q$ are in the same $\mathscr{U}$-galaxy if and only if ${}^*\rho(p, q)$ is finite for every $\rho$ in $\mathscr{S}$.*

*Proof.* Let $Y$ and $\mathscr{S}$ be as stated and assume $p, q \in {}^*Y$. The implication in one direction is contained in Theorem 4.2. Conversely, suppose that ${}^*\rho(p, q)$ is finite for all $\rho$ in $\mathscr{S}$. To prove that $p$ and $q$ are in the same $\mathscr{U}$-galaxy it is necessary to show that $p \equiv_{{}^*A} q$ for every symmetric set $A$ in $\mathscr{U}$. Given such an $A$, the fact that $Y$ is chain connected in $(X, \mathscr{U})$ means that there is an $\equiv_A$ equivalence class $W$ which contains $Y$. Let $A_W = A \cap (W \times W)$ and let $\mathscr{U}_W$ be the restriction of $\mathscr{U}$ to $W$. As in the proof of Theorem 2.3, an application of Lemma 2.1 yields a semimetric $\rho$ on $W$ which satisfies (i) the uniformity defined by $\rho$ on $W$ is weaker than $\mathscr{U}_W$, and (ii) for any $r, s \in {}^*W$, $r \equiv_{{}^*A} s$ if and only if ${}^*\rho(r, s)$ is finite.

Select $w_0$ in $W$ and let $f$ be the function defined on $X$ by

$$f(x) = \begin{cases} w_0 & \text{if } x \notin W \\ x & \text{if } x \in W . \end{cases}$$

Then $f$ is constant on $\equiv_A$ equivalence classes so that $f$ is uniformly continuous as a map from $(X, \mathscr{U})$ to $(W, \mathscr{U}_W)$. It follows that the semimetric $\rho'$ defined on $X$ by

$$\rho'(x, y) = \rho(f(x), f(y))$$

defines a weaker uniformity on $X$ than $\mathscr{U}$. By assumption, this means that ${}^*\rho'(p, q) = {}^*\rho(p, q)$ is finite. Therefore $p \equiv_{{}^*A} q$ by (ii) above, completing the proof.

COROLLARY 4.5. *If $(X, \mathscr{U})$ is chain connected and $\mathscr{S}$ is the set* *of all semimetrics which define weaker uniformities on $X$ than $\mathscr{U}$, then the $\mathscr{U}$-galaxies form the same partition of $^*X$ as do the galaxies determined by $\mathscr{S}$.*

REMARK. As was noted above, if $A$ is in $\mathscr{U}$, then each $\equiv_A$ equivalence class is open and closed in the $\mathscr{U}$-topology on $X$. Therefore if $X$ is connected in the $\mathscr{U}$-topology, then $(X, \mathscr{U})$ must be chain connected. Applying the same reasoning to the uniform space $(^*X, \widetilde{\mathscr{U}})$ shows that any subset of $^*X$ which is connected in the $\widetilde{\mathscr{U}}$-topology must be entirely contained in one $\mathscr{U}$-galaxy.

THEOREM 4.6. *If $(X, \mathscr{U})$ is chain connected, then the following conditions are equivalent:*

(i) *There is a semimetric $\rho$ which defines a uniformity weaker than $\mathscr{U}$ and which satisfies: $p$ and $q$ are in the same $\mathscr{U}$-galaxy in $^*X$ if and only if $^*\rho(p, q)$ is finite:*

(ii) *There is an element $A_0$ of $\mathscr{U}$ which satisfies: for each $A \in \mathscr{U}$ there is an $n \geqq 1$ such that $A_0 \subset A^n$.*

*Proof.* (i) $\rightarrow$ (ii): Let $\rho$ be as in (i) and define

$$A_0 = \{(x, y) \mid \rho(x, y) \leqq 1\}$$

as that $A_0$ is in $\mathscr{U}$. If $A_0$ does not satisfy (ii), then there is an element $A$ of $\mathscr{U}$ such that for no $n \geqq 1$ does $A^n$ contain $A_0$. That is, for each $n \geqq 1$ there exists a pair $x_n$, $y_n$ of elements $X$ which satisfy $\rho(x_n, y_n) \leqq 1$ and $(x_n, y_n) \notin A^n$. Let $\omega$ be an infinite member of $^*N$. Then $^*\rho(^*x_\omega, {}^*y_\omega) \leqq 1$, so that by (i) there is a $^*A$-chain $q_0, \cdots, q_n$ from $^*x_\omega$ to $^*y_\omega$. That is, $(^*x_\omega, {}^*y_\omega)$ is an element of $(^*A)^n = {}^*(A^n)$. But since $\omega$ is not standard, this means that $(x_k, y_k) \in A^n$ holds for infinitely many values of $k$ in $N$. This contradicts the choice of the pairs $(x_k, y_k)$ and proves that $A_0$ satisfies (ii).

(ii) $\rightarrow$ (i): Assume that $A_0$ satisfies (ii). Then for each $A$ in $\mathscr{U}$, $^*A_0 \subset {}^*A^n$ (for some $n$ depending on $A$.) Therefore $p \equiv_{^*A_0} q$ implies $p \equiv_{^*A} q$, for every $p, q \in {}^*X$ and every $A \in \mathscr{U}$. Thus the $\equiv_{^*A_0}$ equivalence classes and the $\mathscr{U}$-galaxies are exactly the same. The existence of the semimetric required in (i) now follows, using Lemma 2.1 and the fact that $(X, \mathscr{U})$ is chain connected.

REMARK. Suppose $(X, \mathscr{U})$ is chain connected and $\mathscr{U}$ is defined by a metric $\rho_0$. If $(X, \mathscr{U})$ satisfies the conditions in Theorem 4.6, then there exists a metric $\rho_1$ which defines $\mathscr{U}$ and also satisfies: $p$ and $q$ are in the same $\mathscr{U}$-galaxy if and only if $^*\rho_1(p, q)$ is finite. That is, $\mathscr{U}$ can be "remetrized" so that the $\mathscr{U}$-galaxies and the galaxies defined by the metric coincide. To construct $\rho_1$, simply choose $\rho$ as in 4.6.i and define

$$\rho_1(x, y) = \max\{\rho(x, y), \min(\rho_0(x, y), 1)\} .$$

The following two examples were developed in collaboration with L. C. Moore, and are based on ideas due to him. In each case the uniformity $\mathscr{U}$ is defined by a metric on $X$. The first example shows that a $\mathscr{U}$-finite point need not be in the same $\mathscr{U}$-galaxy with any standard point, even when $(X, \mathscr{U})$ is complete. The second example shows that even when the original space $(X, \mathscr{U})$ is arcwise connected, the smallest nonstandard hull of $(X, \mathscr{U})$ constructed in [4] need not even be chain connected (or, what is the same, the uniform space obtained by restricting $\widetilde{\mathscr{U}}$ to $\text{fin}_{\mathscr{U}}({}^*X)$ need not be chain connected.)

EXAMPLE 1. In this example $X$ is the set of all pairs $x = (x_1, x_2)$ of positive integers, and $\mathscr{U}$ is the uniformity defined by the metric $\rho$, where

$$\rho(x, y) = \begin{cases} \left|\dfrac{x_2}{x_1} - \dfrac{y_2}{y_1}\right| + \left|\dfrac{x_1}{x_2} - \dfrac{y_1}{y_2}\right| & \text{if } x_1 = y_1 \\ \left|\dfrac{x_2}{x_1} - \dfrac{y_2}{y_1}\right| + \dfrac{x_1}{x_2} + \dfrac{y_1}{y_2} & \text{if } x_1 \neq y_1 . \end{cases}$$

(The metric $\rho$ is obtained in the following way: for each $x$ in $X$ let $\tilde{x}$ be the sequence $\tilde{x} = (a_0, a_1, a_2, \cdots)$, where

$$a_0 = \frac{x_2}{x_1},\ a_{x_1} = \frac{x_1}{x_2}$$

and all other $a_n$ are 0. The distance $\rho(x, y)$ is then just the $l_1$ norm of $\tilde{x} - \tilde{y}$ as an element of the linear space of all sequences which have finite support.)

For an element $(p, q)$ of ${}^*X$ to be $\mathscr{U}$-finite, it is necessary (by Lemma 4.2) that ${}^*\rho((1, 1), (p, q))$ be finite. This implies that $p/q$ and $q/p$ are finite elements of ${}^*R$ (or, what is the same, that $p/q$ is finite but not infinitesimal.) Suppose, conversely, that $q/p$ and $p/q$ are finite. It will be shown that the element $(p, q)$ of ${}^*X$ is $\mathscr{U}$-finite. If either $p$ or $q$ is finite, then the other must be. That is, $(p, q)$ is in $X$. Assume therefore that $p$ and $q$ are both in ${}^*N \sim N$. Given a standard real number $\delta > 0$, a number $r$ in ${}^*N$ may be chosen which satisfies the inequalities

$$r\left[\frac{p}{q^2} + \frac{1}{p}\right] < \delta \leqq (r + 1)\left[\frac{p}{q^2} + \frac{1}{p}\right] . \tag{4.1}$$

For any $k \in N$, the ${}^*\rho$-distance between the elements $(p, q + kr)$ and

$(p, q + kr + r)$ of $^*X$ is equal to

$$\left|\frac{p}{q + kr} - \frac{p}{q + kr + r}\right| + \left|\frac{q + kr}{p} - \frac{q + kr + r}{p}\right|$$

which is bounded above by

$$\frac{rp}{q^2} + \frac{r}{p} < \delta .$$

Now choose the smallest $s$ in $^*N$ which satisfies

$$\frac{p}{q + sr} < \frac{\delta}{4} .$$

The inequalities (4.1), together with the fact that $p/q$ is finite but not infinitesimal, implies that $r/p$ is finite but not infinitesimal. This shows that $s$ is actually in $N$, and the sequence $(p, q), (p, p + r), \cdots, (p, q + sr)$ is a $\delta$-chain in $^*X$ with a finite number of steps.

Since $p/q$ and $r/p$ are each finite but not infinitesimal, there are standard integers $m$, $n$ such that $m/n$ is within $\delta/4$ of

$$\frac{p}{q + sr}$$

and $n/m$ is within $\delta/4$ of the reciprocal

$$\frac{q + sr}{p} .$$

It follows that the $^*\rho$-distance between $(p, q + sr)$ and $(m, n)$ is less than $\delta$. This shows that there is a $\delta$-chain from $(p, q)$ to a standard element of $^*X$, for each standard $\delta > 0$. Therefore $(p, q)$ is $\mathscr{U}$-finite, as claimed.

Given a $\mathscr{U}$-pre-nearstandard element $(p, q)$ of $^*X$, $p/q$ must be finite but not infinitesimal, by Theorem 1.4 and the previous argument. If $(p, q)$ is not standard, then $p$ is infinite. Therefore every standard element of $^*X$ is a $^*\rho$-distance of at least $p/q$ away from $(p, q)$. But $p/q$ is not infinitesimal, so this is a contradiction. Therefore $\text{pns}_{\mathscr{U}}(^*X)$ is simply the set of all standard elements of $^*X$. This shows that $(X, \mathscr{U})$ is complete and that the $\mathscr{U}$-topology on $X$ is discrete.

Also, there are elements of $\text{fin}_{\mathscr{U}}(^*X)$ which are not standard (for example, $(\omega, \omega)$ is one whenever $\omega$ is infinite.) Since the $\mathscr{U}$-topology is discrete, each standard element of $^*X$ comprises a $\mathscr{U}$-galaxy by itself. Thus there are $\mathscr{U}$-finite points which are not in the same $\mathscr{U}$-galaxy with any standard point. In fact it can be shown, by an argument similar to the one used to characterize $\text{fin}_{\mathscr{U}}(^*X)$, that the set $A$ of non-standard, $\mathscr{U}$-finite elements of $^*X$ comprises a single $\mathscr{U}$-galaxy.

Note that if $\omega$ and $\omega'$ are distinct elements of $^*N$, then the $^*\rho$-distance between $(\omega, \omega)$ and $(\omega', \omega')$ is 2. Thus the image under $\pi$ of $\text{fin}_{\mathscr{U}}(^*X)$ in $X_0$ has at least as many elements as $^*N$. Since the enlargement $^*\mathscr{M}$ can be chosen to make the cardinality of $^*N$ arbitrarily large, this shows that the various nonstandard hulls of $(X, \mathscr{U})$ constructed in [4] depend on $^*\mathscr{M}$ as well as on $(X, \mathscr{U})$.

EXAMPLE 2. In this example $X$ consists of a countable set of points $\{a_n \mid n \geqq 0\}$, together with certain arcs joining $a_0$ to the other distinguished points. For each $n \geqq 1$ the arcs joining $a_0$ to $a_n$ form $n$ subspaces $X(n, 1), \cdots, X(n, n)$, each two of which have only the elements $a_0$ and $a_n$ in common. Moreover, if $1 \leqq j \leqq m, 1 \leqq k \leqq n$ and $n \neq m$, then $X(m, j)$ and $X(n, k)$ have only the element $a_0$ in common.

The metric $\rho$ which defines $\mathscr{U}$ is given first on the subspaces

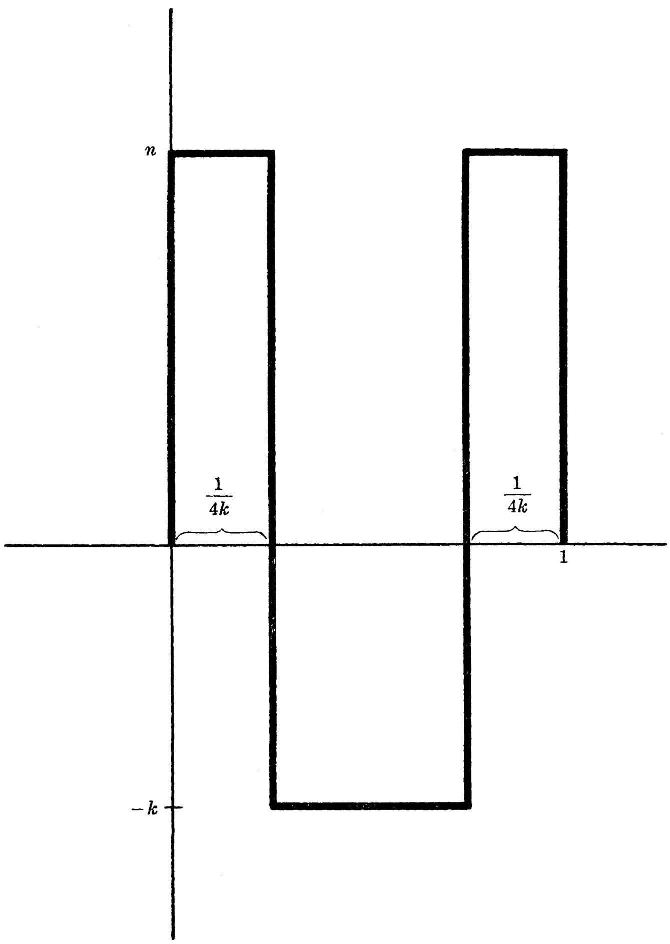


Figure 1.

$X(n, k)$ and then extended to all of $X$. For a given $1 \leqq k \leqq n$, $\rho$ is defined on $X(n, k)$ in such a way as to make the subspace $X(n, k)$ isometric to the subspace of the Euclidean plane pictured in Figure 1. (This subspace consists of the seven line segments obtained by joining adjacent pairs of points in the sequence: $(0, 0)$, $(0, n)$, $(1/4k, n)$, $(1/4k, -k)$, $(1 - 1/4k, -k)$, $(1 - 1/4k, n)$, $(1, n)$, $(1, 0)$.) In each case the isometry is assumed to take $a_0$ to $(0, 0)$ and to take $a_n$ to $(1, 0)$. Therefore there is a function $f$ from $X$ into $R^2$ whose restriction to a given subspace $X(n, k)$ yields the assumed isometry.

The metric $\rho$ is defined on the rest of $X \times X$ as follows. Let $x, y \in X$ and suppose $\rho(x, y)$ is not yet defined. That is, $x \in X(m, j)$ and $y \in X(n, k)$, where the pairs $(m, j)$ and $(n, k)$ are distinct. If $n \neq m$, then $\rho(x, y)$ is defined to be $\rho(x, a_0) + \rho(a_0, y)$. If $n = m$, then $\rho(x, y)$ is defined to be

$$\min \{\rho(x, a_0) + \rho(a_0, y), \rho(x, a_n) + \rho(a_n, y)\} .$$

It will be shown first that for every $x, y \in X$ and $n \geqq 0$

$$\rho(x, y) \leqq \rho(x, a_n) + \rho(a_n, y) . \tag{4.2}$$

If $n = 0$ or if $x$ and $y$ are both elements of the union $X(n, 1) \cup \cdots \cup X(n, n)$, then (4.2) is obvious. Thus assume $x \in X(m, j)$ where $m \neq n$. In that case

$$\rho(x, a_n) = \rho(x, a_0) + \rho(a_0, a_n) . \tag{4.3}$$

If $y \in X(n, k)$ for some $k$, then

$$\rho(a_0, y) \leqq \rho(a_0, a_n) + \rho(a_n, y) .$$

This inequality, together with (4.3) and (4.2) when $n = 0$, proves (4.2) in the present case. By the symmetry of $\rho$, it remains only to consider the case when $y \in X(m, j)$ for some $m \neq n$. In that case

$$\rho(a_n, y) = \rho(a_0, a_n) + \rho(a_0, y) .$$

This, together with (4.3), shows that $\rho(x, a_n) + \rho(a_n, y)$ is bounded below by $\rho(x, a_0) + \rho(a_0, y)$. An application of (4.2) when $n = 0$ completes the proof.

To prove the triangle inequality in general, let $x, y, z \in X$ and assume $z \in X(n, k)$. If neither $x$ nor $y$ is in $X(n, k)$, then

$$\rho(x, z) + \rho(z, y) = \rho(x, b) + \rho(b, z) + \rho(z, c) + \rho(c, y) ,$$

where $b$ and $c$ are each either $a_0$ or $a_n$. Since $b, c, z$ are all in $X(n, k)$, $\rho(b, c) \leqq \rho(b, z) + \rho(z, c)$. This, together with two uses of (4.2), proves the triangle inequality

$$\rho(x, y) \leqq \rho(x, z) + \rho(z, y) \tag{4.4}$$

in this case. By the symmetry of $\rho$ it remains only to consider the case when $x \in X(n, k)$ but $y \notin X(n, k)$. Then

$$\rho(x, z) + \rho(z, y) = \rho(x, z) + \rho(z, b) + \rho(b, y)$$

for $b = a_0$ or $a_n$. The triangle inequality applied to $x, z, b$ (which are all elements of $X(n, k)$) together with one use of (4.2) yields (4.4) in this case, and completes the proof. Thus $\rho$ is a metric on $X$.

In passing to consideration of ${}^*X$, note that there are subsets ${}^*X(\omega, \omega')$ of ${}^*X$ which correspond to the subsets $X(n, k)$ of $X$. In particular, for each $p$ in ${}^*X$ there is at least one pair $(\omega, \omega')$ which satisfies $1 \leqq \omega' \leqq \omega$ and $p \in {}^*X(\omega, \omega')$. Moreover, if $p$ and $q$ are both elements of ${}^*X(\omega, \omega')$, then ${}^*\rho(p, q) = {}^*d({}^*f(p), {}^*f(q))$, where ${}^*d$ is the extension of the Euclidean metric to ${}^*R^2$.

The analysis of $\text{fin}_{\mathscr{U}}({}^*X)$ depends on the following fact.

LEMMA. *If $p$ is $\mathscr{U}$-finite and $p \in {}^*X(\omega, \omega')$, where $\omega' \in {}^*N$, $\omega \in {}^*N \sim N$ and $\omega' \leqq \omega$, then the standard part of the first coordinate of ${}^*f(p)$ is either* 0 *or* 1.

*Proof*. Let $p$, $\omega$ and $\omega'$ be as stated. Since $p$ is $\mathscr{U}$-finite, ${}^*\rho({}^*a_0, p)$ must be finite, by Theorem 4.2. Therefore ${}^*f(p)$ is a finite distance from $(0, 0)$ in ${}^*R^2$, so that the second coordinate of ${}^*f(p)$ must be finite. If $\omega'$ is infinite, this implies that the first coordinate of ${}^*f(p)$ must be one of the numbers: $0, 1/4\omega', 1 - 1/\omega'$, or 1. These numbers have standard part 0 or 1.

Thus it may be assumed that $\omega'$ is finite. Let $A$ be the set of all $q$ in ${}^*X(\omega, \omega')$ such that ${}^*f(q)$ has an infinite second coordinate or has a first coordinate different from 0 or 1. Then if $q \in A$ but $r \in {}^*X \sim A$, it follows that ${}^*\rho(q, r) > 1/8\omega'$. In addition, $A$ has no standard element (since the only standard element of ${}^*X(\omega, \omega')$ is ${}^*a_0$.) Thus there is no $1/8\omega'$-chain from any element of $A$ to any standard element. This shows that no element of $A$ is $\mathscr{U}$-finite. Thus, in this case, ${}^*f(p)$ actually has first coordinate equal to 0 or 1.

Now consider the point ${}^*a_\omega$, where $\omega$ is any infinite element of ${}^*N$. For each standard $k$ in $N$ there is a $1/k$-chain from ${}^*a_\omega$ to ${}^*a_0$ in ${}^*X(\omega, k)$ (since the three segments in ${}^*f({}^*X(\omega, k))$ which lie below the horizontal axis in ${}^*R^2$ have finite length when $k$ is finite.) Therefore ${}^*a_\omega$ is $\mathscr{U}$-finite. However, there cannot be any sequence $q_0, \cdots, q_n$ of $\mathscr{U}$-finite points which satisfy: $q_0 = {}^*a_\omega$, $q_n = {}^*a_0$ and ${}^*\rho(q_i, q_{i+1}) < 1/2$ for all $i = 0, \cdots, n-1$. Otherwise, by the Lemma, there must exist $i, 0 \leqq i \leqq n-1$, such that the first coordinates of ${}^*f(q_i)$ and

$^*f(q_{i+1})$ have standard parts 1 and 0 respectively. But this would imply $^*\rho(q_i, q_{i+1}) > 1/2$, which is a contradiction.

Thus it has been shown that the uniform space resulting from restricting $\tilde{\mathscr{U}}$ to $\text{fin}_{\mathscr{U}}(^*X)$ is not chain connected. The example is completed by noting that since $X$ is essentially a union of polygonal paths from $a_0$, the space $(X, \mathscr{U})$ is arcwise connected.

REMARK. The last example shows that restriction of $\tilde{\mathscr{U}}$ to a $\mathscr{U}$-galaxy need not yield even a chain connected uniform space. In some cases, however, the $\mathscr{U}$-galaxies are exactly the connected components of $^*X$ under the $\tilde{\mathscr{U}}$-topology. For example, let $\mathscr{U}$ be a uniformity defined by a metric $\rho$ on $X$ which satisfies the following convexity assumption: for each $x, y \in X$ and $\delta > 0$ there exists $z \in X$ which satisfies

$$\left|\rho(x, z) - \frac{1}{2}\rho(x, y)\right| < \delta$$

$$\left|\rho(y, z) - \frac{1}{2}\rho(x, y)\right| < \delta \, .$$

(This is equivalent to saying that the completion of $(X, \rho)$ is metrically convex, and it is true, for example, when $X$ is a normed linear space.)

Passing to $^*\mathscr{M}$, and letting $\delta$ be infinitesimal, it follows that for each $p, q \in {}^*X$ there exists $r \in {}^*X$ which satisfies

$$\text{st}(^*\rho(p, r)) = \text{st}(^*\rho(q, r)) = \frac{1}{2}\text{st}(^*\rho(p, q)) \, .$$

Used repeatedly, this shows that whenever $^*\rho(p, q)$ is finite, $p$ and $q$ must be in the same $\mathscr{U}$-galaxy. Moreover, the restriction of $\tilde{\mathscr{U}}$ to any $\mathscr{U}$-galaxy yields a chain connected space. On such a galaxy $Y$ the restriction of $\tilde{\mathscr{U}}$ is defined by the semimetric $\tilde{\rho}$ defined by $\tilde{\rho}(p, q) = \text{st}(^*(p, q))$, as discussed in §1. If $^*\mathscr{M}$ is $\aleph_1$-saturated, then $(Y, \tilde{\rho})$ is a complete semimetric space, by the Remark following Theorem 1.4 (and the fact that $\mathscr{U}$-galaxies are closed in the $\mathscr{U}$-topology.) In fact, it has been shown above that $(Y, \tilde{\rho})$ is convex. As is well known, these facts imply that $Y$ is arcwise connected in the $\tilde{\rho}$-topology. It follows, using the Remark following Corollary 4.5, that the $\mathscr{U}$-galaxies are identical to the connected components of $^*X$ in the $\tilde{\mathscr{U}}$-topology.

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DUKE UNIVERSITY

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# COMPLEMENTATION IN THE LATTICE OF REGULAR TOPOLOGIES

M. JEANETTE HUEBENER

**The present paper is concerned with the lattice of regular topologies on a set, and establishes the following results: a complete, complemented sublattice of the lattice of regular topologies on a set is exhibited and shown to be anti-isomorphic to the lattice of equivalence relations on the set; the lattice of regular topologies on a set is shown to be nonmodular if the cardinality of the set is at least four; the problem of complementation for regular topologies is reduced to considering $T_0$ regular topologies without isolated points; conditions are found which are equivalent to a regular topology having a principal regular complement; then follow some conditions under which the problem can be reduced to considering connected spaces; the final section consists of constructions of complements for certain classes of regular topologies, which classes may or may not be exhaustive.**

Principal regular topologies and relations. Let $(\mathscr{S}, \vee, \wedge)$ be the lattice of all topologies on a set $E$. $(\mathscr{S}, \vee, \wedge)$ is complete, anti-atomic, complemented, and, if $|E|$, the cardinality of $E$, is at least three, it is not modular, [10, pp. 384–5, 389–397]. Next, let $(\mathscr{R}, \vee, \wedge^r)$ be the lattice of all regular topologies on $E$. $(\mathscr{R}, \vee, \wedge^r)$ is complete but not a sublattice of $(\mathscr{S}, \vee, \wedge)$. The greatest lower bound in $\mathscr{R}$ of a collection of topologies in $\mathscr{R}$ is only the least upper bound of all the regular topologies which are weaker than the collection's greatest lower bound in $\mathscr{S}$ [8, pp. 754–755].

The anti-atoms of $\mathscr{S}$ are the ultraspaces on $E$; these are topologies of the form $\mathfrak{S}(x, \mathscr{U}) = P_c(x) \cup \mathscr{U}$ where $\mathscr{U}$ is an ultrafilter on $E$ different from $\mathscr{U}(x) = \{A \subset E: x \in A\}$ and where $P_c(x) = \{A \subset E: x \notin A\}$. Frohlich [5, p. 81, Satz 3] showed that every topology $\tau$ on $E$ is the infimum of the ultraspaces on $E$ which are finer than $\tau$.

The special sublattice of $(\mathscr{S}, \vee, \wedge)$, which is anti-isomorphic to the lattice of preorders on $E$, is called the lattice of principal topologies. From this sublattice Steiner [10, p. 383, Theorem 2.6; pp. 389–397] and van Rooij [16, p. 807] take their complements. Now an ultraspace is said to be principal if its topology is of the form $\mathfrak{S}(x, \mathscr{U}(y))$ where $x \neq y$. A topology $\tau$ is principal if $\tau = 1$, or if $\tau$ is the infimum of the principal ultratopologies finer than $\tau$. These topologies are also characterized [10, pp. 381–2, Theorem 2.3] by the fact that they have a base of open sets which is minimal at each point, i.e. for any $x \in E$ every open set containing $x$ must contain the open set

$$B_x = \{y \in E: \mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau\} .$$

(Throughout the paper $B_x$ in a principal topology $\sigma$ will denote the $\sigma$-open set minimal at the point $x$.) Using this characterization it is easily seen [10, p. 382, Theorem 2.5] that the principal topologies form a sublattice of $(\mathscr{S}, \bigvee, \bigwedge)$. The mapping establishing the anti-isomorphism between this lattice and the lattice of preorders is given by

$$\eta(\tau) = G_\tau = \{(x, y): \mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau\}$$

and

$$\eta^{-1}(G) = \tau_G = \bigwedge \{\mathfrak{S}(x, \mathscr{U}(p)): (x, y) \in G\} .$$

In the lattice of regular topologies there is a sublattice of the lattice of principal topologies which has a familiar structure:

THEOREM 1.1. *A principal topology $\tau$ on $E$ is regular iff its representation satisfies the condition $\mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau$ implies $\mathfrak{S}(y, \mathscr{U}(x)) \geqq \tau$ for any $x, y \in E$.*

*Proof.* Suppose $\tau$ is principal and regular and that $\mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau$. Then $y \in B_x$ and $B_y \subset B_x$. Now $\sim B_y$ is a closed set not containing $y$; accordingly there exists $U \in \tau$ such that $U \supset \sim B_y$ and $U \cap B_y = \varnothing$ which implies that $U = \sim B_y \in \tau$. If $x \in \sim B_y \in \tau$, then $B_y \subset B_x \subset \sim B_y$ which is a contradiction. Hence $x \in B_y$ and $\mathfrak{S}(y, \mathscr{U}(x)) \geqq \tau$.

Conversely, in terms of the base of minimal open sets, the condition, $\mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau$ implies $\mathfrak{S}(y, \mathscr{U}(x)) \geqq \tau$ for any $x, y \in E$, become $y \in B_x$ iff $x \in B_y$. Hence $B_x = B_y$ or $B_x \cap B_y = \varnothing$ for every $x, y \in E$. In which case, if $U = \bigcup \{B_y: y \in U\} \in \tau$ and $x \in \sim U$ then $B_x \cap U = \varnothing$ and it follows that $\sim U = \bigcup \{B_x: x \in \sim U\} \in \tau$. Every open set being closed implies $\tau$ is regular.

COROLLARY 1.2. *A principal topology $\tau$ is regular iff $G_\tau$ is an equivalence relation.*

That the lattice of equivalence relations is complemented is proven *mot a mot* as in Steiner [10, p. 389, Theorem 5.1].

COROLLARY 1.3. *The lattice of principal regular topologies on $E$ is a complete sublattice of $(\mathscr{R}, \bigvee, \bigwedge^r)$ and $(\mathscr{S}, \bigvee, \bigwedge)$.*

Finally, for $|E| \leqq 3$ the lattice $(\mathscr{R}, \vee, \wedge^r)$ is a modular sublattice of $(\mathscr{S}, \vee, \wedge)$. If $|E| \geqq 4$, then the lattice $(\mathscr{R}, \vee, \wedge^r)$ is not modular: Let $a, b, c, d$ be distinct points of $E$. Define each of the following principal regular topologies by its base of minimal open sets

$$\begin{array}{ll} \tau_{(ab)} & \{a, b\}, \{c\}, \{d\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d \\ \tau_{(ab)(cd)} & \{a, b\}, \{c, d\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d \\ \tau_{(ad)(cb)} & \{a, d\}, \{c, b\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d \\ \tau_{(abcd)} & \{a, b, c, d\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d\,. \end{array}$$

Then we have the following diagram of least upper bounds and greatest lower bounds in $(\mathscr{R}, \vee, \wedge^r)$.

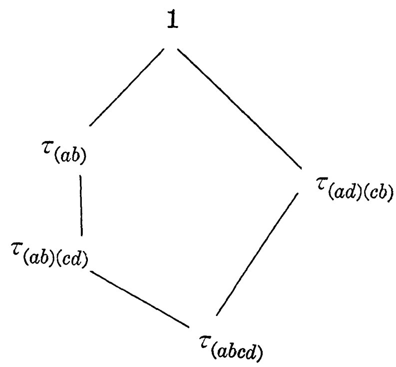

FIGURE 1

**Greatest lower Bounds in $\mathscr{R}$ and continuous functions.** In a paper in 1968 [14, p. 1087, Theorem 1], J. Pelham Thomas characterized the strongest regular topology on a set weaker than a given topology on that set: If $\tau$ is a topology on $E$, then there is a unique regular topology $\tau^*$ weaker than $\tau$, such that, if $Y$ is any regular space, then the continuous maps $(E, \tau) \rightarrow Y$ are the continuous maps $(E, \tau^*) \rightarrow Y$. Furthermore $\tau^*$ is the least upper bound of the regular topologies weaker than $\tau$. In this vein we have the following lemmas.

LEMMA 2.1. *A function $f: (E, 0) \rightarrow (Y, \rho)$ is continuous where $(Y, \rho)$ is a regular space iff $f(E) \subset \mathrm{cl}_\rho\,(f(x))$ for every $x \in E$.*

LEMMA 2.2. *If, for every regular $T_0$ space $(Y, \rho)$, every continuous function $f: (E, \nu) \rightarrow (Y, \rho)$ is constant, then, for every regular space $(Y, \rho)$, every continuous function $f: (E, \nu) \rightarrow (Y, \rho)$ satisfies the condition $f(E) \subset \mathrm{cl}_\rho\,(f(x))$ for every $x \in E$.*

Using the Thomas result we conclude that

COROLLARY 2.3. *In order for $\sigma \bigwedge^r \tau = 0$ it is necessary and sufficient that every continuous function on $(E, \sigma \wedge \tau)$ to a regular $T_0$ space be constant.*

It is now possible to reduce the problem to $T_0$ regular topologies. Let $\tau$ be a regular topology on $E$ and $E^*$ the set of point closures $\{\mathrm{cl}_\tau(x): x \in E\}$. Then $E^*$ is a set of equivalence classes of $E$ and $\varphi: E \to E^*$ given by $\varphi(x) = \mathrm{cl}_\tau(x)$ is the canonical map. If $\tau^*$ is the quotient topology relative to $\varphi$ and $\tau$, that is, the finest topology on $E^*$ such that $\varphi$ is continuous relative to $(E, \tau)$, then $\tau^*$ is a regular $T_0$ topology, lattice-isomorphic to $\tau$ [15, p. 92, Theorem 14.2]; further, $\varphi: (E, \tau) \to (E^*, \tau^*)$ is open and closed [9, p. 155, Theorem 9.3.6], and $(E^*, \tau^*)$ is called the $T_0$ quotient of $(E, \tau)$.

THEOREM 2.4. *If the $T_0$ quotient $(E^*, \tau^*)$ of a regular space $(E, \tau)$ has a (principal) complement in the lattice of regular topologies on $E^*$, then $(E, \tau)$ has a (principal) complement in the lattice of regular topologies on $E$.*

*Proof.* Let $f$ be a choice function on the subsets of $E$, $\sigma^*$ the regular complement for $\tau^*$ and $S = \{y \in E: y \neq f(\mathrm{cl}_\tau(y))\}$. Define $\sigma$ to be the topology on $E$ with the following base

$$\{(\varphi^{-1}B^*) - S: B^* \in \sigma^*\} \cup \{\{y\}: y \in S\} .$$

The topology $\sigma$ is, in fact, regular. Suppose $F$ is closed in $(E, \sigma)$ and $x \notin F$. Then $\sim F = (\varphi^{-1}B^* - S) \cup A$ for some $A \subset S$ and some $B^* \in \sigma^*$. If $x \in S$, then $\{x\} \in \sigma$ and $F \subset E - \{x\} \in \sigma$. If $x \notin S$, then $\varphi x \in B^* \in \sigma^*$ and there exist disjoint sets $U^*, V^* \in \sigma^*$ separating $\varphi(x)$ and $\sim B^*$. In which case, $\varphi^{-1}U^* - S$ and $\varphi^{-1}V^* \cup S$ are $\sigma$-open sets separating $x$ and $F$. Note that $\sigma$ is principal if $\sigma^*$ is.

Next, if $A \in \sigma \wedge \tau$, then $\varphi A \in \tau^*$ and $A = \varphi^{-1}B^*$ for some $B^* \in \sigma^*$. Hence $\varphi: (E, \sigma \wedge \tau) \to (E^*, \sigma^* \wedge \tau^*)$ is open. If $\psi: (E, \sigma \wedge \tau) \to Y$ is any continuous function to a regular $T_0$ space $Y$, then $\psi(\mathrm{cl}_{\sigma\wedge\tau}(x)) = \psi(x)$ for any $x \in E$. Hence $\psi\varphi^{-1}: (E^*, \sigma^* \wedge \tau^*) \to Y$ is a welldefined continuous function. Since $\sigma^* \bigwedge^r \tau^* = 0$ then $\psi\varphi^{-1}$ must be constant, which implies that $\psi$ is constant and hence $\sigma \bigwedge^r \tau = 0$.

Finally $\sigma \vee \tau = 1$. For $x \notin S$ we have $U^* \in \tau^*$ and $V^* \in \sigma^*$ such that $\{\varphi x\} = U^* \cap V^*$ which implies that

$$\{x\} = (\varphi^{-1}U^*) \cap (\varphi^{-1}V^* - S) \in \tau \vee \sigma .$$

**Principal complementation and connectivity.** In order for a regular topology $\tau$ and a principal regular topology $\sigma$ to have a least upper bound of 1, it is necessary and sufficient that the minimal open sets of $\sigma$ be discrete in $\tau$. That they have a greatest lower bound of 0 is characterized in terms of continuous functions. Now a function is continuous on $(E, \sigma \wedge \tau)$ iff it is continuous on both $(E, \sigma)$ and $(E, \tau)$. Relative to continuity on principal regular spaces, we have the following:

LEMMA 3.1. *Let $\sigma$ be a principal regular topology on $E$. A function $f: (E, \sigma) \to (Y, \rho)$, where $\rho$ is a $T_1$ topology, is continuous iff $f$ is constant on each minimal $\sigma$-open set.*

THEOREM 3.2. *If $(E, \tau)$ is a regular $T_0$ space with a disjoint open cover $\{E_\alpha\}_\alpha$ of $E$ and if, for each $\alpha$, the topology $\tau_\alpha = \tau \mid E_\alpha$ has a principal complement $\sigma_\alpha$ in the lattice of regular topologies on $E_\alpha$ then $\tau$ has a principal complement in the lattice of regular topologies on $E$.*

*Proof.* For each $\alpha$ let $B^\alpha$ be some one minimal open set in $\sigma_\alpha$. The set $\bigcup_\alpha B^\alpha$ and, for all $\alpha$, all minimal open sets $B_x$ in $\sigma_\alpha$, different from $B^\alpha$, define a minimal open base for a principal regular topology $\sigma$ on $E$ such that $\sigma \mid E_\alpha = \sigma_\alpha$.

Let $f$ be any function on $E$ to a regular $T_0$ space which is continuous relative to the topology $\sigma \wedge \tau$. Then for any $\alpha$, $f_\alpha = f \mid E_\alpha$ is continuous relative to the topology $(\sigma \wedge \tau) \mid E_\alpha$. But $(\sigma \wedge \tau) \mid E_\alpha \leqq \sigma_\alpha \wedge \tau_\alpha$ so $f_\alpha$ is constant on $E_\alpha$. Since $f$ was continuous relative to $\sigma$ then $f$ must be constant on $\bigcup_\alpha B^\alpha$. Hence $f$ is constant on all of $E$.

Lastly $\sigma \vee \tau = 1$: if $x$ is any point of $E = \bigcup_\alpha B_\alpha$ then $\sigma_\alpha \vee \tau_\alpha = 1$ implies that there are sets $U \in \sigma$ and $V \in \tau$ such that $\{x\} = (U \cap E_\alpha) \cap (V \cap E_\alpha) = U \cap (V \cap E_\alpha) \in \sigma \vee \tau$.

The complementation problem for locally connected regular spaces is then reduced to the complementation problem for connected spaces. Further, the proof of the previous theorem suggests several lines of development.

THEOREM 3.3. *Let $(E, \tau)$ be a regular $T_0$ space whose set $\mathscr{E}$ of components satisfy the following conditions:*

(i) *$\mathscr{E}$ is countable.*

(ii) *For each $C \in \mathscr{E}$ the restriction $\tau \mid C$ has a principal regular complement.*

(iii) *Either $\mathscr{E}$ has finitely many singletons or infinitely many nonsingletons.*

*Then $\tau$ has a principal regular complement.*

*Proof.* Without loss of generality, by (i) the collection of components forms a sequence $\{E_n\}_n$ such that, by (iii) each singleton is followed by a nonsingleton. For each $n$, let $\tau_n = \tau \mid E_n$ and $\sigma_n$ its principal regular complement.

Now for any nonsingleton $E_n$ there must be at least two distinct minimal open sets in $\sigma_n$; otherwise $\tau_n = 1$. But 1 is not connected unless $|E_n| = 1$.

For each $n$, choose $A^n$ and $B^n$ minimal open sets in $\sigma_n$ such that $B^n \neq A^n$ if $|E_n| > 1$. Then the sets

( i ) $B^n \cup A^{n+1}$ for all $n$ such that $|E_n| \neq 1$ and $|E_{n+1}| \neq 1$

(ii) $B^n \cup E_{n+1} \cup A^{n+2}$ for all $n$ such that $|E_{n+1}| = 1$

(iii) $B^{n-1} \cup E_n \cup A^{n+1}$ for all $n$ such that $|E_n| = 1$

(iv) $B_x$ for all minimal $\sigma_n$ open sets with $B_x \neq A^n, B^n, n = 1, \cdots$

define a base of minimal open sets for a principal regular topology $\sigma$ on $E$ such that $\sigma_n = \sigma \mid E_n$ for each $n$.

Let $f$ be any function on $E$ to a regular $T_0$ space which is continuous relative to the topology $\sigma \wedge \tau$. Then $f_n = f \mid E_n$ is continuous relative to the topology $\sigma_n \wedge \tau_n$ for each $n$. Hence $f_n$ is constant on $E_n$ and since $f$ is constant on each set in $\sigma$ then $f$ is constant on all of $E$.

For each $x$ not in some $B^n$ or $A^n$ there are sets $U \in \tau$ and $B_x \in \sigma_n$ such that $\{x\} = (U \cap E_n) \cap B_x = U \cap B_x \in \sigma \vee \tau$. For any $x \in B^n$ there is a neighborhood $U \in \tau$ of $x$ such that $U \cap B^n = \{x\}$ and, since components are closed and $x \notin E_{n\pm 1}, E_{n+2}$, such that $U \cap E_{n\pm 1} = \varnothing$ and $U \cap E_{n+2} = \varnothing$. Hence

$$\begin{aligned} \{x\} &= U \cap (B^n \cup A^{n+1}) \in \tau \vee \sigma \text{ if } |E_n|, |E_{n+1}| \neq 1 ; \\ &= U \cap (B^n \cup E_{n+1} \cup A^{n+2}) \text{ if } |E_{n+1}| = 1 ; \\ &= U \cap (B^{n-1} \cup E_n \cup A^{n+1}) \text{ if } |B_n| = |E_n| = 1 . \end{aligned}$$

Similarly for any $x \in A^n$. Thus $\sigma \vee \tau = 1$.

THEOREM 3.4. *Let $(E, \tau)$ be a regular space and $D$ a dense subset. If $\tau \mid D$ has a complement $\sigma^*$ in the lattice of regular topologies on $D$, then $\tau$ has a complement in the lattice of regular topologies on $E$.*

*Proof*. Define $\sigma$ to be the topology on $E$ with the base $\sigma^* \cup \{\{y\}: y \notin D\}$. Then $\sigma$ is regular, $\sigma \mid D = \sigma^*$; $\sigma$ is principal iff $\sigma^*$ is principal. Now clearly $(\sigma \wedge \tau) \mid D \leqq \sigma \mid D \wedge \tau \mid D$ so $(\sigma \bigwedge^r \tau) \mid D \leqq \sigma \mid D \bigwedge^r \tau \mid D = 0$. In which case, for any nonemply $U \in \sigma \bigwedge^r \tau$ we have $U \supset D$ since $U \cap D = \varnothing$ is impossible. Hence $\sigma \bigwedge^r \tau = 0$. Obviously $\sigma \vee \tau = 1$.

It is now clear that the complementation problem can be reduced to considering spaces without isolated points, because in the following result $(W, \tau \mid W)$ has no isolated points.

COROLLARY 3.5. *Let $(E, \tau)$ be a regular $T_0$ space, $I$ the set of isolated points, $W = \text{int}_\tau (E - I)$ the interior of $E - I$. If $(W, \tau/W)$ has a principal regular complement then there is a principal regular complement for $\tau$.*

**Classes with complements.** In this section our task is to construct principal regular complements for various classes of regular $T_0$ topologies. The first result provides the basic construction used in the following theorem to handle the class of supra-$DN$ spaces. The definition of this class is a generalization of the $DN$ spaces of B. A. Anderson [1, p. 989] and was suggested by Harold Bell as a means of extending methods developed for the $DN$ spaces. The question remains open whether this class exhausts the regular $T_0$ spaces. Subsequent results show an approach to a different class of spaces and to arbitrary products of such spaces.

THEOREM 4.1. *Let $(E, \tau)$ be a regular $T_0$ space, $\xi > |E|$, and $\{S_n: 0 \leqq n < \eta \leqq \xi\}$ a wellordered family of disjoint discrete nonempty subsets of $E$ whose union is dense in $E$. Suppose that for such $n > 0$, any open set containing $\text{cl}_\tau (\bigcup_{\gamma<n} S_\gamma)$ meets $S_n$. Then $\tau$ has a principal regular complement $\sigma$. Moreover there is some point $x \in E$ such that $\text{cl}_{\sigma\wedge\tau} (x) = E$.*

*Proof.* Define $\sigma$ to be the principal regular topology with the base of minimal open sets $\{S_n: n \geqq 0\} \cup \{\{x\}: x \notin \bigcup_{n\geqq 0} S_n\}$. Then for any $x \in E$ we have $\{x\} \in \sigma \vee \tau$.

On the other hand, for each $S_n$ let $x_n$ be any point in $\text{cl}_\tau (S_n)$. Suppose there is an ordinal $n$ such that

$$\text{cl}_{\sigma\wedge^r\tau} (x_n) \neq \text{cl}_{\sigma\wedge^r\tau} (x_0) .$$

Let $m$ be the least such ordinal. Then there are disjoint sets $U^*$, $V^* \in \sigma \bigwedge^r \tau$ such that $\text{cl}_{\sigma\wedge^r\tau} (x_0) \subset U^*$ and $\text{cl}_{\sigma\wedge^r\tau} (x_m) \subset V^*$. Also, for every $\gamma < m$, $\text{cl}_{\sigma\wedge^r\tau} (x_0) = \text{cl}_{\sigma\wedge^r\tau} (x_\gamma)$. But then $\text{cl}_{\sigma\wedge^r\tau} (x_0)$ is a $\tau$-closed set containing all the sets $\text{cl}_{\sigma\wedge^r\tau} (x_\gamma) \supset S_\gamma$ for $\gamma < m$. By the regularity, every $U \in \sigma \wedge^r \tau$ such that $x_0 \in U$ must contain $\text{cl}_{\sigma\wedge^r\tau} (x_0) \supset \text{cl}_\tau (\bigcup_{\gamma<m} S_\gamma)$. So $U^*$ meets $S_m \subset \text{cl}_{\sigma\wedge^r\tau} (x_m) \subset V^*$ which is a contradiction. Hence $\text{cl}_{\sigma\wedge^r\tau} (x_0) = E$ and $\sigma \bigwedge^r \tau = 0$.

DEFINITION. A space $(E, \tau)$ is said to be supra-$DN$ if, for any open set $U$ such that $\text{cl}_\tau (U) - U \neq \emptyset$ there is a discrete set $S \subset U$ such that $\text{cl}_\tau (S) - U \neq \emptyset$.

Note that any first countable space is supra-$DN$.

THEOREM 4.2. *If $(E, \tau)$ is a regular $T_0$ supra-DN space without* *isolated points then $\tau$ has a principal regular complement.*

*Proof.* Let $x_1$ be any point of $E$ and $U_1 = E - \{x_1\} \in \tau$. Then there is a discrete set $S_1 \subset U_1$ such that $\{x_1\} = \mathrm{cl}_\tau(S_1) - U_1$. For the induction, consider any ordinal $n$ between 1 and $\xi$, where $\xi > |E|$; suppose that for each $\beta < n$ the set $S_\beta \subset E - \mathrm{cl}_\tau(\bigcup_{\gamma<\beta} S_\gamma)$ is defined, nonclosed, discrete, and either $\mathrm{cl}_\tau(\bigcup_{\gamma<\beta} S_\gamma) \in \tau$ or any open set containing $\mathrm{cl}_\tau(\bigcup_{\gamma<\beta} S_\gamma)$ meets $S_\beta$. Now for any subset $A \subset E$, either the boundary of $E - \mathrm{cl}_\tau(A)$ is nonempty or $\mathrm{cl}_\tau(A)$ is open. Hence if $\mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma)$ is not open then the boundary of $U_n = E - \mathrm{cl}_\tau(\bigcup_{\tau<n} S_\gamma) \in \tau$ contains some point $x_n$ and $U_n$ contains a discrete set $S_n$ such that $x_n \in \mathrm{cl}_\tau(S_n) - U_n$. So any open set containing $\mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma)$ contains the boundary of $U_n$ and hence, as a neighborhood of $x_n$, meets $S_n$. If, on the other hand, $\mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma) \in \tau$, let $x_n$ be any point of $V_n = E - \mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma)$ and $U_n = V_n - \{x_n\} \in \tau$. Then there is a discrete set $S_n \subset U_n$ such that $\{x_n\} = \mathrm{cl}_\tau(S_n) - U_n$.

Consequently $\mathrm{cl}_\tau(\bigcup_{1\leqq n} S_n) = E$ and $S_0 = \{x_n \colon \mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma) \in \tau\}$ is discrete. Lastly, if $\mathrm{cl}_\tau(\bigcup_{1\leqq\gamma<n} S_\gamma) \in \tau$ then any $\tau$-open set containing $\mathrm{cl}_\tau(\bigcup_{0\leqq\gamma<n} S_\gamma) \supset S_0$, and hence containing $x_n$, meets $S_n$. Otherwise $\mathrm{cl}_\tau(\bigcup_{1\leqq\gamma<n} S_\gamma) \notin \tau$ and any $U \in \tau$ such that $U \supset \mathrm{cl}_\tau(\bigcup_{0\leqq\gamma<n} S_\gamma)$ must meet $S_n$. The conclusion then follows by the previous theorem.

DEFINITION. A space $(E, \tau)$ is said to be Bolzano-Weierstrass compact if every infinite subset of $E$ has a limit point in $E$.

DEFINITION. A space $(E, \tau)$ is said to be locally-B.W.-compact if each point in the space has a fundamental system of neighborhoods each of which is Bolzano-Weierstrass compact.

THEOREM 4.3. *If $(E, \tau)$ is a separable, regular $T_0$ locally-B.W.-compact space without isolated points, then $\tau$ has a principal regular complement.*

*Proof.* Let $Q = \{q_1, q_2, \cdots\}$ be a countable dense subset of $E$. Let $V_1$ be a B.W. compact neighborhood of $x_1 = q_1$. Since $\tau | Q$ is $T_2$ without isolated points, there is a countably infinite discrete $S_1 \subset \mathrm{int}_\tau(V_1) \cap Q$ with $x_1 \in S_1$. For every $x \in S_1$, the $T_2$ regularity of $E$ and the discreteness of the countable set $S_1$ imply that there is an open set $V_x$ such that $x \in V_x \subset \mathrm{cl}_\tau V_x \subset V_1$, $\mathrm{cl}_\tau V_x \cap \mathrm{cl}_\tau S_1 = \{x\}$, and if $x, y \in S_1$ and $x \neq y$, then $\mathrm{cl}_\tau V_x \cap \mathrm{cl}_\tau V_y = \varnothing$. Hence, for each $x \in S_1$, an infinite discrete set $S_x$ may be chosen so that $x \in S_x \subset V_x \cap Q$.

The points of $S_1$ may be denoted by $x_{1n}$ for $n = 1, 2, \cdots$, with $x_{11} = x_1$. The corresponding discrete sets may be denoted by $S_{1n}$. For each $n$, let $y_{1n} \in \mathrm{cl}_\tau(S_{1n}) - S_{1n} \subset \mathrm{cl}_\tau V_{x_{1n}}$.

For each $k > 1$ let $Q_k = Q - \mathrm{cl}_\tau (\bigcup_{p<k} \bigcup_{n=1}^{\infty} S_{pn}) \in \tau \mid Q$. If $Q_k \neq \emptyset$, let $x_k$ be the least element in the order on $Q_k$.

$V_k$ a B.W.-compact neighborhood of $x_k$ in $\sim\mathrm{cl}_\tau (\bigcup_{p<k} \bigcup_{n=1}^{\infty} S_{pn})$

$S_k$ a countably infinite discrete set in $V_k \cap Q_k$ with $x_k \in S_k$

$x_{kn}$ $n = 1, 2, \cdots$ the points of $S_k$ in the induced order

$S_{kn}$ the corresponding countably infinite discrete sets chosen from the intersection of $Q$ and a neighborhood, of $x_{kn}$, whose closure is in $V_k$ with $x_{k1} = x_k \in S_{k1}$ and satisfying $\mathrm{cl}_\tau S_{kn} \cap \mathrm{cl}_\tau S_{kp} = \emptyset$ for $n \neq p$, and

$y_{kn} \in \mathrm{cl}_\tau (S_{kn}) - S_{kn}$.

Clearly $\mathrm{cl}_\tau (\bigcup_{p=1}^{\infty} \bigcup_{n=1}^{\infty} S_{pn}) \supset \mathrm{cl}_\tau (Q) = E$.

Define a principal regular complement $\sigma$ for $\tau$ with a base of minimal open sets consisting of

$$\begin{aligned} &U_1 = S_{11}\,, \\ &U_k = S_{1k} \cup \{y_{1(k-1)}\} \cup S_{k1} && \text{for } k > 1, \\ &U_{pk} = S_{pk} \cup \{y_{p(k-1)}\} && \text{for } p, k > 1, \\ &\{y\} && \text{for all } y \notin (\textstyle\bigcup_k U_k) \cup (\bigcup_{p,k} U_{pk}). \end{aligned}$$

The minimal open sets are discrete in $(E, \tau)$ because $S_{k1}$ was chosen in a closed neighborhood outside $\mathrm{cl}_\tau (\bigcup_{p<k} \bigcup_{m=1}^{\infty} S_{pm})$ which contains $\mathrm{cl}_\tau (S_{1k})$, and because $y_{n(k-1)} \in \mathrm{cl}_\tau (S_{n(k-1)})$ and $\mathrm{cl}_\tau (S_{n(k-1)}) \cap \mathrm{cl}_\tau (S_{nk}) = \emptyset$.

Lastly, if $U \in \tau \wedge \sigma$, $U \neq \emptyset$, then $U \cap (\bigcup_{p,k} S_{pk}) \neq \emptyset$. Let $\bar{\alpha}$ be the least ordinal for which there is a $\beta$ such that $U \cap S_{\bar{\alpha}\beta} \neq \emptyset$ and $\bar{\beta}$ the least such $\beta$. Suppose $\bar{\alpha} \neq 1$. Then $\bar{\beta} \neq 1$ and $y_{\bar{\alpha}(\bar{\beta}-1)} \in U_{\bar{\alpha}\bar{\beta}} \subset U \in \sigma$. But $y_{\bar{\alpha}(\bar{\beta}-1)}$ is a $\tau$-limit point of $S_{\bar{\alpha}(\bar{\beta}-1)}$ so $U \in \tau$ meets $S_{\bar{\alpha}(\bar{\beta}-1)}$ which contradicts the minimality of $\bar{\beta}$. Hence $\bar{\alpha} = 1$. Similarly $\bar{\beta} = 1$ and $S_{11} = U_1 \subset U$ for every $U \in \tau \wedge \sigma$ and $\sigma \bigwedge^r \tau = 0$.

Note that local compactness and countable compactness imply local-B.W.-compactness.

THEOREM 4.4. *For each $i \in \theta$ let $(E_i, \tau_i)$ be a regular $T_0$ space for which there exists a principal regular topology $\sigma_i$ on $E_i$ such that*

(a) $\sigma_i \vee \tau_i = 1$.

(b) *There is a subset $W_i \subset E_i$ such that $U \in \sigma_i \wedge \tau_i$ and $U \neq \emptyset$ imply that $U \supset W_i$.*

(c) *If $U \in \tau_i$ satisfies $U \supset W_i$ then there are $\sigma_i$-isolated points in $U$.*

(d) *The set of $\sigma_i$-nonisolated points is dense in $(E_i, \tau_i)$.*

*If $E = \prod_{i \in \theta} E_i$ and $\tau = \prod_{i \in \theta} \tau_i$ then $(E, \tau)$ has a principal regular complement.*

*Proof.* Well order $\theta$; let $(x_i)_i \in E$. If $x_i$ is isolated in $\sigma_i$ for every $i \in \theta$, then let $B(x_i)_i = \{(x_i)_i\}$. Otherwise, there is a least element $\bar{\iota} \in \theta$ such that $x_{\bar{\iota}}$ is not $\sigma_{\bar{\iota}}$-isolated; let $B(x_i)_i = B_{\bar{\iota}} \times (x_i)_{i \neq \bar{\iota}}$ where $B_{\bar{\iota}}$

is the minimal $\sigma_i$-open set containing $x_i$. The collection $\{B(x_i)_i: (x_i)_i \in E\}$ forms a base of minimal open sets for a principal regular topology $\sigma$ on $E$.

Using hypothesis (a) for the first nonisolated coordinate, it is easily seen that $\sigma \vee \tau = 1$.

Next let $A^1, A^2 \in \sigma \wedge \tau$ be nonempty. Now $A^1, A^2 \in \tau$ implies that there are indices $i_1, i_2, \cdots, i_k \in \theta$ such that $A^1$ and $A^2$ contain rectangular neighborhoods. Hence there are points $(x_i)_i \in A^1$ and $(y_i)_i \in A^2$ such that $x_i = y_i$ for $i \neq i_1, \cdots, i_k$ and, by (d), $x_i$, $y_i$ are $\sigma_i$-nonisolated for $i = i_1, \cdots, i_k$ only. Let $j = \min\{i_1, \cdots, i_k\}$ and $A_j^1 = \{z \in E_j: \{z\} \times (x_i)_{i \neq j} \in A^1\} \in \tau_j$, the inverse image of $A^1$ under the $(x_i)_{i\neq j}$-section; since $x_i$ is $\sigma_i$-isolated for $i < j$ then for any $\sigma_j$-nonisolated point $x \in A_j^1$, $B_x \times (x_i)_{i \neq j} \subset A^1$. In which case, $B_x \subset A_j^1$ and hence $A_j^1 \in \sigma_j$. Similarly $A_j^2 = \{z \in E_j: \{z\} \times (y_i)_{i \neq j} \in A^2\} \in \sigma_j \wedge \tau_j$. Thus by (b), $W_j \subset A_j^1 \cap A_j^2 \in \sigma_j \wedge \tau_j$ and by (c), there is an isolated point $x'_j = y'_j$ in $A_j^1 \cap A_j^2$ which means that

$$(x_i)_{i\neq j} \times x'_j \in A^1 \text{ and } (y_i)_{i \neq j} \times y'_j \in A^2 .$$

Continuing this process and replacing $x_{i_1}, \cdots, x_{i_k}$ and $y_{i_1}, \cdots, y_{i_k}$ locates a point common to $A^1$ and $A^2$. The absence of disjoint sets in $\tau \wedge \sigma$ implies that $\tau \bigwedge^r \sigma = 0$.

In particular, the principal regular complement constructed in Theorem 4.3 satisfies conditions (a), (b) and (d) required of the factor spaces in Theorem 4.4; condition (c) can be accomodated without losing others.

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# THE DIOPHANTINE PROBLEM $Y^2 - X^3 = A$ IN A POLYNOMIAL RING

DENNIS L. JOHNSON

**Let $C[z]$ be the ring of polynomials in $z$ with complex coefficients; we consider the equation $Y^2 - X^3 = A$, with $A \in C[z]$ given, and seek solutions of this with $X, Y \in C[z]$ i.e. we treat the equation as a "polynomial diophantine" problem. We show that when $A$ is of degree 5 or 6 and has no multiple roots, then there are exactly 240 solutions $(X, Y)$ to the problem with $\deg X \leqq 2$ and $\deg Y \leqq 3$.**

It is possible that, $A$ being of degree 6, solutions $(X, Y)$ exist with $\deg X > 2$ or $\deg Y > 3$. We "normalize" the problem so as to remove these from our consideration, and give the following definitions: if $A$ is any polynomial of degree $d$, we shall permit its *formal degree* to be any integer *divisible by* 6 and greater or equal to $d$. Given $A$ of formal degree $6k$, we require the solutions $X, Y$ of the equation to be of formal degrees $2k, 3k$ resp., i.e. $\deg X \leqq 2k$, $\deg Y \leqq 3k$. This problem will be called the *problem of order* $k$. The restriction on the degrees of $X, Y$ causes no loss in generality, for if $k$ is chosen large enough, it will exceed $1/2 \deg X$ and $1/3 \deg Y$. Furthermore, the classification by $k$ has a natural geometric interpretation. We confine our attention to the problem of order 1. The order restriction enables us to projectivize the equation to an equation of degree $6k$, with $\deg A = 6k$, $\deg X = 2k$, $\deg Y = 3k$.

Suppose then that $A$ has formal degree 6, and $(X, Y)$ is a solution of proper formal degree, $\deg X \leqq 2$, $\deg Y \leqq 3$. The projective curve $K: w^3 - 3Xw + 2Y = 0$ has the $z$-discriminant $Y^2 - X^3 = A$, so the function $z: K \to S^2$ (proj. line) has its branches among the roots of $A$, for finite $z$. At $z = \infty$ we introduce $\tilde{z} = 1/z$, $\tilde{w} = w/z = \tilde{z}w$ and get

$$\tilde{z}^3 w^3 - 3\tilde{z}^3 X\left(\frac{1}{\tilde{z}}\right)w + 2\tilde{z}^3 Y\left(\frac{1}{\tilde{z}}\right) = 0 :$$

If $X = a_0 z^2 + \cdots$, $Y = b_0 z^3 + \cdots$, then

$$F = \tilde{w}^3 - 3(a_0 + a_1\tilde{z} + a_2\tilde{z}^2)\tilde{w} + 2(b_0 + b_1\tilde{z} + \cdots) = 0$$

and

$$\frac{\partial F}{\partial \tilde{w}} 3\tilde{w}^2 - 3(a_0 + \cdots) .$$

Now at $\tilde{z} = 0$ (i.e. $z = \infty$) $z$ has a branch point if and only if $\partial F/\partial \tilde{w} = 0$;

i.e. we must have

$$\tilde{w}^3 - 3a_0\tilde{w} + 2b_0 = 0$$

and

$$3\tilde{w}^2 - 3a_0 = 0$$

which is true if and only if $\Delta = -a_0^3 + b_0^2 = 0$ i.e. if and only if $\deg A < 6$. Hence if $\deg A < 6$, we put a "formal root" of $A$ at $\infty$ with multiplicity 6-$\deg A$.

We now assume the roots of $A$ to be *distinct*. This entails $\deg A = 5$ or 6, with no multiple (finite) roots. The roots will be called $z_1, \cdots, z_6$. Note that if either $X$ or $Y$ were zero at $z_\iota$, the other would also be, since $A$ is zero there (for the case $z_\iota = \infty$ just imagine the projective form of $Y^2 - X^3 = A$; the statement then reads that $\deg A < 6$ and if $\deg Y < 3$ then $\deg X < 2$ and conversely). Hence $A$ would have at least a *double* zero at $z_\iota$, (or at $\infty$: $\deg A \leqq 4$) contrary to hypothesis. Hence $X, Y \neq 0$ at $z_\iota$, and $\deg X = 2$ or $\deg Y = 3$. Away from a branch point we may write locally:

$$w_0 = \sqrt[3]{-Y + \sqrt{A}} + \sqrt[3]{-Y - \sqrt{A}}$$

$$w_1 = \omega\sqrt[3]{-Y + \sqrt{A}} + \omega^2\sqrt[3]{-Y - \sqrt{A}}$$

$$w_2 = \omega^2\sqrt[3]{-Y + \sqrt{A}} + \omega\sqrt[3]{-Y - \sqrt{A}}$$

for proper choice of the roots; as we go around $z_\iota, \sqrt{A}$ changes to $-\sqrt{A}$, and we get a root permutation $w_0 \leftrightarrow w_0$, $w_1 \leftrightarrow w_2$. Thus the branching number $b_\iota$ at $z_\iota$ is 1, and the total branching is 6, so the genus is $g = b/2 - r + 1 = 1$, i.e. $K$ is a torus.

We should also prove that $K$ is irreducible; but if $K$ were reducible, factoring as $(w - \alpha)(w^2 + \alpha w + \beta)$ (where $\alpha, \beta$ are polynomials in $z$ by Gauss's lemma) i.e., we have $3X = \alpha^2 - \beta$ and $2Y = -\alpha\beta$, and $A = Y^2 - X^3 = 4\beta^3 + 15\alpha^2\beta^2 + 12\alpha^4\beta - 4\alpha^6 = -(\alpha^2 - 4\beta)(2\alpha^2 + \beta)^2$. It is easy to see that $\deg \alpha \leqq 1$, $\deg \beta \leqq 2$, and hence $\deg(\alpha^2 - 4\beta) \leqq 2$. Since $\deg A \geqq 5$ we see that $\deg (2\alpha^2 + \beta) \geqq 1$, whence $A$ has double roots, contrary to hypothesis.

Thus, any solution $X$, $Y$ gives us an elliptic curve $K$ represented as a 3-sheeted branched covering of $S^2$ with branch points at $z_\iota$, where $z: K \to S^2$ is an elliptic function of degree 3. Furthermore, $w$ is also a function on $K$, and its poles are among those of $z$, and of order $\leqq$ the order of the $z$-poles: for expanding $w_\iota$ at $z = \infty$ we get

$$w_\iota = \omega^\iota\sqrt[3]{-b_0z^3 + \cdots + \sqrt{(b_0^2 - a_0^3)z^6 + \cdots}} + \omega^{2\iota}\sqrt[3]{\text{etc.}}$$

i.e.

$$w_\iota = \left(\omega^\iota \sqrt[3]{-b_0+\sqrt{\Delta}} + \omega^{2\iota}\sqrt[3]{-b_0-\sqrt{\Delta}}\right)z + \text{lower powers of } z$$

i.e. the order of $w$ is $\leqq$ order of $z$ at all places $z=\infty$. (Clearly $w$ has no other poles). Note also that the sum $\Sigma w_\iota$ of the three values of $w$ over any $z$ is zero.

Now suppose conversely that we are given a branched covering of $S^2$ with 6 simple branch points at the roots of $A$; we then have an elliptic curve $K$ and a meromorphic function $z\colon K \to S^2$ with 3 poles (one of which is double if a branch point is at $\infty$) at places $k_1, k_2, k_3$. Now the set of meromorphic functions $w$ on $K$ whose poles are among the $k_\iota$ form a vector space $V$ of dimension 3. Given any such $w$, the sum $w_0+w_1+w_2$ of its 3 values over any $z$ gives us a function which is:

(1) finite for finite $z$

(2) of order $\leqq$ the order of $z$ at $z=\infty$

(3) symmetric in the sheets, so rational in $z$.

Hence $\Sigma w_\iota$ must be *linear* in $z$: $\Sigma w_\iota = a_w z + b_w$, where $a_w$ and $b_w$ are constants depending on $w$. Note that $a_w$ and $b_w$ are clearly *complex-linear* in $w$, i.e. $a, b\colon V \to \boldsymbol{C}$ are linear maps. Furthermore, since both $w=1$ and $w=z$ are in $V$ we have $a$ and $b$ are linearly independent: for

$$a(1)=0 \qquad a(z)=3$$

$$b(1)=3 \qquad b(z)=0$$

and so $a_w=0$, $b_w=0$ defines a one dimensional subspace of $V$ i.e. a $w\neq 0$, defined up to a constant multiple, of degree $\leqq 3$, with its poles among those of $z$, and with $\Sigma w_\iota = 0$. Hence $w$ satisfies some equation

$$w^3-3Pw+2Q=0, \text{ with } P \,\&\, Q \text{ rational in } z\,;$$

but

$$-3P = w_1w_2+w_2w_3+w_3w_1 \text{ is \textit{finite} for } z \text{ finite}\,;$$

hence $P$ is a polynomial; also its degree is $\leqq 2$ since the order of $w_\iota$ is $\leqq$ that of $z$ at $\infty$. Likewise $Q$ is a polynomial of degree $\leqq 3$ in $z$. Finally $w$ is not rational in $z$ since if it were, it would actually be linear, $w=az+b$, and then

$$\Sigma w_\iota = 3w = 3az+3b=0, \quad \text{i.e.} \quad w\equiv 0\,.$$

Hence $w^3-3Pw+2Q=0$ is irreducible, and thus *defines* the curve $K$. Because of this, we must have the branch points as roots of the discriminant $Q^2 - P^3$ $(\neq 0)$; i.e. $A \mid Q^2 - P^3$; $\deg Q^2 - P^3 \leqq 6$, and is $<6$ if and only if as we have seen previously, $\infty$ is a branch point of $K$; in the latter case we also have $\deg A = 5$, and so in every case we have $\deg (Q^2 - P^3) = \deg A$, i.e. $A = k(Q^2 - P^3)$ for some constant $k \neq 0$. If now we replace $w$ by $w/\alpha (\alpha \in \boldsymbol{C})$, we replace $P$ by $P/\alpha^2$ and $Q$ by $Q/\alpha^3$ and $Q^2 - P^3$ by $(Q^2 - P^3)/\alpha^6$; Hence we may choose a scale factor $\alpha$, determined up to a 6th root of unity, and a rescaled $w$ such that $Q^2 - P^3 = A$, i.e. $(P, Q)$ is a solution. Thus we have shown that any 3 sheeted covering of $S^2$ with simple branches at $A = 0$ gives us exactly 6 solutions to the problem (These 6 solutions are distinct since two could be equal if and only if $P$ or $Q \equiv 0$, which is impossible). Furthermore, if we have two different such branched coverings $K_1$, $K_2$, then the corresponding solutions $(P_1, Q_1)$, $(P_2, Q_2)$ must be distinct, since the data $(P_\iota, Q_\iota)$ actually *define* $K$.

Thus the only remaining problem is to enumerate the different coverings possible.

We choose a base point $q \in S^2$, distinct from the roots $z_\iota$, and loops $p_\iota$ $(\iota = 1, \cdots, 6)$ encircling the roots $z_\iota$ acting as free generators of the fundamental group $\pi_1(S^2 - \bigcup_j z_j)$, subject only to the relation $p_1 \cdots p_6 =$ identity. Choosing a numbering 1, 2, 3 of the sheets over $q$, each $p_\iota$ determines a permutation $\pi_\iota$ (in $S_3$) of the sheets, and these completely determine the surface. Since the branches are all simple, these permutations must be *transpositions*: (12), (23) or (31). Also not all the $\pi_\iota$ can be equal, for then two sheets over $q$ would remain unconnected from the third. If we choose $\pi_1, \cdots \pi_5$ arbitrarily then $\pi_6$ is determined by $\pi_1 \pi_2 \cdots \pi_6 = e$. Note however that $\pi_1, \cdots \pi_5$ may not be chosen all equal, since $\pi_6$ would also be same by virtue of the relation. Hence we may choose $\pi_1, \cdots \pi_5$ in $3^5 - 3$ ways, obtaining all possible coverings of the required nature. Two such choices $\pi_\iota, \pi'_\iota$ give the same covering if and only if they differ by a renumbering of the sheets over $q$, i.e. if and only if $\pi'_\iota = g\pi_\iota g^{-1}$ for some $g \in S_3$. Since at least two different transpositions occur among the $\pi_\iota$, conjugation by the elements of $S_3$ produces exactly 6 different equivalent choices of $\pi_\iota$; hence the total number of different surfaces is $(3^5 - 3)/6 = (3^4 - 1)/2 = 40$. Remembering that to each such surface there are 6 solutions, we have:

THEOREM. *If $A$ is a polynomial of degree* 5 *or* 6 *without multiple roots, then there are exactly* 240 *distinct solutions of the equation $Y^2 - X^3 = A$ in polynomials $X$, $Y$ for which* $\deg X \leqq 2$, $\deg Y \leqq 3$.

It should be pointed out that, in principle at least, the determination of the solutions $(X, Y)$ for a given $A$ could be solved by classical elimination theory. For example, if $X = a_0 z^2 + a_1 z + a_2$ and

$Y = b_0z^3 + b_1z^2 + b_2z + b_3$ is a solution to $Y^2 - X^3 = A = \alpha_0z^6 + \cdots + \alpha_6$, then treating the $a_\iota$ and $b_j$ as unknowns, formal manipulation and the equating of coefficients gives us 7 polynomial equations in 7 unknowns which presumably (assuming independence) gives a finite set of solutions for the unknowns $a_\iota$, $b_j$. This also shows us that the $a_\iota$ and $b_j$ are algebraic over the field of the $\alpha_k$. In practice, however, this elimination would probably not be computationally feasible.

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# STRONG LIE IDEALS

ALBERT J. KARAM

**$R$ is 2-torsion free semiprime with $2R = R$. A Lie ideal, $U$, of $R$-strong if $aua \in U$ for all $a \in R, u \in U$. One shows that $U$ contains a nonzero two-sided ideal of $R$. If $R$ has an involution, $*$, (with skew-symmetric elements $K$) a Lie ideal, $U$, of $K$ is $K$-strong if $kuk \in U$ for all $k \in K$, $u \in U$. It is shown that if $R$ is simple with characteristic not 2 and either the center, $Z$, is zero or the dimension of $R$ over the center is greater than 4, then $U = K$. If $R$ is a topological annihilator ring with continuous involution and if $U$ is closed $K$-strong Lie ideal, $U = C \cap K$ where $C$ is a closed two-sided ideal of $R$. A Lie ideal, $U$, of $K$ is $HK$-strong if $u^3 \in U$ for all $u \in U$. A result similar to the above result for $K$-strong Lie ideals can be shown. Let $R$ be a simple ring with involution such that $Z = (0)$ or the dimension of $R$ over $Z$ is greater than 4. Let $\phi$ be a nonzero additive map from $R$ into a ring $A$ such that the subring of $A$ generated by $\{\phi(x): x \in R\}$ is a noncommutative, 2-torsion free prime ring. Suppose $\phi(xy - y^*x^*) = \phi(x)\phi(y) - \phi(y^*)\phi(x^*)$ for all $x, y \in R$. As an application of the above theory, $\phi$ is shown to be an associative isomorphism.**

**1. Introduction.** $R$ will denote a semiprime ring such that $2R = R$ and if $2r = 0$, then $r = 0$. We call the latter property 2-torsion free. $Z$ will denote the center of $R$. If $R$ has an involution, $*$, defined on it, $S$ and $K$ will be the set of symmetric and skew-symmetric elements respectively. The Lie and Jordan products are $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for any $x, y \in R$. If $X, Y \subseteqq R$, $[X, Y]$ will denate the additive subgroup generated by the set $\{[x, y]: x \in X \text{ and } y \in Y\}$. An additive subgroup, $U$, of $R$ is a Lie ideal of $R$ if $[U, R] \subseteqq U$. If $R$ has an involution, we can similarly define a Lie ideal of $K$.

This paper is concerned with the study of different classes of Lie ideals of both $R$ and $K$. A Lie ideal, $U$, of $R$ is said to be $R$-strong if $aua \in U$ for all $a \in R$, $u \in U$. If $U$ is a Lie ideal of $K$, $U$ is $K$-($HK$-) strong if $kuk \in U$ ($u^3 \in U$) for all $k \in K, u \in U$.

In the classical theory of the Lie structure of an associative ring, the main theorem [6; Th. 1.3] states: if $R$ is simple and $U$ is a Lie ideal of $R$, either $U \subseteqq Z$ or $[R, R] \subseteqq U$. We attempt to develop some criteria for differentiating between Lie ideals of $R$ containing $[R, R]$ and $R$ itself. Similar criteria are developed for Lie ideals of $K$. We will have occasion to use the following results of Herstein [6; pp 1, 5, 10, and 28]:

(i) $R$ has no one-sided ideals which are nil of bounded index;

(ii) If $a \in R$ is such that $[a, [a, x]] = 0$ for all $x \in R$, then $a \in Z$;

(iii) Let $R$ be simple with involution and characteristic not 2. If $Z = (0)$ or the dimension of $R$ over $Z$ is greater than 4, then $R = \bar{S} = \bar{K}$ where $\bar{S}$ and $\bar{K}$ are the subrings of $R$ generated by $S$ and $K$ respectively.

If $X \subseteqq R$, $\mathscr{R}(X) = \{a \in R\colon Xa = (0)\}$ and $\mathscr{L}(X) = \{a \in R\colon aX = (0)\}$. The next two lemmas are analogs of a results of Baxter [3; p. 2].

LEMMA 1.1. *If $U$ is a Lie ideal of $R$ such that $u^2 = 0$ for all $u \in U$, then $U = (0)$.*

*Proof.* Let $u \in U, a \in R$. As $[u, a] \in U$, $[u, a]^2 = 0$. Therefore, $uauau = u[u, a]^2 = 0$ and $uR$ is nil of bounded index. By the previously mentioned results, $uR = (0)$. But $R$ is semiprime, so $\mathscr{L}(R) = (0)$. Thus $u = 0$.

LEMMA 1.2. *Let $R$ have an involution, $*$. If $U$ is a Lie ideal of $K$ such that $u^2 = 0$ for all $u \in U$, then $U = (0)$.*

*Proof.* Let $u, v \in U$, then $0 = (u + v)^2 - u^2 - v^2 = uv + vu$. As $[u, v] \in U$, $2uv \in U$. Since $2R = R$, $[uv, K] \subseteqq U$. Thus, for each $k \in K$, $u \circ [uv, k] = 0$, and so, even more $v\{u \circ [uv, k]\} = 0$. Since $u$ and $v$ anti-commute, expansion of this expression yields $uvkuv = 0$. Now $suvs \in K$ for any $s \in S$. So $uv(suvs)uv = 0$. Therefore, given $a \in R$, $a = s + k$ where $s \in S$ and $k \in K$, then $(uv)a(uv)a(uv) = 0$. We conclude that $uvR$ is nil of bounded index. This guarantees $uv = 0$ *for all* $u, v \in U$. Now, $-uku = u[u, k] = 0$. Repeating the previous arguments for $s \in S$ and $k \in K$, we conclude that $u = 0$.

**2. $R$-strong Lie ideals.** In this section $U$ will denote an $R$-strong Lie ideal. If $a, b \in R$ and $u, v \in U$, one can easily show that the following are in $U$: $aub + bua$, $abu + uba$, and $uau$. We associate with $U$ the set $B_U = \{b \in R\colon a \circ b \in U$ for all $a \in R\}$. This set is a Lie ideal of $R$ and $u^2 \in B_U$ for all $u \in U$. The latter can be seen by observing that if we set $b = u$ above, we obtain $au^2 + u^2a \in U$. Thus, via Lemma 1.1, $U \neq (0)$ implies $B_U \neq (0)$.

LEMMA 2.1.

(i) *$B_U$ is an $R$-strong Lie ideal*

(ii) *$u^2xu^2 \in B_U \cap U$ for all $u \in U$, $x \in R$.*

*Proof.*

(i) We know that $B_U$ is a Lie ideal of $R$. For arbitrary $x, y \in R$ and $b \in B_U$, $[x \circ b, y]$ and $[x, b] \circ y$ are in $U$. Thus, by adding and subtracting these terms, we have that $xby - ybx$ and $bxy - yxb$ are in $U$. Now,

$$x(yby) + (yby)x = \{(xy)by - yb(xy)\} + \{yb(yx) - (yx)by\} + \{y(bx + xb)y\} .$$

Since each term on the right is in $U$, $x(yby) + (yby)x \in U$ and $B_U$ is $R$-strong.

(ii) As $u^2 \in B_U$, $u^2xu^2 \in B_U$. Moreover, $u^2xu^2 = u(uxu)u \in U$. Therefore, $u^2xu^2 \in B_U \cap U$.

THEOREM 2.2. *$C = B_U \cap U$ is a nonzero two-sided ideal.*

*Proof.* Note that $C$ is an $R$-strong Lie ideal. Also $C \neq (0)$ since if this were so, for each $u \in U$, $u^2R$ would be a nil right ideal of bounded index. Let $b \in C$ and $x, y \in R$; $xb + bx \in U$. Also

$$(xb + bx)y + y(xb + bx) = \{x(by - yb) - (by - yb)x\} + \{(yx)b + b(yx)\} + \{b(xy) + (yx)b\} .$$

As each term on the right is in $U$, $(x \circ b) \circ y \in U$. Thus, $x \circ b \in C$. Now $2xb = x \circ b + [x, b] \in C$. Since $2R = R$, $Rb \subseteqq C$. Similarly, $bR \subseteqq C$. Thus $C$ is a nonzero two-sided ideal of $R$.

We note that $C$ is the same as the set $L_U = \{u \in U: ua \in U$ for all $a \in R\}$ which was used by Zuev [10] in his study of the Lie structure of $R$.

COROLLARY 2.3. *If $R$ is simple and $U \neq (0)$, $U = R$.*

This corollary allows us to study the $R$-strong structure of the ring as it relates to minimal idempotents of $R$. If $e$ is a minimal idempotent, $eUe$ is an $eRe$-strong Lie ideal. Since $eRe$ is a division ring either $eUe = (0)$ or $eUe = eRe$. We use this fact to prove the next theorem.

THEOREM 2.4. *Let $H$ be the homogeneous component of the socle which contains $e$. Then either $H \subseteqq U$ or $H \subseteqq \mathscr{L}(U) \cap \mathscr{R}(U)$.*

*Proof.* Recall that $H$ is a simple ring. The theorem then follows by considering $H \cap U$.

COROLLARY 2.5. *If $R$ is completely reducible, $U$ is the direct sum of the homogeneous components of the socle which it contains.*

This result is similar to that of Kaplansky [7].

Assume that $R$ has the additional properties that $3R = R$ and $R$ is 3-torsion free. Let $W$ be any Lie ideal of $R$ such that $u^3 \in W$ for all $u \in W$. Let $u, v \in W$. We have $\alpha = 2(v^2u+vuv+uv^2) = (u+v)^3 + (u-v)^3 - 2u^3 \in W$, $\beta = [v, [v, u]] \in W$ and $\gamma = [v^2, u] \in W$. From these we have: $3(v^2u + uv^2) = \alpha + \beta \in W$, $6vuv = \alpha - 2\beta \in W$, $6v^2u = \alpha + 3\gamma \in W$, and $6uv^2 = \alpha - 3\gamma \in W$. We now have enough to show a result similar to Theorem 2.2.

THEOREM 2.6. *Let $W$ be a Lie ideal of $R$ such that $u^3 \in W$ for all $u \in W$. Then either $W$ contains a nonzero two-sided ideal or $u^2 \in Z$ for all $u \in W$.*

*Proof.* Let $a, b \in R$ and $u \in W$. Since $2a[a, u] = [a, [a, u]] + [a^2, u] \in W$ and $2R = R$, $a[a, u] \in W$. Linearization of this expression yields $a[b, u] + b[a, u] \in W$. Upon multiplication by 6 and replacement of $b$ by $v^2$, we obtain $6\{a[v^2, u] + v^2[a, u]\} \in W$. As $6v^2[a, u] \in W$, $6a[v^2, u] \in W$ and this implies $a[v^2, u] \in W$. It immediately follows that $R[v^2, u]R \subseteq W$ of $R[v^2, u]R \neq (0)$, we are finished.

Assume $R[v^2, u]R = (0)$ for all $u, v \in W$, then $[v^2, u]R$ is a nilpotent ideal, hence $[v^2, u] = 0$ for all $u, v \in W$. As $[v^2, a] = [v, va + av] \in W$, $[v^2, [v^2, a]] = 0$. Thus, by remarks in §1, $v^2 \in Z$.

The obvious corollary holds in the case where $R$ is simple.

**3. $K$-strong Lie ideals.** Let $R$ have an involution, $*$, and let $U$ be a $K$-strong Lie ideal. For $u, v \in U$ and $k, l \in K$, the following are in $U$: $kul + luk$, $klu + ulk$, and $uku$. We associate with $U$ the set $B(U) = \{b \in R: ba - a^*b^* \in U \text{ for all } a \in R\}$. This is the analog for Lie ideals of the set which Baxter [3] uses in his study of the Jordan structure of $S$. When there is no confusion, we write $B(U) = B$.

LEMMA 3.1.
(i) *$B$ is a right ideal*
(ii) *$KB \subseteq B$*
(iii) *$u^2 \in B$ for all $u \in U$*

*Proof.* The proofs of ( i ) and ( ii ) are straightforward. We prove (iii). As $u \in U$, $u^2a - a^*(u^2)^* = u^2a - a^*u^2$. Then

$$u^2a - a^*u^2 = \{[u, ua+a^*u]\} + \{u(a - a^*)u\} .$$

The first { } is in $U$ since $ua + a^*u \in K$. The second { } is in $U$ since $(a - a^*) \in K$ and $U$ is $K$-strong.

Now from Lemma 1.2, we know that if $U \neq (0)$, $B \neq (0)$.

For $u \in U$, $k \in K$, $a \in R$ and $b, c \in B$, direct computation leads to the following facts: $ac^*b \in B$, $c^*b \in B$, $bkb^* \in B \cap U$, and $uku \in B \cap U$.

THEOREM 3.2. *Let $R$ be a simple ring with characteristic not 2. If $Z = (0)$ or the dimension of $R$ over $Z$ is greater than 4, then $U = K$.*

The proof of this essentially the same as the proof of Theorem 7 [3; p. 7]. As a corollary, we include a slight extension of a theorem of Baxter [1; p. 74].

COROLLARY 3.3. *Let $R$ be as in the theorem. $S \circ K$, the additive subgroup of $R$ generated by the set $\{s \circ k \colon s \in S$ and $k \in K\}$ is a $K$-strong Lie ideal and hence $S \circ K = K$.*

The following results on $\mathscr{L}(B)$ and $\mathscr{L}(U)$ will be particularly useful in the next section.

THEOREM 3.4. *$\mathscr{L}(B)$ is a self-adjoint two-sided ideal.*

*Proof.* The proof is similar to the proof of Theorem 2 [4; p. 563].

Knowing that $\mathscr{L}(B)$ is a two-sided ideal, we can easily show that $\mathscr{L}(B) \cap B = (0)$ and $\mathscr{L}(B) \cap U = (0)$.

THEOREM 3.5. *$\mathscr{L}(U \cap B) = \mathscr{L}(U)$.*

*Proof.* It suffices to show $\mathscr{L}(U \cap B) \subseteqq \mathscr{L}(U)$. Let $b \in U \cap B$, $k \in K$, and $x \in \mathscr{L}(U \cap B)$. As $bk - kb \in U \cap B$, $xkb = -x(bk - kb) = 0$. Thus, $\mathscr{L}(U \cap B)K \subseteqq \mathscr{L}(U \cap B)$.

Let $u \in U$, then $u^3 \in U \cap B$ so $xu^3 = 0$. Since $u^2k + ku^2 \in U \cap B$, $xu^2ku = x(u^2k + ku^2)u = 0$. Let $a \in R$; $ua^* + au \in K$, therefore $0 = xu^2(ua^* + au)u = xu^2au^2$. If we replace $a$ by $ax$, we have $(xu^2a)^2 = 0$. That is, $xu^2R$ is a nil ideal of bounded index and so $xu^2 = 0$ for any

$u \in U$. Upon linearization we obtain

$$xuv = -xvu \quad \text{for} \quad u, v \in U . \tag{3.5.1}$$

Since $xuvu = -xvu^2 = 0$ and $vkv \in U$, we have

$$xu(vkv)u = 0 . \tag{3.5.2}$$

Let $w \in U$ and $s \in S$; $xuv(ws + sw)vu = 0$. Replacement of $x$ by $xw$, expansion of the expression, and repeated use of (3.5.1) yields, $0 = -xwvuswvu$. By repeated use of (3.5.1) and finally (3.5.2), we have $xwvukwvu = 0$. Given $a \in R$, since $a = s + k$ for some $s \in S$ and $k \in K$, we can write $xwvuawvu = 0$. Replace $a$ by $ax$ to obtain

$$xwvu(ax)wvu = 0 .$$

Then $xwvuR$ is a nilpotent ideal so $xwvu = 0$. As $uk - ku \in U$.

$$0 = xwv(uk - ku) = -xwvku . \tag{3.5.3}$$

Let $s \in S$; $xwv(ws + sw)v = 0$. Moreover, since $xwvwsv = 0$, we have $xwvswv = 0$. From (3.5.3), $xwvkwv = 0$. As before, this implies

$$xwv = 0 . \tag{3.5.4}$$

Immediately, $0 = xw(vk - kv) = -xwkv$. In particular $xwkw = 0$. Since $sws \in K$, $xw(sws)w = 0$. Also, $0 = xw(swk - kws)w = xwswkw$. Again, letting $a = s + k$ for $a \in R$, we have $xwawaw = 0$. Via the same techniques, $xw = 0$ or $x \in \mathscr{L}(U)$. Hence, $\mathscr{L}(U \cap B) \subseteqq \mathscr{L}(U)$.

**4. Topological annihilator rings.** In this section $R$ will denote a semiprime topological annihilator ring with continuous involution such that $2R = R$ and if $\{2x_\alpha\}$ is a net convergent to $0 \in R$, then $\{x_\alpha\}$ is also a net convergent to 0. $U$ will be a closed $K$-strong Lie ideal.

The definition of an annihilator ring says that $\mathscr{L}(R) = \mathscr{R}(R) = (0)$ and if $A(L)$ is a closed right (left) ideal not equal to $R$, then $\mathscr{L}(A) \neq (0)$ $\mathscr{R}(L) \neq (0)$. So if $B = B(U)$, $H = \mathscr{L}(B) \oplus B$ is dense in $R$. It is easy to show that if $U$ is closed, $B$ is closed. If $X \subseteqq R$, $Cl(X)$ will denote to topolopical closure of $X$.

The following results have proofs which are similar to those given by Baxter in [3; p. 4].

THEOREM 4.1.
( i ) *$B$ is a two-sided ideal*
(ii) $\{\mathscr{L}(B)\}^* = \mathscr{L}(B^*)$

(iii) $B = B^*$
(iv) $U \subseteqq B$.

For any $x, y \in R$, we adopt the following notation: $(x, y)_L = xy - y^*x^*$ and $(x, y)_J = xy + y^*x^*$. Using the results of the last theorem, we prove

THEOREM 4.2. *$U = C \cap K$ where $C$ is a closed two-sided ideal.*

*Proof.* Let $V$ be the additive subgroup of $S$ generated by the set $\{(u, a)_J : u \in U \text{ and } a \in R\}$. If we show $(U + V)$ to be a right ideal, since it is self-adjoint, it must be a two-sided ideal.

Since $U \subseteqq B$, $(u, a)_L = ua + a^*u \in U$ for all $a \in R$. Let $c \in R$, then

$$auc + c^*ua^* = ((a, u)_L, c)_L + (u, (-a^*c))_L \in V$$

and

$$auc - c^*ua^* = ((a, u)_L, c)_J + (u, (-a^*c))_J \in V .$$

Since $2R = R$, for any $2d \in R$, $u(2d) = (u, d)_L + (u, d)_J \in U + V$. Thus, $UR \subseteqq U + V$. Also,

$$(u, a)_J(2d) = (u, ad)_L + \{a^*u(-d) + (-d)^*ua\} + (u, ad)_J + \{d^*ua - a^*ud\} \in U + V$$

and $VR \subseteqq U + V$. Thus $(U + V)R \subseteqq U + V$, or the desired conclusion that $(U + V)$ is a two-sided ideal.

Let $C = Cl(U + V)$. $U \subseteqq C \cap K$. Let $x \in C \cap K$. There exists a net $\{u_\alpha + v_\alpha\}$ such that $u_\alpha + v_\alpha \to x$ where $u_\alpha \in U$ and $v_\alpha \in V$. As $x \in K$, $(u_\alpha + v_\alpha)^* = -u_\alpha + v_\alpha \to x^* = -x$. Thus $u_\alpha - v_\alpha \to x$. By subtracing these expressions we obtain $2u_\alpha \to 2x$. Therefore $u_\alpha \to x$. Since $u_\alpha \in U$ and $U$ is closed, $x \in U$. Hence, $C \cap K = U$.

## 5. *HK*-strong Lie ideals.

In this section $U$ is an *HK*-strong Lie ideal. $R$ will have those properties as described in §1. We further assume that $3R = R$ and $R$ is 3-torsion free. *HK*-strong Lie ideals were defined by Herstein [5]. Baxter [2; p. 393] showed that if $R$ is simple with either $Z = (0)$ or the dimension of $R$ over $Z$ greater than 16 with $U \not\subseteqq Z$, then $U = K$. This can be refined by using entirely different techniques.

As before, we associate with $U$ the set $B(U)$. $B$ is a right ideal and $KB \subseteqq B$. However, we are no longer guaranteed that $u^2 \in B$ for all $u \in U$. Hence the possibility that $B = (0)$ does arise.

LEMMA 5.1. *Let $u, v, w \in U$ and $k \in K$.*

(i) $6vuv \in U$
(ii) $6(uvw + wvu) \in U$
(iii) $uv(wk - kw) + (wk - kw)vu \in U$
(iv) $u^2v - vu^2 \in B$.

*Proof.* (i) and (ii) follow in a manner similar to the remarks preceding Theorem 2.6. (iii) holds because $2R = R$ and $3R = R$. Finally (iv) can be verified in the same manner as [6; p. 33].

If $B = (0)$, $u^2v - vu^2 = 0$ for all $u, v \in U$. Let $s \in S$. Since $[u^2, s] = [u, us + su] \in U$, $[u^2, [u^2, s]] = 0$. Also, if $k \in K$, $[u^2, [u, k]] = 0$, therefore $[u^2, [u^2, k]] = [u^2, u \circ [u, k]] = 0$. We know that this implies

$$[u^2, [u^2, a]] = 0$$

for all $a \in R$. Thus, from the first section, $u^2 \in Z$.

We now refine Baxter's theorem.

THEOREM 5.2. *Let $R$ be simple and of characteristic not 2 or 3. If $Z = (0)$ or the dimension of $R$ over $Z$ is greater than 4, then either $U = K$ or $U^2 \in Z$ for all $u \in U$.*

*Proof.* If $B \neq (0)$, by the remarks preceding Lemmas 1.1 and 5.1 we have the alternative result.

We relate the notations of $K$- and $HK$-strong Lie ideals by calling attention to the fact that if $U$ is $HK$-strong, $B \cap U$ is $K$-strong. Clearly $B \cap U$ is a Lie ideal. If $k \in K$ and $u \in B \cap U$, then $[k, [k, u]] = k^2u + uk^2 - 2kuk$. Now, $k^2u + uk^2 \in B \cap U$ by the definition of $B$. Therefore, $kuk \in B \cap U$ since $2R = R$.

Herstein [6; p. 28] has shown that $K^2$ is a Lie ideal of $R$. It is not difficult to show that if $U$ is an $HK$-strong Lie ideal such that $B \cap U = (0)$, then any $x \in B \cap S$ commutes with every element in $K^2$. We need this fact to prove

THEOREM 5.3. *Let $R$ be a topological anninilator ring with properties as described in the previous section. Assume also that $3R = R$ and if $\{3x_\alpha\}$ is a net convergent to $0 \in R$, $\{x_\alpha\}$ is a net converging to 0. If $U$ is a closed $HK$-strong Lie ideal, then either $u^2 \in Z$ for all $u \in U$, $U$ contains the intersection of $K$ with a closed two-sided ideal, or $u^2v - vu^2 \in \mathscr{L}(K)$ for all $u, v \in U$.*

*Proof.* If $B = (0)$, $u^2 \in Z$. Assume $B \neq (0)$ and $B \cap U \neq (0)$.

Since $B \cap U$ is $K$-strong, Theorem 4.2 guarantees the existence of $C$, a closed two-sided ideal, such that $C \cap K = B \cap U \subseteqq U$.

Let $B \cap U = (0)$. As $K^2$ is a Lie ideal of $R$, $t = u^2v - vu^2 \in K^2 \cap (B \cap S)$. Also, by the remarks preceding the theorem, $[t, [t, a]] = 0$ for all $a \in R$. Therefore, $t \in Z$. Let $k \in K$; $tk + kt = tk - k^*t^* \in B \cap U = (0)$. Therefore, $tk = 0$ or $t = u^2v - vu^2 \in \mathscr{L}(K)$.

7. **Application.** We now parallel some of the results obtained by Small [9] and Riedlinger [8] concerning an additive mapping whose multiplicative property is defined relative to an involution. Let $R$ be a simple ring with involution, $*$, and characteristic not 2 such that $Z = (0)$ or the dimension of $R$ over $Z$ is greater than 4. Notice that under these conditions $R$ cannot be commutative. Let $\phi$ be a nozero additive mapping from $R$ into an associative ring $A$. Assume $R' = \overline{\phi(R)}$, the subring of $A$ generated by $\{\phi(r): r \in R\}$, is a noncommutative prime ring such that $2R' = R'$ and $R'$ is 2-torsion free. Let $\phi$ enjoy the further property that $\phi(xy - y^*x^*) = \phi(x)\phi(y) - \phi(y^*)\phi(x^*)$ for all $x, y \in R$. We would like to show that $\phi$ is an associative isomorphism. We will have occasion to use the following theorem by Baxter [1; p. 73] which was slightly modified by Herstein [6; p. 29]: If $R$ is such that $2R = R$ and $\bar{K} = R$, then $S = K \circ K$, the additive subgroup of $R$ generated by the set $\{k \circ l: k, l \in K\}$.

The next lemma is the key to much of what follows.

LEMMA 6.1. $\operatorname{Ker} \phi \cap K = (0)$ .

*Proof.* We show $\operatorname{Ker} \phi \cap K$ to be a $K$-strong Lie ideal. Let $l \in \operatorname{Ker} \phi \cap K$ and $k \in K$. Since $\phi([k, l]) = [\phi(k), \phi(l)] = 0$, $\operatorname{Ker} \phi \cap K$ is a Lie ideal of $K$. Thus $[k, [k, l]] \in \operatorname{Ker} \phi \cap K$ or $\phi([k, [k, l]]) = (0)$. We may expand this and obtain

$$\phi([k, [k, l]]) = \phi(k^2l - 2klk + lk^2) = \phi(k^2l + lk^2) - 2\phi(klk) = 0 .$$

Now, $\phi(k^2l + lk^2) = \phi(k^2)\phi(l) + \phi(l)\phi(k^2) = 0$. Therefore $\phi(klk) = 0$ or $\operatorname{Ker} \phi \cap K$ is a $K$-strong Lie ideal.

By Theorem 3.2 either $\operatorname{Ker} \phi \cap K = (0)$ or $\operatorname{Ker} \phi \cap K = K$. Assume the latter. For $s, t \in S$ and $k, l \in K$, $[\phi(k), \phi(l)] = 0$ and $[\phi(k), \phi(s)] = 0$. As $[s, t] \in K$, $0 = \phi([s, t]) = [\phi(s), \phi(t)]$. Because any $x \in R$ can be written as $x = s + k$, we have $[\phi(x), \phi(y)] = 0$ for all $x, y \in R$. Therefore, $R'$ is commutative, a contradiction. Thus $\operatorname{Ker} \phi \cap K = (0)$.

Let $x, y \in R$, then

$$\phi((xy - y^*x^*)x^* - x(xy - y^*x^*)^*) = \{\phi(x)\phi(y) - \phi(y^*)\phi(x^*)\}\phi(x^*)$$
$$-\phi(x)\{\phi(y^*)\phi(x^*) - \phi(x)\phi(y)\} .$$

If $y = s$, we can write,

$$\phi((xy - y^*x^*)x^* - x(y^*x^* - xy)) = \phi(x^2s - sx^{*2}) = \phi(x^2)\phi(s) - \phi(s)\phi(x^{*2})$$

and

$$\{\phi(x)\phi(y) - \phi(y^*)\phi(x^*)\}\phi(x^*) - \phi(x)\{\phi(y^*)\phi(x^*) - \phi(x)\phi(y)\}$$
$$= (\phi(x))^2\phi(s) - \phi(s)(\phi(x^*))^2 .$$

This can be rewritten as

(6.1.1) $$\{\phi(x^2) - (\phi(x))^2\}\phi(s) = \phi(s)\{\phi(x^{*2}) - (\phi(x^*))^2\}$$

for all $x \in R$ and $s \in S$.

LEMMA 6.2. *For any* $s \in S$ *and*

$$k \in K, \{\phi(s^2) - (\phi(s))^2\} \quad \textit{and} \quad \{\phi(k^2) - (\phi(k))^2\}$$

*are in* $Z'$, *the center of* $R'$.

*Proof.* Set $u$ equal to either $\{\phi(s^2) - (\phi(s))^2\}$ or $\{\phi(k^2) - (\phi(k))^2\}$. From (6.1.1), $\phi(s)u = u\phi(s)$. Consider $2\phi(t_1t_2 \cdots t_n)$ where $t_1 \in S$. We write

$$2\phi(t_1t_2 \cdots t_n) = \phi(t_1t_2 \cdots t_n + t_n \cdots t_2t_1)$$
$$+ \phi(t_1t_2 \cdots t_n - t_n \cdots t_2t_1)$$
$$= \phi(t_1t_2 \cdots t_n + t_n \cdots t_2t_1)$$
$$+ \{\phi(t_1)\phi(t_2 \cdots t_n) - \phi(t_n \cdots t_2)\phi(t_1)\} .$$

By induction, $u$ commutes with $\phi(t_2 \cdots t_n)$ and $\phi(t_n \cdots t_2)$. Since $t_1t_2 \cdots t_n + t_n \cdots t_2t_1 \in S$, $u$ commutes with $\phi(t_1t_2 \cdots t_n + t_n \cdots t_2t_1)$. Thus, $[u, \phi(t_1t_2 \cdots t_n)] = 0$. That is, $u$ commutes with $\phi(\bar{S})$. But under our hypothesis, $\bar{S} = R$. Hence, $u$ commutes with $\phi(R)$ and, indeed, with $\overline{\phi(R)} = R'$. Thus $u \in Z'$.

COROLLARY 6.3.

(6.3.1) $$\{\phi(x^2) - (\phi(x))^2\} \in Z' \quad \textit{for all} \quad x \in R .$$

*Proof.* If $x = s + k$, since $\phi(sk + ks) - \{\phi(s)\phi(k) + \phi(k)\phi(s)\} = 0$, $\{\phi(x^2) - (\phi(x))^2\} = \{\phi(s^2) - (\phi(s))^2\} + \{\phi(k^2) - (\phi(k))^2\} \in Z'$.

Let $x, y \in R$. If we linearize (6.3.1), we obtain

$$\phi(xy + yx) - \{\phi(x)\phi(y) + \phi(y)\phi(x)\} \in Z' .$$

In particular, for $s, t \in S$, $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} \in Z'$. Also, $\phi(st - ts) - \{\phi(s)\phi(t) - \phi(t)\phi(s)\} = 0$. Addition of these terms leads us to $\phi(st) - \phi(s)\phi(t) \in Z'$. Similarly, we can show that $\phi(kl) - \phi(k)\phi(l) \in Z'$ for $k, l \in K$.

For notational convenience, let $\phi(xy) - \phi(x)\phi(y) = x^y$ for any $x, y \in R$. Thus the above says that $s^t, k^l \in Z'$. The definition of $\phi$ tells us that $s^k = -k^s$. Also, we have $k^l = l^k$. Since these terms are in $Z'$, $\phi(s)k^l - l^k\phi(s) = 0$. Upon expansion and rearrangement of terms, we obtain

$$\{\phi(skl - lks)\} - \{\phi(s)\phi(k)\phi(l) - \phi(l)\phi(k)\phi(s)\} = 0 . \tag{6.4.1}$$

We can write $\phi(sk - ks) = \phi(sk)\phi(l) - \phi(l)\phi(ks)$. Replacement of this in (6.4.1) and rearrangement of terms yields

$$s^k\phi(l) - \phi(l)k^s = 0$$

or

$$s^k\phi(l) = \phi(l)k^s = -\phi(l)s^k . \tag{6.4.2}$$

Let $m \in K$, by the above, there exists $z' \in Z'$ such that $\phi(ml+lm)= \phi(m)\phi(l) + \phi(l)\phi(m) + z'$. As a result of (6.4.2) and this relation we have that $s^k\phi(ml + lm) = \phi(ml + lm)s^k$ or $s^k$ commutes with $\phi(K\circ K)$. The preliminary remarks guarantee for us that $K\circ K=S$. So, using an argument exactly like that in Lemma 6.2, we can show

$$s^k \in Z' . \tag{6.4.3}$$

LEMMA 6.4. *$x^y \in Z'$ for all $x, y \in R$.*

The proof follows directly from (6.4.3) and the remarks immediately after Corollary 6.3.

COROLLARY 6.5. *If $Z' = (0)$, $\phi$ is an associative isomorphism.*

*Proof.* As $Z' = (0)$, $\phi(xy) - \phi(x)\phi(y) = 0$. Thus $\phi$ is an associative homomorphism and $\overline{\phi(R)} = \phi(R)$. Moreover, since $R$ is simple, $\phi$ is an associative isomorphism.

Let $z'(\neq 0) \in Z'$. Since $\mathscr{A}(z') = \{r' \in R' : r'z' = 0\}$ is a two-sided ideal in a prime ring, $\mathscr{A}(z') = (0)$.

LEMMA 6.6. *$k^s = s^k = 0$ for all $s \in S, k \in K$.*

*Proof.* From (6.4.2) $s^k\phi(l) = -\phi(l)s^k$ for $l \in K$. By Lemma 6.4, $s^k \in$

$Z'$, therefore $s^k\phi(l) = 0$. Suppose $s^k \neq 0$. By the remarks preceding the lemma, we have $\phi(l) = 0$, that is, $K \subsetneqq \operatorname{Ker}\phi$. Therefore, $\operatorname{Ker}\phi \cap K = K$, a contradiction. We conclude that $0 = s^k = -k^s$.

COROLLARY 6.7. *$\phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for $x, y \in R$.*

We have shown that when $Z' = (0)$, then $\phi$ is an associative isomorphism. Therefore, the following theorem is proved except when $Z' \neq (0)$.

THEOREM 6.8. *$\phi$ is an associative isomorphism.*

*Proof.* From Lemma 6.6, $(s^2)^k - \phi(s)s^k = 0$. Expansion and rearrangement of terms leads to $(s^2)^k - \phi(s)s^k = (s)^{sk} - s^s\phi(k) = 0$. From Lemma 6.4, $(s)^{sk} \in Z'$ so $s^s\phi(k) \in Z'$. Let $l \in K$. There exist $z_1'$ and $z_2'$ in $Z'$ such that $s^s\phi(k) = z_1'$ and $s^s\phi(l) = z_2'$. As $s^s \in Z'$, we can write $0 = [z_1', z_2'] = (s^s)^2[\phi(k), \phi(l)]$ for all $s \in S$ and $k, l \in K$.

If $(s^s)^2 \neq 0$ for some $s \in S$, then by the remarks preceding Lemma 6.6, $[\phi(k), \phi(l)] = 0$ for all $k, l \in K$. As $\phi([k, l]) = [\phi(k), \phi(l)] = 0$, we conclude that $[K, K] \subsetneqq \operatorname{Ker}\phi \cap K = (0)$. This implies $\bar{K} = R$ is commutative, a contradiction. So $(s^s)^2 = 0$ for all $s \in S$. Since the center of a prime ring is an integral domain, $s^s = 0$. Upon linearization of this expression, we obtain $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} = 0$ for all $t, s \in S$.

For $k, l \in K$, $k^l \in Z'$. Thus there exists $z_3' \in Z'$ such that $k^l - z_3' = 0$. Since $k^2 \in S$, $(k^2)^l = 0$ and so $(k^2)^l - \phi(k)\{k^l - z_3'\} = 0$. Expansion and rearrangement of terms leads to $k^{kl} - k^k\phi(l) + z_3'\phi(k) = 0$. In view of Lemma 6.4, there is an element $z_4' \in Z'$ such that $k^{kl} = z_4'$. Therefore we can always find $z_3', z_4', \in Z'$ such that $k^k\phi(l) = z_3'\phi(k) + z_4'$ where $k$ is an arbitrary fixed element in $K$ and $l$ is allowed to vary in $K$. Note that $k^k \in Z'$. For $m \in K$, there are $z_5'$ and $z_6'$ in $Z'$ such that $k^k\phi(m) = z_5'\phi(k) + z_6'$. Thus $0 = (k^k)^2[\phi(l), \phi(m)] = [k^k\phi(l), k^k\phi(m)]$. Via the same argument as above, we can show $k^k = 0$. Linearization of this expression leads to $\phi(kl + lk) - \{\phi(k)\phi(l) + \phi(l)\phi(k)\} = 0$. Now, using this fact and the fact that both $\phi(sk) - \phi(s)\phi(k) = 0$ and $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} = 0$, we have that

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$$

for all $x, y \in R$. From Corollary 6.7, we know

$$\phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x) .$$

Addition of these two expressions yields $\phi(xy) = \phi(x)\phi(y)$ or that $\phi$ is an associative homomorphism. Therefore, $\overline{\phi(R)} = \phi(R)$ and $\operatorname{Ker}\phi = (0)$

since $R$ is simple. Hence $\phi$ is an associative isomorphism.

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# ON LOW DIMENSIONAL MINIMAL SETS

SOON-KYU KIM

**Let $(X, G, f)$ be a topological transformation group. Suppose that the phase space $X$ is compact, separable metric, and locally contractible and the group $G$ is the additive group of all real numbers $R$ with the usual topology. If $X$ is a minimal set of $\dim_L(X) \leqq 2$ then $X$ is a manifold, imposing a further condition on the action when $\dim_L(X) = 2$. Hence $X$ is a singleton, a circle or a torus according to its dimension.**

A topological transformation group is a triple $(X, G, f)$ consisting of a topological space $X$, a topological group $G$, and a continuous map $f$ from $G \times X$ into $X$ such that $f(e, x) = x, f(h, f(g, x)) = f(gh, x)$ for any $x$ in $X$ and any $g, h$ in $G$ and the identity element $e$ of $G$.

The phase space $X$ of a topological transformation group $(X, G, f)$ is called a *minimal set* if for each $x \in X$ the closure of the orbit of $x$ is $X$ itself. A *locally contractible* space $X$ is a space such that for each $x \in X$ and for any open set $U$ containing $x$ there is an open set $V$ containing $x$, which is contractible in $U$ to the point $x$.

Chu [3] has shown that if the phase space $X$ is a compact Hausdorff minimal set and $\dim_L(X) \leqq n$, then $H^n(A, L) = 0$ for every proper closed subset $A$ of $X$ under any connected topological group $G$. Here $\dim_L(X)$ is the cohomology dimension of $X$ in the sense of Cohen ([2], [4]) and $L$ is a principal ideal domain. The Alexander-Spanier cohomology theory is used here. Using this result, Chu has answered questions that were raised by Gottschalk [6]. He proved that the universal curve of Menger and the universal curve of Sierpinski are not minimal sets under any connected topological group.

Chu has also shown that some cohomological natures of a minimal set are similar to those of a generalized manifold. We try to see whether certain minimal sets are actually generalized manifolds. In this regard, we have some results in low dimensions as mentioned in the abstract.

We use the section theorem of Bebutov and Hájek and the umbrella theorem of Bing-Borsuk that we state here.

*The section theorem* ([11: p. 332] and [8: p. 210])

Given a topological transformation group $(X, R, f)$ with $X$ separable metric and a non-fixed point $x_0$ in $X$ there exist sections $S \ni x_0$ generating arbitrary small neighborhoods of $x_0$ in $X$. If $X$ is locally compact or locally connected, then $S$ may be taken compact or connected respectively. Furthermore, if $X$ is compact and locally connected, then $S$ may be taken locally connected.

*The umbrella theorem* ([1: Cor. 5.3]).

In an $n$-dimensional locally contractible separable metric space $X$ the set of all centers of $n$-dimensional umbrellas (see [1] for definition) contained in $X$ is of the first category of Baire.

We note that if the phase group is discrete then $X$ is not necessarily a homogeneous space hence not a manifold ([5], [6: p. 139]).

## 1. Zero and one dimensional minimal sets.

THEOREM 1. *Let $(X, R, f)$ be a topological transformation group with $X$ a locally connected compact separable metric space and $R$ the additive real group. Suppose $X$ is a minimal set of* $\dim_L(X) = 0$ *or* 1. *Then $X$ is a singleton or a circle.*

*Proof.* Since $X$ is necessarily connected, $X$ is a point if $\dim_L(X)=0$. Let the dimension of $X$ be 1. Since each point $x \in X$ is not a fixed point, by the section theorem of Bebutov and Hájek there is a section generating arbitrary small neighborhoods of $x$ in $X$. That is, there exist $\delta > 0$, $\varepsilon < 0$ and a set $S'_x$ in $f(\overline{S(x, \delta)}, [-\varepsilon, \varepsilon])$ such that for each $y \in f(\overline{S(x, \delta)}, [-\varepsilon, \varepsilon])$ there exists a unique $t_y \in R$ such that $|t_y| \leqq \varepsilon$ and $f(y, t_y) \in S'_x$, where $S(x, \delta)$ is a $\delta$-neighborhood of $x$ and $\overline{S(x, \delta)}$ is the closure of $S(x, \delta)$. Furthermore, $x$ is in $S'_x$. There is a homeomorphism $h\colon S'_x \times [-\varepsilon, \varepsilon] \to f(S'_x, [-\varepsilon, \varepsilon]) \subset X$ defined by $h(s, t) = f(s, t)$, $s \in S'_x$, $t \in [-\varepsilon, \varepsilon]$.

Let $S_x = \{f(y, t_y) \in S'_x | y \in S(x, \delta)\}$. Then $S_x \times (-\varepsilon, \varepsilon)$ is homeomorphic to an open neighborhood of $x$ in $X$. So we may regard $S_x \times (-\varepsilon, \varepsilon)$ as a neighborhood of $x$ in $X$. Since the dimension of $S_x \times (-\varepsilon, \varepsilon)$ is 1, the dimension of $S_x$ is 0 by [4: p. 222]. Since $S_x$ may be taken connected, $S_x$ is the point $x$ itself. Hence $(-\varepsilon, \varepsilon)$ is a neighborhood of $x$ in $X$. This proves that each point $x$ in $X$ has an interval neighborhood. Since $X$ is compact, $X$ is a circle.

**2. Two dimensional minimal sets.** Let $(X, R, f)$ be again a topological transformation group (continuous flow). If a minimal set of $\dim_L(X) = 2$ is a manifold then it is either a torus or a Klein bottle since its Euler characteristic has to vanish [12: p. 197]. Since a Klein bottle cannot be a minimal set by a result of Kneser [10: p. 153] (we are told this by Arthur J. Schwartz), $X$ must be a torus.

The following seems plausible.

*Conjecture.* Let $(X, R, f)$ be a topological transformation group with $X$ a locally contractible compact separable metric space. Suppose $X$ is a minimal set and $\dim_L(X) = 2$. Then $X$ is a manifold, hence a torus.

If we further assume that $X$ is almost periodic, then $X$ is a homogeneous space [6: p. 343]. By an *almost periodic* topological transformation group we mean that for given $\varepsilon > 0$ there exists a relative dense subset of numbers $\{\tau_n\}$ such that for all $x \in X$, $d(f(x, t), f(x, t + \tau_n)) > \varepsilon$ for all $t \in R$ and each $\tau_n$, where $d$ is a complete metric on $X$. A set $Y$ of real numbers is called *relative dense* if there exists a $T > 0$ such that $Y \cap (t - T, t + T) \neq \emptyset$ for all $t \in R$. It is known that such a space $X$ is a torus by a result of [6: p. 39] and Lie group theory. And Bing and Borsuk showed that such a space is a manifold [1: p. 110]. But we give here a proof because the method of Theorem 1 can also be applied to prove this result and we hope that the technique used in the proof is useful to prove that each point $x$ in $X$ has a Euclidean neighborhood without assuming almost periodicity of the action, thus proving the conjecture.

THEOREM 2. *Let $(X, R, f)$ be a topological transformation group with $X$ a locally contractible compact separable metric space and $R$ the additive real group. Suppose $X$ is an almost periodic minimal set of* $\dim_L(X) = 2$. *Then $X$ is a manifold (hence a torus).*

*Proof.* Note again that $X$ is necessarily connected. Since each $x$ in $X$ is not a fixed point, by the section theorem of Bebutov and Hájek there is a section generating arbitrary small neighborhoods of $x$ in $X$. That is, as in Theorem 1, $x$ has an open neighborhood of the form $S_x \times (-\varepsilon, \varepsilon)$ in $X$. Here a section $S_x$ may be taken connected, locally connected and locally compact. Since the dimension of $S_x \times (-\varepsilon, \varepsilon)$ is 2, the dimension of $S_x$ is at least 1 (in fact, it is 1 [4]). Since $S_x$ is locally compact, connected and locally connected, there is a non-degenerate arc $\alpha_y$ in $S_x$ which contains $y$ for each $y \in S_x$. Then $\alpha_x[0, 1] \times [-\varepsilon, \varepsilon]$ is a closed 2-cell in $X$, and $x \times 0 = x \in \alpha_x[0, 1] \times [-\varepsilon, \varepsilon]$.

Suppose $\alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$ contains a limit point $x_0$ of $X - (\alpha_x[0, 1] \times [-\varepsilon, \varepsilon])$. Take an open set $V_0$ of $x_0$ in $\alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$ such that $\bar{V}_0$ is compact and $\bar{V}_0 \subset \alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$. Let $V$ be an open neighborhood in $X$ such that $V_0 = \alpha_x(0, 1) \times (-\varepsilon, \varepsilon) \cap V$. Since $X$ is locally contractible, there is an open neighborhood $U$ of $x_0$ in $X$ such that $U$ is contractible in $V$ to the point $x_0$ and $U \cap (X - (\alpha_x[0, 1] \times [-\varepsilon, \varepsilon]) \neq \emptyset$; i.e., there is a continuous map $H$: $U \times [0, 1] \to V$ such that $H(y, 0) = y$, $H(y, 1) = x_0$ for each $y \in U$. Then for a point $z \in U \cap (X - (\alpha_x[0, 1] \times [-\varepsilon, \varepsilon]))$, $H_z$: $[0, 1] \to V$ is a path from $z$ to $x_0$ and $H_z[0, 1] \cap (\alpha_x[0, 1] \times [-\varepsilon, \varepsilon]) \subset V \cap (\alpha_x[0, 1] \times [-\varepsilon, \varepsilon]) = V_0 \subset \bar{V}_0 \subset \alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$. Therefore, there is a path from $x$ to $x_0$ which misses $\alpha_x[0, 1] \times [-\varepsilon, \varepsilon] - \alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$.

Considering the path is ordered from $z$ to $x_0$, there is a point $x_0' \in \bar{V}_0$ such that $x_0'$ is the first point of the path $H_z$ which meets $\bar{V}_0$.

Therefore, there is 2-dimensional umbrella with $x'_0$ as its center (see [1] for definition). Then each point of $X$ is a center of a 2-dimensional umbrella by the homogeneity of $X$ that follows by the assumptions. This contradicts the umbrella theorem of Bing and Borsuk.

Thus an open 2-disk $\alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$ is an open set in $X$. If $x \in \alpha_x(0, 1)$ then $\alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$ is an open neighborhood of $x$ in $X$. Otherwise to get an open neighborhood of $x$ that is an open 2-disk we appeal to the minimality of $X$ (or homogeneity of $X$ in this case). For if $x$ has no open neighborhood that is an open 2-disk then there is no element of $R$ that sends $x$ into the open set $\alpha_x(0, 1) \times (-\varepsilon, \varepsilon)$. This contradicts the minimality of $X$.

Therefore, $X$ is a compact 2-manifold. Hence $X$ is a torus by the remark that we made in the beginning of the section.

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# A PHRAGMÉN-LINDELÖF THEOREM WITH APPLICATIONS TO $\mathscr{M}(u, v)$ FUNCTIONS

THOMAS L. KRIETE, III AND MARVIN ROSENBLUM

**A well-known theorem of Paley and Wiener asserts that if $f$ is an entire function, its restriction to the real line belongs to the Hilbert space $\mathscr{F}^*L^2(-\tau, \tau)$ (where $\mathscr{F}$ is the Fourier-Plancherel operator) if and only if $f$ is square integrable on the real axis and satisfies $|f(z)| \leqq Ke^{\tau|\mathrm{Im}\, z|}$ for some positive $K$. The "if" part of this result may be viewed as a Phragmén-Lindelöf type theorem. The pair $(e^{i\tau x}, e^{i\tau x})$ of inner functions can be associated with the above mentioned Hilbert space in a natural way. By replacing this pair by a more general pair $(u, v)$ of inner functions it is possible to define a space $\mathscr{M}(u, v)$ of analytic functions similar to the Paley-Wiener space. For a certain class of inner functions (those of "type $\mathfrak{C}$") it is shown that membership in $\mathscr{M}(u, v)$ is implied by an inequality analogous to the exponential inequality above.**

**A second application of our results is to star-invariant subspaces of the Hardy space $H^2$. It is well known that if $u$ is an inner function on the circle and $f$ is in $H^2$, then in order for $f$ to be in $(uH^2)^\perp$ it is necessary for $f$ to have a meromorphic pseudocontinuation to $|z| > 1$ satisfying**

$$|f(z)|^2 \leqq K\frac{1 - |u(z)|^2}{1 - |z|^2}, \quad |z| > 1 .$$

**If $u$ is inner of type $\mathfrak{C}$, it is proved that this necessary condition is also sufficient.**

Let $\Gamma = \{e^{i\theta}: 0 < \theta < 2\pi\}$ be the unit circle and

$$R = \{x : -\infty < x < \infty\}$$

the real line considered as point sets in the complex plane $\boldsymbol{C}$. Let $D$ and $D_-$ be the interior and exterior of the unit circle and let $\Omega$ and $\Omega_-$ be the open upper and open lower half-planes in $\boldsymbol{C}$. A function $\Phi$ is *outer* on $D$ or $\Omega$ if $\Phi$ is holomorphic on $D$ or $\Omega$ and of the form

$$\Phi(z) = \exp\int_\Gamma \frac{e^{i\xi} + z}{e^{i\xi} - z} k_1(e^{i\xi})\, \sigma(d\xi), \quad z \in D ,$$

or

$$\Phi(z) = \exp\frac{1}{\pi i}\int_R \frac{1 + tz}{t - z} k_2(t)dt, \quad z \in \Omega ,$$

where $k_1, k_2$ are real with $k_1 \in L^1(\Gamma)$, $k_2 \in L^1(R)$, and $\sigma$ is normalized Lebesgue measure on $\Gamma$. A function $F$ on $D$ or $\Omega$ is in $\mathfrak{N}^+$ if $F$ is holomorphic on $D$ or $\Omega$ and if there exists an outer function $\Phi$ that is not identically zero and such that $\Phi F$ is a bounded holomorphic function on $D$ or $\Omega$. If $F$ is in $\mathfrak{N}^+$ on $D$ or $\Omega$, then $f(e^{i\theta}) = \lim F(re^{i\theta})$ exists for almost all $e^{i\theta} \in \Gamma$, or

$$f(x) = \lim_{y \downarrow 0} F(x + iy)$$

exists for almost all $x$ in $R$. Such $f$ form the class $\mathscr{N}^+$ of functions on $\Gamma$ and $R$ respectively. We shall systematically use capital letters $F, G, \cdots$ for functions in $\mathfrak{N}^+$ and lower case letters $f, g, \cdots$ for the corresponding functions in $\mathscr{N}^+$.

Every outer function is in $\mathfrak{N}^+$. A function $U$ in $\mathfrak{N}^+$ is *inner* if $|u| = 1$ a.e.. Every function $F$ in $\mathfrak{N}^+$ has a factorization of the form $F = UG$, where $U$ is inner and $G$ is outer.

Suppose $U$ and $V$ are inner functions, say, on $\Omega$. $\mathscr{M}(u, v, R)$ is the set of functions $f$ on $R$ such that $uf$ and $vf^*$ are in $\mathscr{N}^+$ on $R$. ($f^*$ is the complex conjugate of $f$). $\mathscr{M}(u, v, \Gamma)$ is similarly defined. As shown in [5] one can associate with each $f$ in $\mathscr{M}(u, v, R)$ a unique function $F$ separately meromorphic in $\Omega$ and $\Omega_-$ such that $UF \in \mathfrak{N}^+$, $V\tilde{F} \in \mathfrak{N}^+$, and

$$f(x) = \lim_{y \downarrow 0} F(x + iy) = \lim_{y \downarrow 0} F(x - iy) \tag{1}$$

for almost all $x$ in $R$, where $\tilde{F}(z) = F^*(z^*)$, $z \in \Omega$. If $F$ is meromorphic in $\Omega$, then an extension of $F$ to a meromorphic function on $\Omega \cup \Omega_-$ satisfying (1) is said to be a *meromorphic pseudocontinuation* (relative to $R$) of $F$. Similarly, to each $f$ in $\mathscr{M}(u, v, \Gamma)$ one associates a unique $F$ meromorphic in $D \cup D_-$ such that $UF \in \mathfrak{N}^+$, $V\tilde{F} \in \mathfrak{N}^+$, and

$$f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta}) \tag{2}$$

for almost all $e^{i\theta} \in \Gamma$ where $\tilde{F}(z) = F^*(z^{*-1})$, $z \in D$. Meromorphic pseudocontinuation is defined relative to $\Gamma$ in a manner analogous to the $R$ definition.

Considerations about $\mathscr{M}(u, v, R)$ may be motivated by examining the special case when $U(z) = V(z) = e^{iz\tau}$, $\tau \geqq 0$. Then

$$\mathscr{M}(u, v, R) \cap L^2(R)$$

is the class of functions that are the restrictions to $R$ of entire functions of exponential type $\leqq \tau$ such that $\int_R |F(x)|^2 dx < \infty$. Such entire $F$ can be characterized by this integral condition together with the inequality

$$| F(z) |^2 < K | y |^{-1} | \sinh (2 \tau y) |$$

for all $z \in \Omega \cup \Omega_-$, where $K > 0$. The object of this paper is to extend this type of function-theoretic characterization to more general $\mathscr{M}(u, v)$ classes. The above mentioned application to star-invariant subspaces arises from the fact that $M(1, v) \cap L^2(R) = H^2(\Omega) \ominus vH^2(\Omega)$, where $H^2(\Omega)$ is the Hardy space of the upper half-plane. In § 3 and 4 applications are given to factorization problems for nonnegative operator-valued functions and to generalized Paley-Wiener representations.

## 1. A Phragmén-Lindelöf Theorem.

In this section we shall derive a Phragmén-Lindelöf type theorem for certain functions holomorphic on $D$, and then transcribe the result to obtain a like theorem for functions on $\Omega$. A rather different Phragmén-Lindelöf type theorem is discussed by Helson in [2, p. 33].

Recall that a Blaschke product $B$ on $D$ has a representation

$$(3) \qquad B(z) = \prod_{j \geq 1} B_j(z),\ B_j(z) = \frac{z_j^*}{|z_j|} \frac{z_j - z}{1 - z_j^* z},\ z \in D\,,$$

where $\sum_{j \geq 1}(1 - | z_j |) < \infty$. We take $z_j^*/| z_j | = 1$ if $z_j = 0$. The *support* supp $B$ of $B$ is the intersection of $\Gamma$ with the closure of $\{z_j\}_{j \geq 1}$. A singular inner function $S$ has a representation

$$(4) \qquad S(z) = \exp\Big(-\int_\Gamma \frac{e^{i\xi} + z}{e^{i\xi} - z} \mu(d\xi)\Big),\ z \in D\,,$$

where $\mu$ is a positive singular measure on $\Gamma$. The *support* supp $S$ is the closed support of the measure $\mu$.

Any inner function $U$ on $D$ can be factored in the form $U = cBS$, where $c \in C$, $| c | = 1$, $B$ is a Blaschke product and $S$ is a singular inner function. The *support* supp $U$ of $U$ is supp $B \cup$ supp $S$.

A closed set $N$ on $\Gamma$ is a *Carleson set* if $N$ has zero Lebesgue measure and if the complement of $N$ in $\Gamma$ is a union of open arcs $I_j$ of lengths $\varepsilon_j$ such that $\sum_{j \geq 1} \varepsilon_j \log \varepsilon_j > -\infty$.

THEOREM 1.1. (Carleson [1]). *A closed subset $N$ of $\Gamma$ is a Carleson set on $\Gamma$ if and only if there exists an outer function $G$ on $D$ that satisfies a Lipschitz condition and such that*

$$g(e^{i\theta}) \stackrel{\text{def}}{=} \lim_{r \uparrow 1} G(re^{i\theta})$$

*vanishes on $N$.*

DEFINITION 1.2. An inner function $U$ on $D$ is of *type* $\mathfrak{C}$ if

(i) supp $U$ is a Carleson set, and

(ii) $\sum_{j\geqq 1}$ [dist $(z_j,$ supp $U)] < \infty$,

where $\{z_j\}_{j\leqq 1}$ are the zeros of $U$ in $D$ repeated according to multiplicity.

LEMMA 1.3. *Let $B$ be the Blaschke product given by* (3) *and suppose $B$ is of type $\mathfrak{C}$. If $G$ is a Lipschitz outer function on $D$ such that $g(e^{i\theta}) = \lim_{r\uparrow 1} G(re^{i\theta})$ vanishes on supp $B$, then*

$$\sum_{j\geqq 1}(1 - |z_j|^2)\int|(1 - z_j^* e^{i\theta})^{-1}g(e^{i\theta})|^2\sigma(d\theta) < \infty \,. \tag{5}$$

*Proof.* Since $G$ is Lipschitz there exists $K > 0$ such that

$$|g(e^{i\theta})| \leqq K|e^{i\theta} - \lambda|$$

for all $e^{i\theta}$ in $\Gamma$ and $\lambda$ in supp $B$. Thus for $\lambda$ in supp $B$,

$$\begin{aligned}&(1 - |z_j|^2)\int|(1 - z_j^* e^{i\theta})^{-1}g(e^{i\theta})|^2\sigma(d\theta)\\ &\leqq (1 - |z_j|^2)K^2\int|(1 - z_j^* e^{i\theta})^{-1}(e^{i\theta} - \lambda)|^2\sigma(d\theta)\,.\end{aligned}$$

Applying Parseval's equality to the Fourier series for the function $(1 - z_j^* e^{i\theta})^{-1}(e^{i\theta} - \lambda)$ shows that this last expression is equal to

$$K^2(|z_j - \lambda|^2 + (1 - |z_j|^2))\,.$$

Since $\sum_{j\geqq 1}(1 - |z_j|^2) < \infty$ and we are free to let $\lambda$ vary over supp $B$ this inequality implies (5).

The following theorem is our Phragmén-Lindelöf result for functions on $D$.

THEOREM 1.4. *Let $U$ be an inner function of type $\mathfrak{C}$ on $D$. Suppose $F$ is holomorphic in $D$ and there exists $M > 0$ such that*

$$|F(z)|^2 \leqq M(1 - |z|^2)^{-1}(1 - |U(z)|^2),\ z\in D\,. \tag{6}$$

*Then $F \in \mathfrak{N}^+$.*

*Proof.* $U$ has the factorization $U = cBS$, where $|c| = 1$, $B$ is a Blaschke product of type $\mathfrak{C}$ and $S$ is a singular inner function of type $\mathfrak{C}$. We have

$$\begin{aligned}&(1 - |z|^2)^{-1}(1 - |U(z)|^2)\\ &= (1 - |z|^2)^{-1}(1 - |B(z)|^2) + |B(z)|^2(1 - |z|^2)^{-1}(1 - |S(z)|^2)\\ &\leqq (1 - |z|^2)^{-1}(1 - |B(z)|^2) + (1 - |z|^2)^{-1}(1 - |S(z)|^2),\ z\in D\,.\end{aligned} \tag{7}$$

If $B$ is given by (3), then

$$1 - |B(z)|^2 = 1 - |B_1(z)|^2 + \sum_{n \geqq 2} \left| \prod_{j=1}^{n-1} B_j(z) \right|^2 (1 - |B_n(z)|^2)$$
$$\leqq \sum_{j \geqq 1} (1 - |B_j(z)|^2) .$$

Thus

$$(8) \qquad (1 - |z|^2)^{-1}(1 - |B(z)|^2) \leqq \sum_{j \geqq 1} (1 - |z_j|^2)\,|1 - z_j^* z|^{-2} .$$

If $S$ is given by (4), then

$$|S(z)|^2 = \exp\left\{-2\int_\Gamma (1 - |z|^2)\,|e^{i\xi} - z|^{-2}\,\mu(d\xi)\right\}, \quad z \in D .$$

Applying the elementary inequality $(1 - e^{-ah})/h) \leqq a$ if $a,\ h \geqq 0$, with $h = 1 - |z|^2$ and $a = 2\int_\Gamma |e^{i\xi} - z|^{-2}\,\mu(d\xi)$ yields

$$(9) \qquad (1 - |z|^2)^{-1}(1 - |S(z)|^2) \leqq 2\int_\Gamma |e^{i\xi} - z|^{-2}\,\mu(d\xi), \quad z \in D .$$

Suppose now that (6) holds and let $G$ be a Lipschitz outer function such that $g(e^{i\theta}) = \lim_{r \uparrow 1} G(re^{i\theta})$ vanishes on supp $U$. We have from (6) – (9) that

$$|G(z)F(z)|^2 \leqq M \sum_{j \geqq 1} (1 - |z_j|^2)\,|1 - z_j^* z|^{-2}\,|G(z)|^2$$
$$+ 2M\int_\Gamma |e^{i\xi} - z|^{-2}\,|G(z)|^2\,\mu(d\xi), \quad z \in D .$$

But for some $K > 0$

$$|G(z)|^2 \leqq K^2\,|e^{i\xi} - z|^2 \text{ if } e^{i\xi} \in \operatorname{supp} U ,$$

and $\mu$ is supported on supp $S \subseteqq$ supp $U$. Thus for all $z \in D$

$$|G(z)F(z)|^2 \leqq M \sum_{j \geqq 1} (1 - |z_j|^2)\,|1 - z_j^* z|^{-2}\,|G(z)|^2 + 2MK^2\,\mu(\Gamma) .$$

It now follows from Lemma 1.3 that

$$\sup_{0 \leqq r < 1} \int_\Gamma |G(re^{i\theta})F(re^{i\theta})|^2\,\sigma(d\theta) \qquad < \infty \qquad ,$$

so $GF \in H^2$. It is easy to multiply $G$ by an outer function $G_1$ and obtain $G_1GF$ bounded, and so $F$ is in $\mathfrak{N}^+$.

We shall next recast Theorem 1.4 for functions holomorphic on $\Omega$. Any inner function $U$ on $\Omega$ has a factorization $U = cBSV^a$, where $c \in \boldsymbol{C}$, $|c| = 1$, $B$ is a Blaschke product on $\Omega$, $S$ is a singular function on $\Omega$, and $V^a(z) = e^{iaz}$, where $0 \leqq a \in \boldsymbol{R}$. Then supp $B$ is defined to be the set of limit points on $\boldsymbol{R} \cup \{\infty\}$ of the zeros of $B$, and supp $S$ is defined to be the support of the singular measure in the representation for $S$ analogous to (4), (Hoffman [3] p. 132-133). We define supp $V^a$ to be empty if $a = 0$, and $\{\infty\}$ if $a > 0$. The *support* supp $U$ of $U$ is supp $B \cup$ supp $S \cup$ supp $V^a$.

A closed subset $N$ of the extended real line $R \cup \{\infty\}$ is a *Carleson set* if $N \cap R$ has Lebesgue measure zero, $\infty \in N$, and the complement of $N$ in $R \cup \{\infty\}$ is a union of open intervals

$$I_j = (a_j, b_j), \ -\infty \leqq a_j < b_j \leqq \infty, \ j = 1, 2, \cdots$$

such that $\sum_{j>1} \delta_j \log \delta_j > -\infty$, where

$$\delta_j = \frac{b_j - a_j}{(1 + b_j^2)^{1/2}(1 + a_j^2)^{1/2}}, \quad j = 1, 2, \cdots .$$

We understand in the above that $\infty/\infty = 1$.

Now let $\alpha$: $\bar{D} \to \bar{\Omega} \cup \{\infty\}$ be the mapping defined by

$$\alpha(z) = i(1 + z)(1 - z)^{-1}$$

if $z \neq 1$ and $\alpha(1) = \infty$, and let $\beta$ be the inverse of $\alpha$. Then if $z_1, z_2 \in \bar{\Omega}$,

$$|\beta(z_1) - \beta(z_2)|^2 = 4\frac{|z_1 - z_2|^2}{|z_1 + i|^2 |z_2 + i|^2}.$$

Moreover $\beta$ maps $(-\infty, \infty]$ onto $\Gamma$ and $N$ is a Carleson set on $R \cup \{\infty\}$ if and only if $\beta(N) \cup \{1\}$ is a Carleson set on $\Gamma$. If $U$ is inner on $\Omega$ then $U \circ \alpha$ is inner on $D$ and supp $(U \circ \alpha) = \beta$ (Supp $U$). Furthermore if $\{z_j\}_{j>1}$ is the sequence of zeros of $U$, then $\{\beta(z_j)\}_{j\geqq 1}$ is the sequence of zeros of $U \circ \alpha$.

DEFINITION 1.5. Let $U$ be an inner function on $\Omega$. $U$ is of *type* $\mathfrak{C}$ if supp $U \cup \{\infty\}$ is a Carleson set on $R \cup \{\infty\}$ and

$$\sum_{j \geqq 1}\left(\inf_{\lambda \in \operatorname{supp} U} \frac{|z_j - \lambda|^2}{(1 + \lambda^2)(1 + |z_j|^2)}\right) < \infty ,$$

where $\{z_j\}_{j\geqq 1}$ is the sequence of zeros of $U$ in $\Omega$ repeated according to multiplicity.

The following lemma follows from the above discussion.

LEMMA 1.6. *Let $U$ be inner on $\Omega$. Then $U$ is of type $\mathfrak{C}$ if and only if $U \circ \alpha$ is of type $\mathfrak{C}$ on $D$.*

We can now recast Theorem 1.4 for the half-plane.

THEOREM 1.7. *Let $F$ be holomorphic in $\Omega$ and suppose that $U$ is inner of type $\mathfrak{E}$ in $\Omega$. Suppose that there exists $K > 0$ such that*

$$|F(z)|^2 \leqq K(\operatorname{Im} z)^{-1}(1 + |z|^2)(1 - |U(z)|^2) \text{ for } z \in \Omega. \tag{10}$$

*Then $F \in \mathfrak{N}^+$ on $\Omega$.*

*Proof.* Set $G = F \circ \alpha$, so $G$ is meromorphic on $D$ and

$$|G(z)|^2 \leqq K[\operatorname{Im} \alpha(z)]^{-1}(1 + |\alpha(z)|^2)(1 - |U(\alpha(z))|^2), \quad z \in D.$$

We can replace $1 + |\alpha(z)|^2$ by $|i + \alpha(z)|^2$ and the inequality still holds but for a different constant $K$. Now

$$\operatorname{Im} \alpha(z) = (1 - |z|^2)|1 - z|^{-2}$$

and

$$|i + \alpha(z)|^2 = 4|1 - z|^{-2},$$

so

$$|G(z)|^2 \leqq K'(1 - |z|^2)^{-1}(1 - |U(\alpha(z)|^2), \; z \in D.$$

But by Lemma 1.6 $U \circ \alpha$ is of type $\mathfrak{E}$, and thus Theorem 1.4 implies that $G \in \mathfrak{N}^+$ on $D$. We then deduce that $F = G \circ \beta$ is in $\mathfrak{N}^+$ on $\Omega$.

2. The classes $\mathscr{M}(u, v, \Gamma)$ and $\mathscr{M}(u, v, R)$. Suppose $U$ is inner in $D$. Then $U$ has a meromorphic pseudocontinuation to a function $U$ on $D \cup D_-$ that is given by

$$U(z) = \begin{cases} U(z), & z \in D \\ 1/U^*(z^{*-1}), & z \in D_-. \end{cases} \tag{11}$$

If supp $U \neq \Gamma$, then $U$ on $D$ has a single valued meromorphic continuation to $D_-$ that coincides with $U$ as given by (11). If $F$ is meromorphic on $D_-$ then $\tilde{F}(z) = F^*(z^{*-1})$ defines $\tilde{F}$ to be meromorphic on $D$. Of course $\tilde{F}$ need not be a pseudocontinuation of $F$.

Analogous definitions are made for $\Omega$. Suppose $U$ is inner on $\Omega$. Then $U$ has a meromorphic pseudocontinuation on $\Omega \cup \Omega_-$ given by

$$U(z) = \begin{cases} U(z) & z \in \Omega \\ 1/U^*(z^*) & z \in \Omega_-. \end{cases} \tag{12}$$

If $F$ is meromorphic on $\Omega$, then $\tilde{F}(z) = F^*(z^*)$ defines $\tilde{F}$ to be meromorphic on $\Omega_-$.

We say that $F$ is $\mathfrak{N}_0^+$ on $D$ if $F \in \mathfrak{N}^+$ on $D$ and $F(0) = 0$. $\mathscr{N}_0^+$ is defined to be the set of all $f$ such that $f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta})$ a.e., where $F \in \mathfrak{N}_0^+$ on $D$.

Suppose $U, V$ are inner functions on $D$. $\mathscr{M}_0(u, v, \Gamma)$ is the set of all functions $f$ on $\Gamma$ such that $uf \in \mathcal{N}^+$ and $vf^* \in \mathcal{N}_0^+$. $\mathcal{M}_0(u, v, \Gamma)$ can be characterized as follows: $f \in \mathcal{M}_0(u, v, \Gamma)$ if and only if there exists a function $F$ separately meromorphic in $D$ and $D_-$ and such that

$$(13) \qquad f(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta}) = \lim_{r\downarrow 1} F(re^{i\theta}) \quad \text{a.e.},$$

with

$$(14) \qquad UF \in \mathfrak{N}^+ \text{ on } D \text{ and } V\tilde{F} \in \mathfrak{N}_0^+ \text{ on } D\,.$$

In case $U$ and $V$ are of type $\mathfrak{C}$ we can deduce (14) from an inequality involving $F$, $U$ and $V$.

THEOREM 2.1. *Suppose $U$ and $V$ are of type $\mathfrak{C}$, and $F$ is meromorphic in $D$ and has a meromorphic pseudocontinuation to a function $F$ on $D \cup D_-$. Further suppose there exists $K > 0$ such that*

$$(15) \qquad |F(z)|^2 \leqq K(1 - |z|^2)^{-1}(|U(z)|^{-2} - |V(z)|^2), \quad |z| \neq 1.$$

*Then* $f(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta}) \in \mathcal{M}_0(u, v, \Gamma)$.

*Proof.* If $F$ satisfies (15) on $D$ then

$$|U(z)F(z)|^2 \leqq K(1 - |z|^2)^{-1}(1 - |U(z)V(z)|^2)\,,$$

so $UF \in \mathfrak{N}^+$ by Theorem 1.4.

If $F$ satisfies (15) on $D_-$, then for all $z \in D$,

$$|V(z)\tilde{F}(z)|^2 \leqq K|z|^2(1 - |z|^2)^{-1}(1 - |U(z)V(z)|^2)$$

so $V\tilde{F} \in \mathfrak{N}^+$ by 1.4. But we also deduce that $V(0)\tilde{F}(0) = 0$, so $V\tilde{F} \in \mathfrak{N}_0^+$. It therefore follows from the characterization of $\mathcal{M}_0(u, v, \Gamma)$ given in (13) and (14) that $f \in \mathcal{M}_0(u, v, \Gamma)$.

In case $f \in L^2(\Gamma)$, i.e., in case $\int |f|^2 d\sigma < \infty$, we have a stronger result.

THEOREM 2.2. *Assume that $U$, $V$ are inner of type $\mathfrak{C}$ on $D$ and $f \in L^2(\Gamma)$. Then $f \in \mathcal{M}_0(u, v, \Gamma)$ if and only if there exists a function $F$ satisfying the hypotheses of Theorem* 2.1 *such*

$$f(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta}) \text{ a.e..}$$

*Proof.* It follows from Theorem 2.1 that if $F$ satisfies (15) then $f \in \mathcal{M}_0(u, v, \Gamma)$. Conversely, suppose $f \in \mathcal{M}_0(u, v, \Gamma) \cap L^2(\Gamma)$. Then $uf \in \mathcal{N}^+ \cap L^2(\Gamma) = H^2$ and $vf^* \in \mathcal{N}_0^+ \cap L^2(\Gamma) \subseteqq H^2$ with $\int vf^* d\sigma = 0$.

Thus $uf$ and $v\chi^*f^*$ are in $(uvH^2)^\perp \cap H^2$, where $\chi(e^{i\theta}) = e^{i\theta}$.

Now any $g \in (uvH^2)^\perp \cap H^2$ is the boundary value function of

$$G(z) = \int (1 - ze^{-i\xi})^{-1} (1 - u^*(e^{i\xi})v^*(e^{i\xi}) U(z) V(z)) g(e^{i\xi}) \sigma(d\xi), \quad z \in D .$$

But then it follows from the Schwarz inequality that

$$(16) \qquad | G(z) |^2 \leqq K(1 - | z |^2)^{-1} (1 - | U(z) V(z) |^2), \quad z \in D ,$$

where $K = \int | g |^2 d\sigma$.

By applying (16) to $g = uf$ and $g = v\chi^*f^*$ we obtain

$$(17) \qquad | U(z) F(z) |^2 \leqq K(1 - | z |^2)^{-1} (1 - | U(z) V(z) |^2), \quad z \in D ,$$

and

$$(18) \qquad | V(z)\tilde{F}(z) |^2 \leqq K | z |^2 (1 - | z |^2)^{-1} (1 - | U(z) V(z) |^2), \quad z \in D ,$$

where $K = \int_\Gamma | f |^2 d\sigma$.

It is easily seen that (17) and (18) together is equivalent to (15).

COROLLARY 2.3. *Assume that $V$ is inner of type $\mathfrak{C}$ on $D$ and $f \in H^2$ on $\Gamma$. Then $f \in (vH^2)^\perp$ if and only if there exists a meromorphic function $F$ on $D \cup D_-$ such that*

$$(19) \qquad f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta}) \text{ a.e.,}$$

*for which there exists $K > 0$ with*

$$| F(z) |^2 \leqq K (1 - | z |^2)^{-1} (1 - | V(z) |^2), \quad z \in D \cup D_- .$$

*Proof.* Note that $(vH^2)^\perp \cap H^2 = \mathscr{M}_0(1, v, \Gamma)$, and use 2.2.

COROLLARY 2.4. *Assume that $U$, $V$ are inner of type $\mathfrak{C}$ on $D$ and $f \in L^2(\Gamma)$. Then $f \in \mathscr{M}(u, v, \Gamma)$ if and only if there exists a function $F$ meromorphic in $D$ with pseudocontinuation $F$ such that* (19) *holds and there exists $K > 0$ such that*

$$| F(z) |^2 \leqq K (1 - | z |^2)^{-1} (| U(z) |^{-2} - | zV(z) |^2), \quad z \in D .$$

*Proof.* Note that $\mathscr{M}(u, v, \Gamma) = \mathscr{M}_0 (u, \chi v, \Gamma)$.

The same kind of problem can be considered on $\Omega$ with minor modifications in the proofs.

THEOREM 2.5. *Suppose $F$ is meromorphic on $\Omega$ and has a mero-*

*morphic pseudocontinuation to a function $F$ on $\Omega \cup \Omega_-$. Assume that $U$ and $V$ are inner functions of type $\mathfrak{C}$ on $\Omega$. Further suppose that there exists $K > 0$ such that*

$$| F(z) |^2 \leqq K(\operatorname{Im} z)^{-1} (1 + | z |^2) (| U(z) |^{-2} - | V(z) |^2), \quad z \in \Omega \cap \Omega_- .$$

Then $f(x) = \lim_{y \downarrow 0} F(x + iy) \in \mathscr{M}(u, v, R)$.

THEOREM 2.6. *Assume that $U$, $V$ are inner of type $\mathfrak{C}$ on $\Omega$ and $f \in L^2(R)$. Then $f \in \mathscr{M}(u, v, R)$ if and only if there exists a function satisfying the hypotheses of Theorem 2.5 such that*

$$f(x) = \lim_{y \downarrow 0} F(x + iy) \text{ a.e..}$$

**3. Factorization of nonnegative functions.** In this section we shall reformulate an operator factorization theorem of the type set down in [5] in terms of inequalities of the type discussed in § 1 and 2. Throughout $\mathscr{C}$ is a complex separable Hilbert space and $B(\mathscr{C})$ the space of bounded operators on $\mathscr{C}$. We shall restrict ourselves to considerations involving $\Omega$ rather than $D$ in order to simplify the exposition. Following [5] we say that a holomorphic function $F$ on $\Omega$ taking values in $B(\mathscr{C})$ is in $\mathfrak{N}^+_{B(\mathscr{C})}$ if there exists a nonzero complex-valued outer function $\Phi$ such that $\Phi F$ is a bounded holomorphic function on $\Omega$ that takes values in $B(\mathscr{C})$. Any $F$ in $\mathfrak{N}^+_{B(\mathscr{C})}$ has strong boundary values a.e., that is, the limit $\lim_{y \downarrow 0} F(x + iy) = f(x)$ exists a.e. in the strong operator topology.

We say that a holomorphic function $G$ in $\mathfrak{N}^+_{B(\mathscr{C})}$ has a *meromorphic pseudocontinuation* $G$ if $G$ is meromorphic in $\Omega_-$ and the strong limits $\lim_{y \uparrow 0} G(x - iy)$ and $\lim_{y \uparrow 0} G(x + iy)$ exist and are a.e. equal. For such $G$ we define $\tilde{G}$ by $\tilde{G}(z) = G^*(z^*)$, $z \in \Omega \cup \Omega_-$.

THEOREM 3.1. *Let $U$ be a complex-valued inner function on $\Omega$ and $F$ a meromorphic function on $\Omega$ taking values in $B(\mathscr{C})$ such that $UF \in \mathfrak{N}^+_{B(\mathscr{C})}$. Then $F(x + iy)$ has strong boundary values $f(x)$ a.e. as $y \downarrow 0$. Assume that $\langle f(x)c, c\rangle \geqq 0$ a.e. for each $c$ in $\mathscr{C}$.*

*Then $F$ has a factorization $F(z) = \tilde{G}(z)G(z)$, $z \in \Omega$, where $G$ is in $\mathfrak{N}^+_{B(\mathscr{C})}$ and has a meromorphic pseudocontinuation $G$ such that $U\tilde{G} \in \mathfrak{N}^+_{B(\mathscr{C})}$. If there is real interval $I$ such that $f(.)$ is a.e. bounded on $I$ and $U$ is analytically continuable across $I$, then $G$ is analytically continuable across $I$.*

*Proof.* This theorem is a summary of results proved in [5].

THEOREM 3.2. *Theorem 3.1 may be modified as follows:*

(i) *The hypothesis "$UF \in \mathfrak{N}^+_{B(\mathscr{C})}$" may be replaced by the stronger hypothesis "there exists $K > 0$ such that*

$$(20) \qquad \| F(z) \|^2 \leqq K(\operatorname{Im} z)^{-1} (1 + |z|^2)(|U(z)|^{-2} - |U(z)|^2)$$

*for all $z$ in $\Omega$".*

(ii) *If in addition one assumes that $\int_{-\infty}^{\infty} \langle f(x)c, c\rangle \, dx < \infty$ for all $c$ in $\mathscr{C}$, then $G$ can be chosen to in addition satisfy*

$$(21) \qquad |\langle G(z)c, c\rangle|^2 \leqq K_c(\operatorname{Im} z)^{-1} (1 + |z|^2)(1 - |U(z)|^2), \; c \in \mathscr{C}$$

*for some $K_c > 0$ ($K_c$ depends on $c$) and all $z \in \Omega \cup \Omega_-$.*

*Proof.* The proof of 1.4 shows that (20) implies that $UF \in \mathfrak{N}^+_{B(\mathscr{C})}$.

Assume the hypotheses of (ii). Now $f = g^*g$, where $g(x)$ are the strong boundary values of $G(x + iy)$ as $y \downarrow 0$ and $y \uparrow 0$. We have $|\langle g(\cdot)c, c\rangle|^2 \leqq \| g(\cdot)c \|^2 \| c \|^2 = \langle f(\cdot)c, c\rangle \| c \|^2$ for all $c$ in $\mathscr{C}$, so $\langle g(\cdot)c, c\rangle \in L^2(R)$ for all $c$ in $\mathscr{C}$. (21) now follows from Theorem 2.6 and the fact that $\langle g(\cdot)c, c\rangle \in \mathscr{M}(1, u, R)$.

As an example suppose $F(\cdot)$ is an entire function taking values in $B(\mathscr{C})$ such that $\langle F(x)c, c\rangle \geqq 0$ whenever $c \in \mathscr{C}$ and $x \in R$, and there exists $\tau \geqq 0$ and $K > 0$ with

$$\| F(z) \|^2 \leqq Ky^{-1}(1 + |z|^2) \sinh 2\tau y, \; z = x + iy \in \Omega \, .$$

Then $F$ is factorable, $F(z) = \tilde{G}(z)G(z)$, where $G(\cdot)$ is an entire function taking values in $B(\mathscr{C})$. This follows from Theorems 3.1 and 3.2 (i) with $U(z) = e^{i\tau z}$. $G(\cdot)$ is entire by the last statement in Theorem 3.1. It also is deducible from Theorem 3.6 of [5].

If in addition to above $F(\cdot)$ satisfies $\int_{-\infty}^{\infty} \langle F(x)c, c\rangle \, dx < \infty$, then by (21) $G$ satisfies

$$|\langle G(z)c, c\rangle|^2 \leqq K_c y^{-1}(1 + |z|^2)(1 - e^{-\tau y}) \, ,$$

for all $z = x + iy$ with $y \neq 0$ and $c \in \mathscr{C}$. $K_c$ is a constant depending on $c$.

**4. A Fourier type transform and the Paley-Wiener representation.** As before let $U$ and $V$ be inner functions in $\Omega$ and denote the space $\mathscr{M}(u, v, R) \cap L^2(R)$ by $\mathscr{M}^2(u, v, R)$. This space is easily seen to be a Hilbert subspace of $L^2(R)$. As noted in the introduction $\mathscr{M}^2(e^{ix\tau}, e^{ix\tau}, R)$ is the restriction to the real axis of a classical Paley-Wiener space of entire functions. That

$$\mathscr{M}^2(e^{ix\tau}, e^{ix\tau}, R) = \mathscr{F}^* L^2(-\tau, \tau) \, ,$$

(where $\mathscr{F}$ is the Fourier-Plancherel operator on $L^2(R)$), is the content of a well known theorem of Paley and Wiener.

In [4] one of the present authors generalized this theorem to give an integral representation for any of the spaces $\mathscr{M}^2(u, v, R)$. In this section we combine this result with Theorem 2.6. First we shall set down some basic facts from [4]. For simplicity we assume that $U$ and $V$ have no zeros and are normalized so that $U(i)$ and $V(i)$ are positive. $U$ then has a factorization $U(z) = S(z)e^{i\alpha z}$ where $S$ is a singular inner function in $\Omega$ and $\alpha \geqq 0$. Using the usual representation for singular inner functions we can combine the two factors in the following convenient form:

$$U(z) = \exp\left(i\int_{R^*}\frac{1+tz}{t-z}\,\mu(dt)\right) \tag{22}$$

where $\mu$ is a finite positive measure on the extended real numbers $R^* = R \cup \{\infty\}$ whose restriction to $R$ is singular and with $\mu(\{\infty\}) = \alpha$. In the integrand, and elsewhere below, we always take $(z\,\infty)/\infty = z$ for any complex $z$. $V$ has a similar representation with corresponding measure $\gamma$.

Let $\tau$ be the total variation of $\mu$ and suppose that $a$ is an $R^*$-valued measurable function defined on $[0, \tau]$ such that $m(a^{-1}(E)) = \mu(E)$ for every subinterval $E$ of $R^*$. For example, we could take $a(t) = \inf\{x \in R^*: \mu((-\infty, x]) \geqq t\}$. Extend the definition of $a$ to $[0, \infty)$ by setting $a(t) = \infty$ if $t > \tau$. For each $t \geqq 0$ let

$$U_t(z) = \exp\left(i\int_0^t \frac{1 + za(x)}{a(x) - z}\,dx\right).$$

It is clear from (22) and a change of variables that $U_\tau = U$. Moreover, $U_t$ is an inner function for each $t$ and $U_s$ divides $U_t$ if $0 \leqq s < t$.

In a like manner one can associate $\sigma$, $b: [0, \sigma] \to R^*$ and $V_t$ (analogous to $\tau$, $a$ and $U_t$) with the inner function $V$. Note that $V_\sigma = V$. $U_t$ and $V_t$ have pseudo-continuations to $\Omega_-$ given by (12). For any $z$ in $\Omega \cup \Omega_-$ let

$$H_z^+(t) = V_t(z)\,\frac{b(t) - i}{b(t) - z}$$

and

$$H_z^-(t) = U_t(z)^{-1}\,\frac{a(t) + i}{a(t) - z}\,, \quad t \geqq 0\,.$$

Now let $H^2(\Omega)$ and $H^2(\Omega_-)$ denote the usual Hardy spaces of functions analytic in $\Omega$ and $\Omega_-$ respectively, which can also be considered as orthogonal complements of each other in $L^2(R)$. It was shown in [4] that the mappings $W_1$ and $W_2$ given by

$$(W_1 g)(z) = (2\pi)^{-1/2}\int_0^\infty H_z^+(t)g(t)\,dt, \ \operatorname{Im} z > 0$$

and

$$(W_2 g)(z) = (2\pi)^{-1/2}\int_0^\infty H_z^-(t)g(t)dt, \ \operatorname{Im} z < 0\ ,$$

are isometries from $L^2(0, \infty)$ onto $H^2(\Omega)$ and $H^2(\Omega_-)$ respectively.

Let $E$: $L^2(-\infty, 0) \to L^2(0, \infty)$ be the operator $(Eg)(t) = g(-t)$. The $W_2E \oplus W_1$ can be considered as a unitary operator from

$$L^2(-\infty, 0) \oplus L^2(0, \infty) = L^2(R)$$

onto $H^2(\Omega_-) \oplus H^2(\Omega) = L^2(R)$. This operator takes $L^2(-s, t)$ onto $\mathscr{M}^2(u_s, v_t, R)$ for all $s, t \geqq 0$. If $\mu$ and $\gamma$ are supported on the singleton $\{\infty\}$ or, equivalently, if $a(t) = b(t) = \infty$ a.e., then $W_2E \oplus W_1$ is the adjoint of the Fourier-Plancherel operator. Combining this with Theorem 2.6 yields the following result.

THEOREM 4.1. *Let $U$ and $V$ be inner functions of type $\mathfrak{C}$. Let $F$ be analytic in $\Omega \cup \Omega_-$ and suppose that the two sided boundary function $f(x) = \lim_{|y|\to 0} F(x + iy)$ exists* a.e. *and lies in $L^2(R)$. Let $s, t \geqq 0$. Then the following are equivalent.*

(i)

$$|F(z)|^2 \leqq K(\operatorname{Im} z)^{-1}(1 + |z|^2)(|U_s(z)|^{-2} - |V_t(z)|^2),\ z \in \Omega \cup \Omega_-\ .$$

(ii) *There exist* a.e. *unique functions $g_1$ in $L^2(0, t)$ and $g_2$ in $L^2(0, s)$ such that*

$$F(z) = (2\pi)^{-1/2}\int_0^t H_z^+(z)g_1(x)\,dx$$
$$+ (2\pi)^{-1/2}\int_0^s H_z^-(x)g_2(x)dx,\ \operatorname{Im} z \neq 0\ .$$

*Moreover,* $\|f\|_2^2 = \|g_1\|_2^2 + \|g_2\|_2^2$.

*Added in proof.* We refer the reader to the papers.

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and,

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# NOTES ON RELATED STRUCTURES OF A UNIVERSAL ALGEBRA

WILLIAM A. LAMPE

**The related structures of a universal algebra $\mathfrak{A}$ that are studied here are the subalgebra lattice of $\mathfrak{A}$, the congruence lattice of $\mathfrak{A}$, the automorphism group of $\mathfrak{A}$, and the endomorphism semigroup of $\mathfrak{A}$. Characterizations of these structures known, and E. T. Schmidt proved the independence of the automorphism group and the subalgebra lattice. It has been conjectured that the first three of the structures listed above are independent, i.e., that the congruence lattice, subalgebra lattice, and automorphism group are independent. One result in this paper is a proof of a special case of this conjecture. Various observations concerning the relationship between the endomorphism semigroup and the congruence lattice are also in this paper. In the last section a problem of G. Grätzer is solved, namely that of characterizing the endomorphism semigroups of simple unary algebras. (An algebra is simple when the only congruences are the trivial ones.)**

The characterizations of the various related structures are as follows: the congruence lattice is an arbitrary algebraic lattice the subalgebra lattice is an arbitrary algebraic lattice; the automorphism group is an arbitrary group; the endomorphism semigroup is an arbitrary semigroup with identity. The "independence of the automorphism group and the subalgebra lattice" is more precisely phrased as: for each pair $\langle \mathfrak{G}, \mathfrak{L} \rangle$, where $\mathfrak{G}$ is a group and $\mathfrak{L}$ is an algebraic lattice with more than one element, there is an algebra $\mathfrak{A}$ with $\mathfrak{G}$ isomorphic to the automorphism group of $\mathfrak{A}$ and with $\mathfrak{L}$ isomorphic to the subalgebra lattice of the same algebra $\mathfrak{A}$. All statements about the independence of related structures will be phrased in this way.

Mentioned above was a proof of a special case of the independence of the triple consisting of the automorphism group, the subalgebra lattice, and the congruence lattice. As a corollary one gets a proof of a special case of the independence of the pair consisting of the automorphism group and the congruence lattice. E. T. Schmidt published what was supposed to be a proof of the independence of this pair of structures. But, his proof [10] was incorrect. (See e.g. Exercise 31 of chapter 2 of [2]). The author has just completed a proof of the independence of this pair [8].

The terminology essentially conforms to that in [2]. $\omega$(or $\omega_A$) will denote the equality relation on the set $A$, and $\iota$(or $\iota_A$) will denote the total relation. $\Theta(a_0, a_1)$ will represent the smallest congrucence collapsing $a_0$ and $a_1$. $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ will denote a lattice. $\mathfrak{C}(\mathfrak{A}) = \langle\mathscr{C}(\mathfrak{A}); \subseteqq\rangle$ will denote the congruence lattice of $\mathfrak{A}$. $\mathfrak{S}(\mathfrak{A}) = \langle\mathscr{S}(\mathfrak{A}); \subseteqq\rangle$ will denote the subalgebra lattice of $\mathfrak{A}$. $\mathfrak{G}(\mathfrak{A}) = \langle G(\mathfrak{A}); \circ\rangle$ will denote the automorphism group of $\mathfrak{A}$. $\mathfrak{E}(\mathfrak{A}) = \langle E\mathfrak{A}); \circ\rangle$ will denote the endomorphism semigroup of $\mathfrak{A}$.

An important algebra for dealing with endomorphism semigroup and automorphism group problems is the algebra of left multiplications $\mathfrak{L}(\mathfrak{S})$ of the semigroup $\mathfrak{S}$. The operations are all left multiplication maps and the endomorphisms are all right multiplication maps. As in Cayley's Theorem, the semigroup of right multiplications of $\mathfrak{S}$ is isomorphic to $\mathfrak{S}$.

Many of the details of the proofs which are left out can be found in the author's dissertation [6]. The various characterizations mentioned above can be found in [1], [2], [3]. E. T. Schmidt's result on the independence of the automorphism group and subalgebra lattice is found in [11].

## 1. The property restricting the representation of $\langle\mathfrak{G}, \mathfrak{L}_0, \mathfrak{L}_1\rangle$ as $\langle\mathfrak{G}(\mathfrak{A}), \mathfrak{S}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A})\rangle$.

Let $\mathfrak{A} = \langle A; F\rangle$ be an algebra. The lattice $\mathfrak{L}$ is assumed to be an algebraic lattice. Let $a \in L$, and let $(x_i | i \in I)$ be a family of elements of $L$.

Essentially the property mentioned in the heading is: there exist $a_0, a_1 \in A$ such that for any $x \neq a_0$ and for any congruence $\Theta$, if $a_0 \equiv x(\Theta)$, then $a \equiv a_1(\Theta)$. We will give a generalization of this property and a property of the congruence lattice equivalent to the more general property. Also, the class of algebraic lattices having the equivalent property will be discussed.

Let $a_0, a_1 \in A$ with $a_0 \neq a_1$.

(**) There exists a partition $\{A_0, A_1\}$ of $A$ such that $a_i \in A_i$ and for any $\langle x, y\rangle \in A_0 \times A_1$, $\Theta(a_0, a_1) \leqq \Theta(x, y)$.

( * ) If $a \leqq \bigvee (x_i | i \in I)$, then $a \leqq x_i$ for some $i$.

Notice that the originally stated condition is a special case of (**) where $A_0 = \{a_0\}$. Obviously, if an element $a$ of $\mathfrak{L}$ has property (*), then $a$ is complete-join irreducible. Also, $a$ has property (*) if and only if $a$'s dual ideal is completely prime.

PROPOSITION 1. *If property* (**) *is satisfied for* $\langle a_0, a_1\rangle$, *then* $\Theta(a_0, a_1\rangle$ *satisfies property* (*) *in the congruence lattice of* $\mathfrak{A}$.

REMARK. This statement was first observed by G. Grätzer.

*Proof.* Suppose that $(\Phi_i \mid i \in I)$ is a family of congruences and that $\Theta(a_0, a_i) \subseteqq \bigvee (\Phi_i \mid i \in I)$. There exists a sequence $a_0 = z_0, \cdots, z_n, = a_1$, with $z_j \in A$ such that $z_j \equiv z_{j+1}(\Phi_{i_j})$ for some $i_j \in I$. Since $a_0 \in A_0$, $a_1 \in A_1$, and $\{A_0, A_1\}$ is a partition of $A$, there is a $k$ such that $z_k \in A_0$ and $z_{k+1} \in A_1$. So $\Theta(a_0, a_1) \subseteqq \Theta(z_k, z_{k+1}) \subseteqq \Phi_{i_k}$.

PROPOSITION 2. *If there is a congruence $\Theta$ different from $\omega$ having property (*), then $\Theta = \Theta(a_0, a_1)$ for some $a_0, a_1$ in $A$ with $a_0 \neq a_1$ and property (**) is satisfied for $\langle a_0, a_1 \rangle$.*

*Proof.* Always $\Theta = \bigvee(\Theta(x, y) \mid x \equiv y(\Theta))$. Since $\Theta$ has property (*), $\Theta = \Theta(x, y)$ for some $x, y \in A$. Fix $a_0, a_1$ such that $\Theta = \Theta(a_0, a_1)$. Set

$$B_0 = \{x \mid \Theta(x, a_0) \not\supseteqq \Theta(a_0, a_1)\} ,$$
$$B_1 = \{y \mid \Theta(y, a_1) \not\supseteqq \Theta(a_0, a_1)\} ,$$
$$B_2 = \{z \mid \Theta(a_0, z) = \Theta(a_1, z)\} .$$

Set $A_0 = B_0$ and $A_1 = B_1 \cup B_2$. It follows that $A_0 \cap A_1 = \varnothing$. Clearly, $a_0 \in A_0$ and $a_1 \in A_1$. Also $A = A_1 \cup A_2$.

Let $x_0 \in A_0$ and $x_1 \in A_1$ and consider $\Theta(x_0, x_1)$. First suppose that $x_1 \in B_1$. Thus, $\Theta(x_0, a_0) \not\supseteqq \Theta(a_0, a_1)$ and $\Theta(x_1, a_1) \not\supseteqq \Theta(a_0, a_1)$. Now, since $\Theta(a_0, a_1) \subseteqq \Theta(x_0, a_0) \vee \Theta(x_0, x_1) \vee \Theta(x_1, a_1)$ and since $\Theta(a_0, a_1)$ has (*), we have that $\Theta(a_0, a_1) \subseteqq \Theta(x_0, x_1)$. Now suppose that $x_1 \in B_2$. So $\Theta(a_0, a_1) \subseteqq \Theta(a_0, x_1) \subseteqq \Theta(a_0, x_0) \vee \Theta(x_0, x_1)$, and thus, $\Theta(a_0, a_1) \subseteqq \Theta(x_0, x_1)$.

Combining these two propositions with the congruence lattice characterization theorem, we get the following statement.

PROPOSITION 3. *If $\mathfrak{L}$ is an algebraic lattice, the following are equivalent*:

( i ) *there exists $a \neq 0, a \in L$, such that $a$ has property* (*);

(ii) *there exists an algebra $\mathfrak{A} = \langle A; F \rangle$ with $\mathfrak{C}(\mathfrak{A})$, the congruence lattice of $\mathfrak{A}$, isomorphic to $\mathfrak{L}$, and there are $a_0, a_1 \in A$, $a_0 \neq a_1$, such that (**) is satisfied for $\langle a_0, a_1 \rangle$*;

(iii) *for any algebra $\mathfrak{A} = \langle A; F \rangle$ with $\mathfrak{C}(\mathfrak{A})$ isomorphic to $\mathfrak{L}$, there are $a_0, a_1 \in A$, $a_0 \neq a_1$, such that (**) is satisfied for $\langle a_0, a_1 \rangle$.*

Let $\mathscr{K}$ be the class of algebraic lattices having an $a \neq 0$ with property (*). Several simple observations can be made. The five element modular non-distributive lattice is not in $\mathscr{K}$ since none of the dual ideals generated by a nonzero element is prime. Every distributive algebraic lattice with a complete-join irreducible element is in $\mathscr{K}$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are algebraic lattices, then $\mathfrak{L}_1 + \mathfrak{L}_2 \in \mathscr{K}$ (where +

denotes ordinal sum). (The zero of $\mathfrak{L}_2$ is a nonzero element in $\mathfrak{L}_1 + \mathfrak{L}_2$ having (*)). Every algebraic lattice $\mathfrak{L}$ is both a complete sublattice of and a homomorphic image of a member of $\mathscr{K}$ since $\mathfrak{C}_1 + \mathfrak{L} \in \mathscr{K}$. ($\mathfrak{C}_n$ denotes the $n$-element chain.) Also, observe that for a family $(\mathfrak{L}_i | i \in I)$ of algebraic lattices, $\Pi(\mathfrak{L}_i | i \in I) \in \mathscr{K}$ if and only if there exists at least one $j \in I$ with $\mathfrak{L}_j \in \mathscr{K}$.

**2. The construction for representing $\langle \mathfrak{G}, \mathfrak{L}_0, \mathfrak{L}_1 \rangle$ as $\langle \mathfrak{G}(\mathfrak{A}), \mathfrak{S}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A}) \rangle$.** First we need some notation. Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra and $X \subseteqq A$. Set $F(\mathfrak{A}, X) = \{\varphi | \varphi$ is an endomorphism of $\mathfrak{A}, \{x\} = x\varphi^{-1}$ for all $x \in X\}$, and set $\mathfrak{F}(\mathfrak{A}, X) = \langle F(\mathfrak{A}, X); \circ \rangle$. In other words, an endomorphism $\varphi$ is in $F(\mathfrak{A}, X)$ if $(A - X)\varphi \subseteqq A - X$ and $x\varphi = x$ for $x \in X$. Clearly, $\mathfrak{F}(\mathfrak{A}, X)$ is a nonempty semigroup with identity.

$\mathscr{S}(\mathfrak{A})$ is the subalgebra system of $\mathfrak{A}$. Recall that $\mathfrak{S}(\mathfrak{A})$ is the subalgebra lattice, that $\mathfrak{C}(\mathfrak{A})$ is the congruence lattice, and that $\mathfrak{E}(\mathfrak{A})$ is the endomorphism semigroup.

THEOREM 1. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are algebras, that $\mathfrak{A}$ is simple, that there is a $U \subseteqq A$, $|U| = 2$, $U \subseteqq D$ for every $D \in \mathscr{S}(\mathfrak{A})$, and that there is an $\langle a_0, a_1 \rangle \in B^2$ with $a_0 \neq a_1$ for which property (**) is satisfied. There exists an algebra $\mathfrak{A}'$ such that:*

(i) *$\mathfrak{S}(\mathfrak{A}')$ is isomorphic to $\mathfrak{S}(\mathfrak{A})$;*
(ii) *$\mathfrak{C}(\mathfrak{A}')$ is isomorphic to $\mathfrak{C}(\mathfrak{B})$;*
(iii) *$\mathfrak{E}(\mathfrak{A}')$ is isomorphic to $\mathfrak{F}(\mathfrak{A}, U)$.*

*Proof.* Let $\mathfrak{A} = \langle A; F \rangle$ and $\mathfrak{B} = \langle B; G \rangle$ and $U = \{u_0, u_1\}$ and let $\langle a_0, a_1 \rangle \in B^2$ have (**) and let $a_0 \neq a_1$. Assume that $A$ and $B$ are disjoint. For each $x \in B \cup U$ define a nullary operation $f_x$ whose value is $x$. Let $\{A_0, A_1\}$ be a partition of $B$ for satisfying (**). Define four unary operations as follows:

$$g_1(x) = \begin{cases} u_0, & x \in A \cup A_0 \\ u_1, & x \in A_1 \end{cases};$$

$$g_2(x) = \begin{cases} a_0, & x = u_0 \\ a_1, & \text{otherwise} \end{cases};$$

$$g_3(x) = \begin{cases} a_0, & x \in A \\ a_1, & x \in B \end{cases};$$

$$g_4(x) = \begin{cases} x, & x \in A - U \\ u_0, & x \in B \cup \{u_1\} \\ a_0, & x = u_0 \end{cases}.$$

For $x \in A'$, set $\hat{x} = x$ if $x \in B$ and set $\hat{x} = a_0$ if $x \in A$. Let $x_i \in A'$.

Extend the operations of $F$ and $G$ to $A'$ as follows: For $f \in G$ set $f(a_0, \cdots, a_{n-1}) = f(\hat{a}_0, \cdots, \hat{a}_{n-1})$. For $f \in F$, if all $a_i \in A$, then keep the value of $f$ in $\mathfrak{A}$, and set $f(a_0, \cdots, a_{n-1}) = u_0$, otherwise. Set $F' = F \dot{\cup} G \dot{\cup} \{f_x \mid x \in B \dot{\cup} U\} \dot{\cup} \{g_i \mid i = 1, 2, 3, 4\}$. Set $\mathfrak{A}' = \langle A'; F' \rangle$.

For each $D \in \mathscr{S}(\mathfrak{A})$, set $\bar{D} = D \cup B$. For each $\varphi \in F(\mathfrak{A}, U)$, define $\bar{\varphi}$ by letting $x\bar{\varphi} = x\varphi$ if $x \in A$ and $x\bar{\varphi} = x$ if $x \in B$. For $\Theta \in \mathscr{C}(\mathfrak{A})$ define $\Theta^*$ by letting $\Theta^* = \omega_A$, the equality relation on $A$, if $\Theta(a_0, a_1) \not\subseteq \Theta$ and $\Theta^* = \iota_A \cup \{\langle x, b\rangle \mid x \in A, b \equiv a_0(\Theta)\} \cup \{\langle b, x\rangle \mid x \in A, b \equiv a_0(\Theta)\}$ in case $\Theta(a_0, a_1) \subseteq \Theta$. Now set $\bar{\Theta} = \Theta \cup \Theta^*$. To complete the proof one shows that $D \to \bar{D}$, $\varphi \to \bar{\varphi}$, and $\Theta \to \bar{\Theta}$ are isomorphisms. The lengthy, but routine, calculations are left to the reader.

In the proof above the operation $g_4$ guarantees that an endomorphism $\sigma$ of $\mathfrak{A}'$ has the property that $A\sigma \subseteq A$. The operations $g_1, g_2, g_3$ guarantee that $a_0 \equiv a_1$ iff $u_0 \equiv u_1$ iff $a_0 \equiv a_1 \equiv u_0 \equiv u_1$. That $\mathfrak{A}$ is simple guarantees that if $x, y \in A$ and $x \neq y$ and $x \equiv y$ then $x \equiv u_0 \equiv u_1$. Finally $g_3$ guarantees that if $x \in A$ and $y \in B$ and $x \equiv y$ then $a_0 \equiv a_1$.

## 3. Representing $\langle \mathfrak{G}, \mathfrak{L}_0, \mathfrak{L}_1 \rangle$ as $\langle \mathfrak{G}(\mathfrak{A}), \mathfrak{S}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A}) \rangle$.

LEMMA 1. *If $\mathfrak{A} = \langle A; F \rangle$ is an algebra, then there is an algebra $\mathfrak{A}' = \langle A; F \rangle$ such that:*

(i) *$\mathfrak{A}'$ is simple;*
(ii) *$D$ is a subalgebra of $\mathfrak{A}$ iff $D$ is a subalgebra of $\mathfrak{A}'$;*
(iii) *$E(\mathfrak{A}) = \{\varphi \mid \varphi \in E(\mathfrak{A}), \varphi$ is $1 - 1$ or $\varphi$ is constant$\}$.*

REMARKS. Roughly (iii) says $E(\mathfrak{A}')$ is as big as is possible given (i) and (ii).

Suppose $\mathfrak{S}$ is a semigroup in which every element is right cancellative or a right zero. Every endomorphism of $\mathfrak{L}(\mathfrak{S})$ is $1 - 1$ or constant. By applying this lemma to $\mathfrak{L}(\mathfrak{S})$ we get an easier proof that $\langle \mathfrak{S}; \mathfrak{C}_2 \rangle$ is representable. (See [3].)

*Proof.* Add an additional operation $g$ defined as follows:

$$g(x, y, u, v) = \begin{cases} u, & \text{if } x \neq y \\ v, & \text{if } x = y \,. \end{cases}$$

Suppose $x \neq y$ and $\Theta$ is any congruence of $\mathfrak{A}'$ with $x \equiv y(\Theta)$. Let $u, v \in A$. Thus, $u = g(x, y, u, v) \equiv g(y, y, u, v) = v(\Theta)$. So $\Theta = \iota$, and (i) is established.

The rest is routine.

The operation used in the above lemma was used in [5] in a different context, but in each case the purpose of the operation is to "fill out" subalgebras in a direct power. This 4-ary function is equivalent to the ternary discriminator function [12] [9] in that each can be expressed as a polynomial in the other.

A modification of the above 4-ary function is used in Lemma 6. It does not appear that the modified 4-ary function is equivlent to a ternary function.

LEMMA 2. *If* $\mathfrak{A}$ *is any algebra, then there is an algebra* $\mathfrak{A}' = \langle A'; F' \rangle$ *and* $U \subsetneqq A'$ *with* $|U| = 2$ *such that:*

( i ) $\mathfrak{S}(\mathfrak{A})$ *is isomorphic to* $\mathfrak{S}(\mathfrak{A}')$;

(ii) $U \subsetneqq D$ *for all* $D \in \mathscr{S}(\mathfrak{A}')$;

(iii) $\mathfrak{F}(\mathfrak{A}, U)$ *is isomorphic to* $\mathfrak{S}(\mathfrak{A})$.

*Proof.* Add two elements $u_0, u_1$. Let $u_0$ and $u_1$ each be the value of a nullary operation. Extend every operation $f$ of $\mathfrak{A}$ by setting $f(x_0, \cdots, x_{n-1}) = u_0$ if $x_j \in U$. The rest is obvious.

The next lemma is a theorem due to E. T. Schmidt [11]. Recall that $\mathfrak{G}(\mathfrak{A})$ is the automorphism group of $\mathfrak{A}$.

LEMMA 3. *If* $\mathfrak{G}$ *is any group and* $\mathfrak{L}$ *is any algebraic lattice with* $|L| > 1$, *then there is an algebra* $\mathfrak{A}$ *with* $\mathfrak{G}$ *isomorphic to* $\mathfrak{G}(\mathfrak{A})$ *and* $\mathfrak{L}$ *isomorphic to* $\mathfrak{S}(\mathfrak{A})$.

THEOREM 2. *If* $\mathfrak{G}$ *is any group, if* $\mathfrak{L}_0$ *and* $\mathfrak{L}_1$ *are algebraic lattices such that* $|L_0| > 1$, *and if there is an* $a \neq 0$, $a \in L_1$, *with property* (*), *then there is an algebra* $\mathfrak{A}$ *such that:*

( i ) $\mathfrak{G}$ *is isomorphic to* $\mathfrak{G}(\mathfrak{A})$;

(ii) $\mathfrak{L}_0$ *is isomorphic to* $\mathfrak{S}(\mathfrak{A})$;

(iii) $\mathfrak{L}_1$ *is isomorphic to* $\mathfrak{C}(\mathfrak{A})$.

REMARKS. A best possible representation theorem would, of course, have the restriction that $|L_1| > 1$. Also, if $|L_0| = 1$, then it is necessary that $|G| = 1$. Of course any triple of the form $\langle 1, \mathfrak{C}_1, \mathfrak{L} \rangle$ is representable. ($\mathfrak{C}_1$ is the one element chain.)

*Proof.* Let $\mathfrak{B}$ be the algebra given by Lemma 3 when applied to $\mathfrak{G}$ and $\mathfrak{L}_0$. Let $\mathfrak{B}'$ be the algebra given by Lemma 2 applied to $\mathfrak{B}$. Let $\mathfrak{B}''$ be the algebra given by Lemma 1. Let $\mathfrak{C}$ be the algebra constructed in the proof of the congruence lattice characterization theorem [2], [4] or [7]. Let $\mathfrak{A}$ be the algebra given by Theorem 1 when applied to $\mathfrak{B}''$ and $\mathfrak{C}$. The rest is routine.

COROLLARY 1. *If* $\mathfrak{G}$ *is any group and* $\mathfrak{L}$ *is any algebraic lattice with* $a \neq 0, a \in L$, *having property* (*), *then there is an algebra* $\mathfrak{A}$ *with* $\mathfrak{G}$ *isomorphic to* $\mathfrak{G}(\mathfrak{A})$ *and* $\mathfrak{L}$ *isomorphic to* $\mathfrak{C}(\mathfrak{A})$.

COROLLARY 2. *If* $\mathfrak{L}_0$ *is any algebraic lattice and* $\mathfrak{L}_1$ *is an algebraic lattice with an* $a \neq 0 (a \in L_1)$ *such that a has property* (*), *then there is an algebra* $\mathfrak{A}$ *with* $\mathfrak{S}(\mathfrak{A})$ *isomorphic to* $\mathfrak{L}_0$ *and* $\mathfrak{C}(\mathfrak{A})$ *isomorphic to* $\mathfrak{L}_1$.

4. Necessary conditions for $\langle \mathfrak{S}, \mathfrak{L} \rangle$ to be representable as $\langle \mathfrak{E}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A}) \rangle$. Recall that if $\mathfrak{S}$ is a semigroup, $\mathfrak{L}(\mathfrak{S})$ is the algebra of left multiplications of $\mathfrak{S}$. $\mathfrak{A} = \langle A; F \rangle$ is some universal algebra. The basic thing established in this section is a relationship between $\mathfrak{C}(\mathfrak{L}(\mathfrak{E}(\mathfrak{A})))$ and $\mathfrak{E}(\mathfrak{A})$. If $\varphi$ is an endomorphism, then set $x \equiv y(\varepsilon_\varphi)$ iff $x\varphi = y\varphi$. $\varepsilon_\varphi$ is a congruence.

Let $\mathfrak{S} = \langle S; \cdot \rangle$ be a semigroup with identity, and let $x, s \in S$. The right multiplication map for $s$ is defined by $x\rho_s = xs$.

Thus, if $\varphi \in E(\mathfrak{A})$, then we have the congruence $\varepsilon_\varphi$ on $\mathfrak{A}$ and the mapping $\rho_\varphi$ on $E(\mathfrak{A})$. So we have the equivalence relation $\varepsilon_{\rho_\varphi}$ on $E(\mathfrak{A})$. Observe that since $\rho_\varphi$ is an endomorphism of $\mathfrak{L}(\mathfrak{E}(\mathfrak{A}))$, $\varepsilon_{\rho_\varphi}$ is a congruence of $\mathfrak{L}(\mathfrak{E}(\mathfrak{A}))$.

The proof of the next lemma involves only routine calculations.

LEMMA 4. *If* $\varepsilon_\psi = \bigcap (\varepsilon_{\varphi_i} \mid i \in I)$, *then* $\varepsilon_{\rho_\psi} = \bigcap (\varepsilon_{\rho_{\varphi_i}} \mid i \in I)$.

COROLLARY. $\varepsilon_\varphi \to \varepsilon_{\rho_\varphi}$ *is a mapping, and this mapping preserves arbitrary existing meets. In particular, it is order preserving.*

This mapping need not be $1-1$.

LEMMA 5. *If* $\varepsilon_{\rho_\varphi} = \iota$ ($\varphi$ *is a right zero*), *then* $\varepsilon_\psi \subseteq \varepsilon_\varphi$ *for every endomorphism* $\psi$.

*Proof*. Trivial.

$\varepsilon_{\rho_\varphi}$ can be $\iota$ and $\varepsilon_\varphi$ need not be $\iota$. $\varepsilon_{\rho_\varphi} = \iota$ means $\varphi$ is a right zero in $\mathfrak{E}(\mathfrak{A})$, but $\varphi$ need not be a constant map. But $\varphi$ is a constant map iff $\varepsilon_\varphi = \iota$. On the other hand, if $\varepsilon_\varphi = \iota$, then $\varepsilon_{\rho_\varphi} = \iota$ (i.e., if $\varphi$ is a constant map, then $\varphi$ is a right zero). Also, there is a $\varphi$ with $\varepsilon_\varphi = \omega$ and $\varepsilon_{\rho_\varphi} = \omega$ (the identity map).

To summarize we state the following theorem.

THEOREM 3. *Suppose* $\mathfrak{S} = \langle S; \cdot \rangle$ *is a semigroup with identity and* $\mathfrak{L} = \langle L; \vee, \wedge \rangle$ *is an algebraic lattice. Set* $\mathscr{K} = \{\varepsilon_{\rho_s} \mid s \in S\}$. *If*

*$\langle\mathfrak{S}, \mathfrak{L}\rangle$ is representable, then there is a subset $H$ of $L$ and there are two mappings $\alpha$ from $S$ onto $H$ and $\beta$ from $H$ onto $\mathscr{K}$ such that the following hold:*

(i) $(s\alpha)\beta = \varepsilon_{\rho_s}$ *for all* $s \in S$;

(ii) $\beta$ *preserves arbitrary existing meets;*

(iii) *if* $\varepsilon_{\rho_s} = \iota$, *then* $s\alpha$ *is the maximum element of* $H$ *and* $|\iota\beta^{-1}| = 1$;

(iv) $0 \in H$ *(and* $0\beta = \omega$*)*;

(v) *if* $1 \in H$, *then* $\iota \in \mathscr{K}$.

COROLLARY. *If $\langle\mathfrak{S}, \mathfrak{C}_n\rangle$ is representable and $\mathfrak{C}_n$ is the $n$-element chain, then $\mathscr{K} \cup \{\iota\}$ is a chain of length $\leqq n$.*

**5. More on the class of representable pairs.** Throughout this section, $\mathfrak{S} = \langle S; \cdot\rangle$ will be a semigroup with identity and $\mathfrak{L}$ will be an algebraic lattice. The ordinal sum of the lattices will be denoted by $+$. $\mathfrak{C}_n$ is the $n$ element chain. $\mathfrak{A} = \langle A; F\rangle$ is an algebra.

In the preceding section, a necessary condition for $\langle\mathfrak{S}, \mathfrak{L}\rangle$ to be representable as $\langle\mathfrak{E}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A})\rangle$ was given. Roughly the condition states that $\mathfrak{S}$ gives a lower bound on the cardinality of $L$, namely, $|\mathscr{K}|$, and an upper bound on the meet struture of part of $\mathfrak{L}$. This suggests that one could take a representable pair and expand the lattice and expect the result to be a representable pair. A few such expansions are given here.

Sort of a multiplication formula for members of the class of all representable pairs is given.

One could question whether or not there exist a semigroup with identity and an algebraic lattice which are in some vague sense completely "incompatible." Theorem 4 gives a negative answer.

First we will state the theorems, and then we will give sketches of their proofs.

THEOREM 4. *If $\mathfrak{S}$ is any semigroup with identity and $\mathfrak{L}$ is any algebraic lattice, then there is an algebra $\mathfrak{A}$ with $\mathfrak{S}$ isomorphic to $\mathfrak{E}(\mathfrak{A})$ and $\mathfrak{L}$ isomorphic to a sublattice of $\mathfrak{C}(\mathfrak{A})$.*

This follows from Theorem 7.

THEOREM 5. *If $\langle\mathfrak{S}; \mathfrak{L}\rangle$ is representable, then $\langle\mathfrak{S}; \mathfrak{L} + \mathfrak{C}_1\rangle$ is representable.*

COROLLARY 1. *If $\langle\mathfrak{S}; \mathfrak{C}_k\rangle$ is representable, then $\langle\mathfrak{S}; \mathfrak{C}_n\rangle$ is representable for any $n \geqq k$.*

COROLLARY 2. *If every member of $\mathfrak{S}$ is right cancellative or is* *a right zero and* $n \geqq 2$, *then* $\langle \mathfrak{S}, \mathfrak{C}_n \rangle$ *is representable.*

See [3], or see the remarks after Lemma 1.

THEOREM 6. *If* $\langle \mathfrak{S}_0, \mathfrak{L}_0 \rangle$ *is representable and* $\mathfrak{L}_1$ *is any algebraic lattice, then both* $\langle \mathfrak{S}_0, \mathfrak{L}_0 + \mathfrak{L}_1 + \mathfrak{C}_1 \rangle$ *and* $\langle \mathfrak{S}_0, \mathfrak{L}_1 + \mathfrak{L}_0 + \mathfrak{C}_1 \rangle$ *are representable.*

THEOREM 7. *If* $\langle \mathfrak{S}_0, \mathfrak{L}_0 \rangle$ *is representable and* $\mathfrak{L}_1$ *is any algebraic lattice, then* $\langle \mathfrak{S}_0, (\mathfrak{L}_0 \times \mathfrak{L}_1) + \mathfrak{C}_1 \rangle$ *is representable.*

This is a special case of Theorem 8.

THEOREM 8. *If* $\langle \mathfrak{S}_0, \mathfrak{L}_0 \rangle$ *and* $\langle \mathfrak{S}_1, \mathfrak{L}_1 \rangle$ *are representable, then* $\langle \mathfrak{S}_0 \times \mathfrak{S}_1, (\mathfrak{L}_0 \times \mathfrak{L}_1) + \mathfrak{S}_1 \rangle$ *is representable.*

Note that each of the "$+ \mathfrak{C}_1$"'s gives us a nonzero element in the resulting lattice that has property (*). (See §1.)

In Theorem 6 one can easily do without the "$+ \mathfrak{C}_1$" in the first pair (i.e., one can show $\langle \mathfrak{S}_0, \mathfrak{L}_0 + \mathfrak{L}_1 \rangle$ is representable) in case $\mathfrak{L}_1$ already had a non-zero element satisfying property (*). A similar comment can be made for the other pair in Theorem 6 in case $\mathfrak{L}_0$ already had a non-zero element satisfying (*). To do the same for Theorem 7 or 8 would seem to require that both $\mathfrak{L}_0$ and $\mathfrak{L}_1$ have such an element.

*Proof of Theorem* 5. Let $\mathfrak{A}$ represent $\langle \mathfrak{S}; \mathfrak{L} \rangle$. Let $U = \{u, v\}$ be a two element set disjoint from $A$. Set $A' = A \cup U$. Extend each $f \in F$ to $A'$ by setting $f(x_0, \cdots, x_{n-1}) = u$ if there is an $x_i \in U$. Let $u, v$ each be the value of a nullary operation. Define a unary operation $p$ and a binary operation $g$ as follows:

$$p(x) = \begin{cases} x, & \text{if } x \in A\,; \\ v, & \text{if } x = u\,; \\ u, & \text{if } x = v\,; \end{cases}$$

$$g(x, y) = \begin{cases} x, & \text{if } x, y \in A \text{ or if } y = u\,; \\ y, & \text{if } x = u\,; \\ v, & \text{if } x \text{ or } y = v\,. \end{cases}$$

Let $\mathfrak{A}' = \langle A'; F \cup \{p, g, u, v\} \rangle$. For each $\varphi \in E(\mathfrak{A})$ define $\bar{\varphi}$ on $A'$ by $x\bar{\varphi} = x\varphi$ if $x \in A$ and $x\bar{\varphi} = x$ if $x \in U$. For each $\Theta \in \mathscr{C}(\mathfrak{A})$ set $\bar{\Theta} = \Theta \cup \omega_U$.

$\varphi \to \bar{\varphi}$ is an isomorphism from $\mathfrak{E}(\mathfrak{A})$ onto $\mathfrak{E}(\mathfrak{A}')$. $\Theta \to \bar{\Theta}$ is an embedding of $\mathfrak{C}(\mathfrak{A})$ into $\mathfrak{C}(\mathfrak{A}')$. $\mathscr{C}(\mathfrak{A}') = \{\bar{\Theta} \mid \Theta \in \mathscr{C}(\mathfrak{A})\} \cup \{\iota_{A'}\}$. The details are almost identical to the details in [3].

*Proof of Theorem* 8. Let $\mathfrak{A}_0 = \langle A_0; F_0 \rangle$ and $\mathfrak{A}_1 = \langle A_1; F_1 \rangle$ be algebras with $\mathfrak{E}(\mathfrak{A}_i)$ isomorphic to $\mathfrak{S}_i$ and $\mathfrak{C}(\mathfrak{A}_i)$ isomorphic to $\mathfrak{L}_i$. Assume $A_0 \cap A_1 = \varnothing$. Let $A_2 = A_0 \cup A_1 \cup \{u, v\}$ where $u \neq v$ and $u$, $v \notin A_0 \cup A_1$. Let $x_0, \cdots, x_{n-1} \in A_2$, and let $f \in F_i$. Extend $f$ to $A_2$ by setting $f(x_0, \cdots, x_{n-1}) = u$ if there exists $x_j \notin A_i$. Let $a_i \in A_i$ and define two unary operations $g_0$ and $g_1$ by $g_0(a_0) = g_0(u) = u$ and $g_0(a_1) = g_0(v) = v$ and $g_1(a_i) = a_i$ and $g_1(u) = v$ and $g_1(v) = u$. Define a binary $g_2$ on $A_2$ by setting $g_2(x, v) = g_2(v, x) = x$ and $g_2(x, y) = u$ otherwise. Take each of $u$ and $v$ as the value of a nullary operation. Let

$$\mathfrak{A}_2 = \langle A_2; F_0 \dot{\cup} F_1 \dot{\cup} \{g_0, g_1, g_2, u, v\} \rangle = \langle A_2; F_2 \rangle .$$

For each $\psi = \langle \varphi_0, \varphi_1 \rangle \in E(\mathfrak{A}_0) \times E(\mathfrak{A}_1)$ define a mapping $\bar{\psi}$ on $A_2$ by $x\bar{\psi} = x\varphi_i$ if $x \in A_i$ and $x\bar{\psi} = x$ if $x = u$ or $x = v$. For each $\Phi = \langle \Theta_0, \Theta_1 \rangle \in \mathscr{C}(\mathfrak{A}_0) \times \mathscr{C}(\mathfrak{A}_1)$ set $\bar{\Phi} = \Theta_0 \cup \Theta_1 \cup \omega_U$. To complete the proof, one would show that $\mathscr{C}(\mathfrak{A}_2) = \{\bar{\Phi} \mid \bar{\Phi} \in \mathscr{C}(\mathfrak{A}_0) \times \mathscr{C}(\mathfrak{A}_1)\} \cup \{\iota_{A_2}\}$, that $\Phi \rightarrow \bar{\Phi}$ is an embedding of $\mathfrak{C}(\mathfrak{A}_0) \times \mathfrak{C}(\mathfrak{A}_1)$ into $\mathfrak{C}(\mathfrak{A}_2)$, and that $\psi \rightarrow \bar{\psi}$ is an isomorphism of $\mathfrak{E}(\mathfrak{A}_0) \times \mathfrak{E}(\mathfrak{A}_1)$ onto $\mathfrak{E}(\mathfrak{A}_2)$. A few of the details follow.

Let $\sigma$ be an endomorphism of $\mathfrak{A}$. Note that $x\sigma = x$ for $x = u$ or $x = v$ since $u$ and $v$ are the vaules of nullary operations. Let $a_i \in A_i$. Now $g_0(a_0\sigma) = g_0(a_0)\sigma = u\sigma = u$. Thus, $a_0\sigma \in A_0$ or $a_0\sigma = u$. Suppose $a_0\sigma = u$. Then $u = a_0\sigma = g_1(a_0)\sigma = g_1(a_0\sigma) = g_1(u) = v$. Since $u \neq v$, it follows that $a_0\sigma \in A_0$. Similarly, $a_1\sigma \in A_1$. Thus, $\sigma = \overline{\langle \sigma|_{A_0}, \sigma_{A_1} \rangle}$.

Suppose $a_i \in A_i$ and $a_0 \equiv a_1(\Psi)$ and suppose $\Psi \in \mathscr{C}(\mathfrak{A}_2)$. Then $u \equiv v(\Psi)$ since $u = g_0(a_0)$, $v = g_0(a_1)$ and $g_0(a_0) \equiv g_0(a_1)(\Psi)$. For $x \in A_0 \cup A_1$ it happens that $x \equiv u(\psi)$ iff $x \equiv v$ because $g_1(x) = x$, $g_1(u) = v$ and $g_1(v) = u$. So if $x \in A_0 \cup A_1$ and $y \in \{u, v\}$ and $x \equiv y(\Psi)$, then $u \equiv v(\Psi)$. If $u \equiv v(\Psi)$, then $\Psi = \iota_{A_2}$ because for any $x \in A_2, x \equiv u(\Psi)$. (This is because $u = g_2(u, x)$ and $x = g_2(v, x)$.) $\Psi|_{A_i} \in \mathscr{C}(\mathfrak{A}_i)$. Thus, if $\Psi \neq \iota_{A_2}$, then $\Psi = \bar{\Phi}$ for some $\bar{\Phi} \in \mathscr{C}(\mathfrak{A}_0) \times \mathscr{C}(\mathfrak{A}_1)$, namely, for $\bar{\Phi} = \langle \Psi|_{A_0}, \Psi|_{A_1} \rangle$.

*Proof* (*of Theorem* 6). Let $\mathfrak{A}_i = \langle A_i; F_i \rangle$, for $i = 0, 1$, be algebras with $A_0 \cap A_1 = \varnothing$. We shall prove the theorem by showing that $\langle \mathfrak{E}(\mathfrak{A}_0), \mathfrak{C}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1) + \mathfrak{C}_1 \rangle$ and $\langle \mathfrak{E}(\mathfrak{A}_1), \mathfrak{C}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1) + \mathfrak{C}_1 \rangle$ are representable. First we consider the case with $\mathfrak{E}(\mathfrak{A}_0)$.

Let $u \neq v$ and $u, v \notin A_0 \cup A_1$, and let $A_2 = A_0 \dot{\cup} A_1 \dot{\cup} \{u, v\}$. For $f \in F_i$ extend $f$ to $A_2$ as in Theorem 8. Define the unary operations $g_0, g_1$ as in the proof of Theorem 8. Define the binary operation $g_2$ by setting $g_2(a_0, v) = g_2(v, a_0) = a_0$ for $a_0 \in A_0$ and $g_2(x, y) = u$ otherwise. Let $a_0 \in A_0$ and $a_1, b_1 \in A_1$. Define the binary operation $g_3$ by $g_3(a_1, b_1) =$

$v$, if $a_1 \neq b_1$, and $g_3(x, y) = u$ otherwise. Define the binary operation $g_4$ by $g_4(a_1, y) = y$ and $g_4(x, y) = u$ otherwise. Take each member of $A_1 \cup \{u, v\}$ as the value of nullary operations. Set

$$\mathfrak{A}_2 = \langle A_2; F_0 \dot{\cup} F_1 \dot{\cup} \{g_0, \cdots, g_4\} \dot{\cup} \{u, v\} \dot{\cup} A_1 \rangle = \langle A_2; F_2 \rangle .$$

For each $\varphi \in E(\mathfrak{A}_0)$ let $\bar{\varphi}$ be defined by $x\bar{\varphi} = x\varphi$ if $x \in A_0$ and $x\bar{\varphi} = x$ if $x \notin A_0$. For each $\Theta \in \mathscr{C}(\mathfrak{U}_0) + \mathscr{C}(\mathfrak{A}_1)$ define $\bar{\Theta}$ by $\bar{\Theta} = \Theta \cup \omega_{A_1} \cup \omega_{\{u,v\}}$ if $\Theta \in \mathscr{C}(\mathfrak{A}_0)$ and $\bar{\Theta} = \Theta \cup \iota_{A_0 \cup \{u,v\}}$ if $\Theta \mathscr{C}(\mathfrak{A}_1)$.

To complete the proof for this pair, one would show that $\varphi \rightarrow \bar{\varphi}$ is an isomorphism of $\mathfrak{G}(\mathfrak{A}_0)$ onto $\mathfrak{G}(\mathfrak{A}_2)$, that $\Theta \rightarrow \bar{\Theta}$ is an embedding of $\mathfrak{C}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1)$ into $\mathfrak{C}(\mathfrak{A}_2)$, and that $\mathscr{C}(\mathfrak{A}_2) = \{\bar{\Theta} \mid \Theta \in \mathscr{C}(\mathfrak{A}_0) + \mathscr{C}(\mathfrak{A}_1)\} \cup \{\iota_{A_2}\}$. A few of the details follow.

As in the proof of Theorem 8, for $\sigma \in E(\mathfrak{A}_2)$, $A_0\sigma \subseteqq A_0$ Clearly $x\sigma = x$ for $x \in A_2 - A_0$ since every element is a nullary constant.

Let $\Theta \in \mathscr{C}(\mathfrak{A}_2)$. For $x \in A_2, x \equiv u$ iff $x \equiv v$ as in the proof of Theorem 8. Let $a_i, b_i \in A_i$. If $u \equiv v(\Theta)$, then $a_0 \equiv u$ because $g_2(u, a_0) = u$ and $g_2(v, a_0) = a_0$. If $a_1 \neq b_1$ and $a_1 \equiv b_1(\Theta)$, then $u \equiv v$ because $g_3(a_1, b_1) = u$ and $g_3(b_1, b_1) = v$. Let $x \in A_2 - A_1$ and let $z \in A_2$. If $a_1 \equiv x(\Theta)$, then $z \equiv u(\Theta)$ because $g_4(a_1, z) = z$ and $g_4(x, z) = u$.

We now turn to considering the case for $\langle \mathfrak{G}(\mathfrak{A}_1), \mathfrak{C}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1) + \mathfrak{C}_1 \rangle$. We may *now* assume without loss of generality that $\mathfrak{G}(\mathfrak{A}_0)$ is the one element group and that there are no nullary operations in $\mathfrak{A}_0$. That this assumption can be made is verified in [6] and [7].

Let $w, r, s \notin A_2$. Let $A_3 = A_2 \cup \{w, r, s\}$. For $f \in F_0$ or $F_1$, change the value of $f(x_0, \cdots, x_{n-1})$ to $w$ where in the above case it was $u$, i.e., in the case when there is an $x_i$ not an element of the appropriate $A_i$. Extend the $g_i$ in the following way: $g_0(r) = g_0(s) = v$; $g_0(w) = w$; $g_1(w) = w$; $g_1(r) = r$; $g_1(s) = s$; still keep $g_4(a_1, y) = y$, but let $g_4(x, y) = w$ otherwise; keep $g_3(a_1, b_1) = v$ for $a_1 \neq b_1$ and $g_3(x, y) = u$ otherwise except let $g_3(w, w) = w$; $g_2(w, q) = g_2(v, w) = g_2(w, w) = w$ and $g_2(x, y) = u$ for any other new pair. Define three new operations as follows. Let $x, y \in A_0$, $z \in A_0 \cup \{u, v, w\}$ and let $a_1 \in A_1$. Set $g_5(w, w) = w$, $g_5(w, x) = u$, $g_5(y, x) = v$, $g_5(z_1, z) = a_1$ and $g_5(z, a_1) = z$. Set $g_6(r) = s$, $g_6(s) = r$ and $g_6(x) = x$ otherwise. For $x, y \in A_3$ set $g_7(r, x) = g_7(x, r) = r$ and $g_7(x, s) = g_7(s, x) = x$, if $x \neq r$, and $g_7(x, y) = x$ otherwise. Take $w, r, s$ as values of nullary operations but don't take $A_1 \cup \{u, v\}$ as nullaries. Set

$$\mathfrak{A}_3 = \langle A_3; F_0 \dot{\cup} F_1 \dot{\cup} \{g_0, \cdots, g_7\} \dot{\cup} \{w, r, s\} \rangle .$$

For $\varphi \in E(\mathfrak{A}_1)$ define $\bar{\varphi}$ on $A_3$ as follows; $x\bar{\varphi} = x\varphi$ if $x \in A_1$, $x\bar{\varphi} = x$ if $x = w, r$, or $s$, and for $x \in A_0 \cup \{u, v, w\}$ set $x\bar{\varphi} = x$ if $\varphi$ is $1 - 1$ and $x\bar{\varphi} = w$ if not. Let $\Theta \in \mathscr{C}(\mathfrak{A}_0) + \mathscr{C}(\mathfrak{A}_1)$. For $\Theta \in \mathscr{C}(\mathfrak{A}_0)$ set $\bar{\Theta} = \Theta \cup \omega_{A_1 \cup \{u,v,w,r,s\}}$, and for $\Theta \in \mathscr{C}(\mathfrak{A}_1)$ set $\bar{\Theta} = \Theta \cup \iota_{A_0 \cup \{u,v,w\}} \cup \omega_{\{r,s\}}$. The outline of the rest of the proof is clear. Some details follow, particularly concerning endomorphisms.

Let $\Theta \in \mathscr{C}(\mathfrak{A}_3)$. All the statements made about $\Theta$ in the previous case still hold with one change. Here if $a_1 \in A_1$ and $x \in A_3 - A_1$ and $x \equiv a_1$, then for all $z \in A_3$, $z \equiv w$ (instead of $u$). Some more should now be said. If $x \in A_0$ and $w \equiv x(\Theta)$, then $u \equiv v$ because $g_5(w, x) = u$ and $g_5(x, x) = v$. If $u \equiv v(\Theta)$, then one gets $w \equiv u$ using $g_2$. Similar to the case with $u$ and $v$, for any $x \in A_3$, $x \equiv r$ iff $x \equiv r \equiv s$(use $g_6$). Using $g_7$ we have that if $r \equiv s(\Theta)$ and $z \in A_3$, then $z \equiv r(\Theta)$.

Note that there can be no constant endomorphisms because there are three nullary operations with different values. Let $\sigma \in E(\mathfrak{A}_3)$. Let $x \in A_0 \cup A_1 \cup \{w\}$, and let $y \in A_0 \cup A_1 \cup \{u, v, w\}$. Using $g_1$, $x\sigma \neq u$ or $v$, and using $g_6$, $y\sigma \notin \{r, s\}$. Thus, $(A_0 \cup A_1 \cup \{w\})\sigma \subseteqq A_0 \cup A_1 \cup \{w\}$. Let $a_i \in A_i$. Now $w\sigma = w$. If $a_1\sigma = w$, then we would have $\sigma$ is a constant endomorphism because the congruence relation induced by $\sigma$ would be $\iota_{A_3}$. So $a_1\sigma \in A_0 \cup A_1$. Now, as before, $A_1\sigma \subseteqq A_1$. Similarly, one gets $(A_0 \cup \{w\})\sigma \subseteqq A_0 \cup \{w\}$. Using the congruence structure and the fact that $w\sigma = w$, either $A_0\sigma \subseteqq A_0$ or $(A_0 \cup \{u, v, w\})\sigma = \{w\}$. Clearly, if $A_0\sigma \subseteqq A_0$, then $a_0\sigma = a_0$. When $A_0\sigma \subseteqq A_0$, using the congruence structure and $g_2$, one gets $u\sigma = u$ and $v\sigma = v$. Finally, the congruence structure requires that if $\sigma$ is not $1-1$ on $A_1$, then $\sigma$ must be constant on $A_0 \cup \{u, v, w\}$. And if $\sigma$ is constant on $A_0 \cup \{u, v, w\}$, then $\sigma$ would have the value $w$ there.

6. **Concerning $\langle \mathfrak{S}, \mathfrak{C}_3 \rangle$.** From §4 we know that a necessary condition for the representability of $\langle \mathfrak{S}, \mathfrak{C}_3 \rangle$ is that $|\{\varepsilon_{\rho_s} | s \in S\} \cup \{\iota\}| \leqq 3$. A stronger condition is proved to be sufficient. The representability of $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ has been characterized [3] (or see the remarks after Lemma 1), and $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ is representable iff $|\{\varepsilon_{\rho_s} | s \in S\} \cup \{\iota\}| \leqq 2$.

The method for proving the next lemma is very similar to that in Lemma 1. Recall the definition of $\varepsilon_\varphi$.

LEMMA 6. *Let $\mathfrak{A} = \langle A; F \rangle =$ be an algebra, and let $\Theta \neq \omega, \Theta \in \mathscr{C}(\mathfrak{A})$. There is an algebra $\mathfrak{A}' = \langle A; F' \rangle$ so that:*

(i) $\mathscr{S}(\mathfrak{A}) = \mathscr{S}(\mathfrak{A}')$;

(ii) $\mathscr{C}(\mathfrak{A}') = \{\omega, \Theta, \iota\}$;

(iii) *$\varphi \in E(\mathfrak{A}')$ iff $\varphi \in E(\mathfrak{A})$, $\varepsilon_\varphi = \omega, \Theta$, or $\iota$ and following conditions are satisfied:*

(a) *if $\varepsilon_\varphi = \Theta$, then $\varepsilon_{\varphi^2} = \Theta$ or $\iota$;*

(b) *if $\varepsilon_\varphi = \omega$ and $\psi$ is any map with $\varepsilon_\psi = \Theta$, then $\varepsilon_{\varphi \circ \psi} = \Theta$.*

REMARK. Obviously, one could not improve upon condition (a), but perhaps a proof could be given with (b) changed to read" $\cdots$, then $\varepsilon_{\varphi \circ \psi} = \omega, \Theta$ or $\iota$." Notice that all automorphisms are kept.

*Proof.* Add one 4-ary operation $g$ defined as follows:

$$g(x, y, u, v) = \begin{cases} u, & \text{if } x \not\equiv y(\Theta), u \not\equiv v(\Theta) \text{ or} \\ & \text{if } x \equiv y(\Theta), u \equiv v(\Theta) \text{ and} \\ & \quad x \neq y, u \neq v \\ v, & \text{otherwise} . \end{cases}$$

Set $\mathfrak{A}' = \langle A; F \cup \{g\}\rangle$. Clearly, (i) holds.

Proving that $\Theta \in \mathscr{C}(\mathfrak{A}')$ involves only routine calculation. So let $\Phi \in \mathscr{C}(\mathfrak{A}')$ with $\omega \neq \Phi$. So there exist $x, y$ with $x \neq y$ and $x \equiv y(\Phi)$. Suppose $x \not\equiv y(\Theta)$. We will show that $\Phi = \iota$. Let $u \neq v$. First assume $u \not\equiv v(\Theta)$. Then $u = g(x, y, u, v) \equiv g(y, y, u, v) = v(\Phi)$. Now assume $u \equiv v(\Theta)$. Since $x \not\equiv y(\Theta)$, there is a $z \in \{x, y\}$ with $z \not\equiv u(\Theta)$ and $z \not\equiv v(\Theta)$. From above $u \equiv z(\Phi)$ and $v \equiv z(\Phi)$. Thus, $u \equiv v(\Phi)$. So $\Phi = \iota$. Now suppose for every $u, v$, with $u \equiv v(\Phi)$ that $u \equiv v(\Theta)$. Thus, $\Phi \subseteqq \Theta$. (We are still assuming $\Phi \neq \omega$, that $x \neq y$, and that $x \equiv y(\Phi)$.) We will show that in this case $\Phi = \Theta$. Let $u \equiv v(\Theta)$ with $u \neq v$. Then $u = g(x, y, u, v) \equiv g(y, y, u, v) = v(\Phi)$. So $\Theta \subseteqq \Phi$ and $\Theta = \Phi$. Thus $\mathscr{C}(\mathfrak{A}') = \{\omega, \Theta, \iota\}$ and (ii) holds.

Obviously, if $\varphi \in E(\mathfrak{A}')$, then $\varphi \in E(\mathfrak{A})$ and $\varepsilon_\varphi \in \{\varphi, \Theta, \iota\}$. Suppose $\varepsilon_\varphi = \Theta$. Since $\varphi^2 \in E(\mathfrak{A}')$, then $\varepsilon_{\varphi^2} = \Theta$ or $\iota$.

It is a routine calculation to show that if $\varepsilon_\varphi = \iota$ and $\varphi \in E(\mathfrak{A})$, then $\varphi \in E(\mathfrak{A}')$.

Let $\varphi \in E(\mathfrak{A})$ with $\varepsilon_\varphi = \Theta$ and with $\varepsilon_{\varphi_2} = \Theta$ or $\iota$. Consider $g(x, y, u, v)$. There are two possibly troublesome cases. One is if $g(x, y, u, v) = u$ and $g(x\varphi, y\varphi, u\varphi, v\varphi) = v\varphi$. The other is if $g(x, y, u, v) = v$ and $g(x\varphi, y\varphi, u\varphi, v\varphi) = u\varphi$. The latter is the easiest to dispense with. If $g(x\varphi, y\varphi, u\varphi, v\varphi) = u\varphi$ and $u\varphi \neq v\varphi$, then $x\varphi \neq y\varphi$. Thus, $x \not\equiv y(\Theta)$ and $u \not\equiv v(\Theta)$. So $g(x, y, u, v) = u$ and $g(x, y, u, v)\varphi = u\varphi$. So now assume $g(x\varphi, y\varphi, u\varphi, v\varphi) = v\varphi$ and $g(x, y, u, v) = u$. Thus, either $x \not\equiv y(\Theta)$ and $u \not\equiv v(\Theta)$ or $x \equiv y(\Theta)$ and $u \equiv v(\Theta)$. Suppose $x \not\equiv y$ and $u \not\equiv v$. Then $x\varphi \neq y\varphi$ and $u\varphi \neq v\varphi$. Now $x\varphi \equiv y\varphi(\Theta)$ iff $u\varphi \equiv v\varphi(\Theta)$. Indeed, suppose that $x\varphi \equiv y\varphi(\Theta)$. Then $(x\varphi)\varphi = (y\varphi)\varphi$ and $\varepsilon_{\varphi^2} \neq \Theta$. So by assumption $\varepsilon_{\varphi^2} = \iota$. Thus, $(u\varphi)\varphi = (v\varphi)\varphi$, and $u\varphi \equiv v\varphi(\Theta)$. Similarly if $u\varphi \equiv v\varphi$, then $x\varphi \equiv y\varphi$. Thus, either $x\varphi \not\equiv y\varphi(\Theta)$, $u\varphi \not\equiv v\varphi(\Theta)$ or $x\varphi \equiv y\varphi(\Theta), u\varphi \equiv v\varphi(\Theta), x\varphi \neq y\varphi, u\varphi \neq v\varphi$. In any case, $g(x\varphi, y\varphi, u\varphi, v\varphi) = u\varphi \not\equiv v\varphi$. So $x \equiv y(\Theta)$ and $u \equiv v(\Theta)$. In this case $u\varphi = v\varphi$. Therefore, $g(x, y, u, v)\varphi = u\varphi = v\varphi = g(x\varphi, y\varphi, u\varphi, v\varphi)$. Thus, $\varphi \in E(\mathfrak{A}')$.

Suppose $\varphi$ is $1 - 1$ and $\varphi \in E(\mathfrak{A}')$. Let $\psi$ be any map with $\varepsilon_\psi = \Theta$. Consider $\varepsilon_{\varphi \circ \psi}$. Suppose $\varepsilon_{\varphi \circ \psi} = \omega$. Then, $\varepsilon_\psi \neq \iota$. So there exist $x, y$ such that $x \not\equiv y(\Theta)$. Since $\Theta \neq \omega$, there exist $u, v$ with $u \neq v$ and $u \equiv v(\Theta)$. Since $u \neq v$, it follows that $(u\varphi)\psi \neq (v\varphi)\psi$. Thus, $u\varphi \not\equiv v\varphi(\Theta)$. Similarly, since $x \neq y, x\varphi \not\equiv y\varphi(\Theta)$. This implies that

$g(x, y, u, v)\varphi = v\varphi \neq u\varphi = g(x\varphi, y\varphi, u\varphi, v\varphi)$. But since $\varphi$ is an endomorphism, we have that $\varepsilon_{\varphi\circ\psi} \neq \omega$. Suppose $\varepsilon_{\varphi\circ\psi} = \iota$. By a similar argument we would get that $\varepsilon_{\varphi\circ\psi} \neq \iota$ unless $\Theta = \iota$. So $\varepsilon_{\varphi\circ\psi} = \Theta$.

Let $\varphi \in E(\mathfrak{A})$ with $\varepsilon_\varphi = \omega$. Let $\psi$ be any map with $\varepsilon_\psi = \Theta$.

Suppose $\varepsilon_{\varphi\circ\psi} = \Theta$. Routine computation shows that $\varphi \in E(\mathfrak{A}')$. The crucial point in these computations is that the assumption $\varepsilon_{\varphi\circ\psi} = \Theta$ implies $\varphi$ presves both $\Theta$ and not-$\Theta$. Therefore (iii) holds.

Recall that if $\mathfrak{S} = \langle S; \cdot\rangle$ is a semigroup with identity, then $\mathscr{K} = \{\varepsilon_{\rho_s} | \in S\}$. $\mathfrak{L}(\mathfrak{S})$ is the algebra of left multiplications. $E(\mathfrak{L}(\mathfrak{S})) = \{\rho_s | s \in S\}$.

THEOREM 9. *Let $\mathfrak{S} = \langle S; \cdot\rangle$ be a semigroup with identity.*

(A) *If $\langle\mathfrak{S}; \mathfrak{C}_3\rangle$ is representable, then $|\mathscr{K} \cup \{\iota\}| \leqq 3$.*

(B) *If $|\mathscr{K} \cup \{\iota\}| \leqq 3$ and if for right cancellative $r$ and for $m$ that is neither right cancellative nor a right zero $r\cdot m$ is also neither right cancellative nor a right zero, then $\langle\mathfrak{S}; \mathfrak{C}_3\rangle$ is representable.*

REMARK. If $|\mathscr{K} \cup \{\iota\}| = 2$, the rest of (B) holds trivially. So the sufficient condition includes all those representable pairs derived from Corollary 2 to Theorem 5.

*Proof.* For part (A) see the corollary to Theorem 3.

Suppose the hypotheses of (B) hold. If $|\mathscr{K} \cup \{\iota\}| = 2$, then $\langle\mathfrak{S}; \mathfrak{C}_2\rangle$ is representable. By Theorem 5, $\langle\mathfrak{S}; \mathfrak{C}_3\rangle$ is representable. Suppose then that $\mathscr{K} \cup \{\iota\} = \{\omega, \Theta, \iota\}$ and that $\omega \neq \Theta \neq \iota$. Suppose $\varepsilon_{\rho_m} = \Theta$. Since

$$\varepsilon_{(\rho_m)^2} = \varepsilon_{\rho_{m^2}} ,$$

it follows that

$$\varepsilon_{(\rho_m)^2} = \Theta \text{ or } \iota .$$

Let $\varepsilon_{\rho_r} = \omega$ and $\varepsilon_{\rho_m} = \Theta$. Since $r\cdot m$ is neither right cancellative nor a right zero, it follows that

$$\varepsilon_{\rho_r\circ\rho_m} = \varepsilon_{\rho_{rm}} = \Theta .$$

Now apply Lemma 6 to $\Theta$ and $\mathfrak{L}(\mathfrak{S})$.

7. $\langle\mathfrak{S}, \mathfrak{C}_2\rangle$ **for unary algebras.** In [3] G. Grätzer characterized the endomorphism semigroups of simple algebras. He also showed that not all such semigroups were isomorphic to endomorphism semigroups of simple unary algebras. Since previous representations involving congruence lattices and endomorphism semigroups had needed

only unary algebras, he raised the question, "What semigroups are isomorphic to the endomorphism semigroups of simple unary algebras?" The answer to that question is that there are hardly any such semigroups.

Every endomorphism $\varphi$ induces a congruence relation which we have denoted by $\varepsilon_\varphi$. The difference with unary algebras is that every endomorphism also induces another congruence. Throughout $\mathfrak{A} = \langle A; F\rangle$ will denote a unary algebra. For $\varphi \in E(\mathfrak{A})$ and $x, y \in A$ set $x \equiv y(\Theta_\varphi)$ iff there exist natural numbers $i, j$ such that $x\varphi^i = y\varphi^j (x\varphi^0 = x)$. $\Theta_\varphi$ is the "extra" congruence. To prove that the substitution property holds for $\Theta_\varphi$, one needs that each operation of $\mathfrak{A}$ is unary or nullary.

Lemma 7. *If $\varphi$ is $1-1$ and $\Theta_\varphi = \omega$ or $\iota$, then $\varphi$ is onto or $A = \{a\varphi^n \mid n = 0, 1\cdots\}$ for some $a \in A$.*

*Proof.* $x \equiv x\varphi^n(\Theta_\varphi)$ for any natural number $n$ (by using the numbers $n, 0$). In particular $x \equiv x\varphi(\Theta_\varphi)$. Thus, if $\Theta_\varphi = \omega$, then $x = x\varphi$, and therefore, $\varphi$ is the identity map. Therefore, we can assume $\Theta_\varphi = \iota$, and this implies $x \equiv y(\Theta_\varphi)$ for any $x, y \in A$. Thus, for some $i, j$, $x\varphi^i = y\varphi^j$. If $i \leqq j$, then since $\varphi$ is $1-1$, we have that $x = y\varphi^{i-j}$. If $j \leqq i$, then $y = x\varphi^{i-j}$. Thus, $x \in \{y\varphi^n \mid n = 0, 1, \cdots\}$ or $y \in \{x\varphi^n \mid n = 0, 1, \cdots\}$. Suppose $\varphi$ is not onto. Then there is an $a$ such that $a \neq x\varphi$ for all $x \in A (x \neq a)$. Thus, $a \notin \{x\varphi \mid n = 0, 1, \cdots\}$ for any $x \in A (x \neq a)$. Now since $x \in \{a\varphi^n \mid n = 0, 1, \cdots\}$ or $a \in \{x\varphi^n \mid n = 0, 1, \cdots\}$ for all $x \in A$, we have $x \in \{a\varphi^n \mid n = 0, 1, \cdots\}$ for all $x \in A$.

Lemma 8. *If $\Theta_\varphi = \omega$ or $\iota$ and $\Theta$ is $1-1$ but not onto, then $\mathfrak{A}$ is not simple.*

*Proof.* By Lemma 7, $A = \{a\varphi^n \mid n = 0, 1\cdots\}$ for some $a \in A$. For $n > 1$, $a\varphi^n = (a\varphi^{n-1})\varphi$. Since $\varphi$ is not onto $a \neq x\varphi$ for any $x \in A$. Suppose $a\varphi^i = a\varphi^j$ and $i \neq j$. We may assume $i < j$. Since $\varphi$ is $1-1$, $a = a\varphi^{j-i}$. Since $j - i \geqq 1$, $a = (a\varphi^{j-i-1})\varphi$. Thus, $a\varphi^i \neq a\varphi^j$ if $i \neq j$. Now set $E = \{a\varphi^n \mid n = 0, 2, 4, \cdots\}$ and $D = \{a\varphi^n \mid n = 1, 3, 5, \cdots\}$. By the above, $D \cap E = \varnothing$. Clearly, $D \cup E = A$. Let $\Phi$ be the equivalence relation whose only two classes are $D$ and $E$. $\Phi$ is a congruence. Since $\omega \neq \Phi \neq \iota$, $\mathfrak{A}$ is not simple.

For a *simple* algebra $\mathfrak{A}$ any right zero of $\mathfrak{E}(\mathfrak{A})$ is necessarily a constant mapping (unless $\mathfrak{E}(\mathfrak{A})$ is the one element group). See § 4.

Corollary. *If $\mathfrak{A}$ is a simple unary algebra, then $E(\mathfrak{A})$ consists of automorphisms and constant mappings.*

G. Grätzer [3] characterized the automorphism group of a simple unary algebra as a cyclic group of order $p$ where $p = 1$ or $p$ is a prime number. He also showed that if $p \neq 1$, then $A = \{a\alpha \mid \alpha \in G(\mathfrak{A})\}$ for any $a \in A$.

LEMMA 9. *If* $\mathfrak{A}$ *is simple,* $|G(\mathfrak{A})| \neq 1$, *and there exists a right zero in* $\mathfrak{E}(\mathfrak{A})$, *then* $|A| = 2$ *and* $\mathfrak{E}(\mathfrak{A}) = \langle A^A; \circ \rangle$.

*Proof.* Let $\{a\} = A\varphi$. Let $f$ be an operation. Then $a = (f(a))\varphi = f(a\varphi) = f(a)$. If $x \in A$, then $x = a\alpha$ for some $\alpha \in G(\mathfrak{A})$. Thus, $f(x) = f(a\alpha) = f(a)\alpha = a\alpha = x$. Therefore, $E(\mathfrak{A}) = A^A$ and all equivalence relations are congruence relations. $|G(\mathfrak{A})| \geqq 2$ implies $|A| \geqq 2$. If $|A| > 2$, then there are more than two equivalence relations on $A$. Thus $|A| = 2$.

LEMMA 10. *If* $\mathfrak{A}$ *is simple and* $|G(\mathfrak{A})| = 1$, *then* $|E(\mathfrak{A})| \leqq 2$.

*Proof.* Suppose there exist two constant endomorphisms $\varphi_0, \varphi_1$. Let $\{a_0\} = A\varphi_0$ and $\{a_1\} = A\varphi_1$. As in the proof of Lemma 3, $f(a_0) = a_0$ and $f(a_1) = a_1$ for any operation $f$. If $|A|$ were two, then every operation would be the identity function and $|G(\mathfrak{A})| = 2$. Thus, $|A| > 2$. Set $x \equiv y(\Phi)$ iff $x = y$ or $x, y \in \{a_0, a_1\}$. Since every operation restricted to $\{a_0, a_1\}$ is the identity function, $\Phi$ is a congruence. Since $|A| > 2, \Phi \neq \iota$. Since $\Phi \neq \omega$, $\mathfrak{A}$ is not simple.

THEOREM 10. *Let* $\mathfrak{S} = \langle S; \cdot \rangle$ *be a semigroup with identity.* $\mathfrak{S}$ *is isomorphic to the endomorphism semigroup of a simple unary algebra (i.e.,* $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ *is representable by a unary algebra) if and only if* $\mathfrak{S}$ *is one of the semigroups listed below*:

(i) *the group of order* $p, p = 1$ *or* $p$ *is a prime*;

(ii) *the two element semi-lattice*;

(iii) *a four element semigroup isomorphic to* $\langle A^A; \circ \rangle$ *where* $|A| = 2$. *Moreover, if* $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ *is representable by a unary algebra and* $|S| \neq 1$, *then* $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ *is representable using a unary algebra with one operation.*

*Proof.* It follows from the corollary to Lemma 8 and Lemmas 9 and 10 that the endomorphism semigroup of a simple unary algebra is one of those listed in (i) – (iii).

To complete the proof, we will represent $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ for each $\mathfrak{S}$ listed in (i) – (iii).

In case $\mathfrak{S}$ is the one element group, let $A$ be a two element set. Set $\mathfrak{A} = \langle A; A^A \rangle$. Clearly, $\mathfrak{A}$ has the required properties.

In case $\mathfrak{S}$ is $\langle A^A; \circ \rangle$ where $|A| = 2$, Let $\mathfrak{A} = \langle A; F \rangle$ where $f$ is the identity map. Obviously, $\mathfrak{A}$ has the required properties.

In case $\mathfrak{S}$ is the two element semi-lattice, let $A = \{a, b\}$ with $a \neq b$. Set $f(a) = f(b) = b$, and set $\mathfrak{A} = \langle A; f\rangle$. Since $|A| = 2, \mathfrak{A}$ is simple. The endomorphisms are exactly the identity map $\sigma$ and $\psi$ where $\psi = f$. Since $\sigma\circ\psi = \psi\circ\sigma = \psi = \psi\circ\psi$, the endomorphism semigroup is the two element semi-lattice.

In case $\mathfrak{S}$ is the group of order $p$ where $p$ is a prime, set $A = \{0, \cdots, p-1\}$. Let $f(x) = x + 1 \pmod{p}$, and set $\mathfrak{A} = \langle A; f\rangle$. Since $p$ is a prime it is easy to check that $\mathfrak{A}$ is simple. For $x \in A$ define the mapping $\varphi_x$ by $y\varphi_x = y + x$. Clearly, $x \rightarrow \varphi_x$ is an isomorphism from the cyclic group of order $p$ onto $\mathfrak{E}(\mathfrak{A})$.

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# THE REDUCING IDEAL IS A RADICAL

T. W. PALMER

**For any *-algebra $\mathfrak{A}$ the reducing ideal $\mathfrak{A}_R$ of $\mathfrak{A}$ is the intersection of the kernels of all the *-representations of $\mathfrak{A}$. Although the reducing ideal has been called the *-radical, and obviously satisfies $(\mathfrak{A}/\mathfrak{A}_R)_R = \{0\}$, it has not previously been shown to satisfy another of the fundamental properties of an abstract radical except in the case of hermitian Banach *-algebras where it equals the Jacobson radical. In this paper we prove two extension theorems for *-representations. The more important one states that any essential *-representation of a *-ideal of a $U^*$-algebra (*a fortiori*, of a Banach *-algebra) has a unique extension to a *-representation of the whole algebra. These theorems show in particular that $(\mathfrak{A}_R)_R = \mathfrak{A}_R$ if $\mathfrak{A}$ is either a commutative *-algebra or a $U^*$-algebra. The somewhat stronger statements which are actually proved, together with previously known properties of the reducing ideal, show that the reducing ideal defines a radical subcategory of each of the following three semi-abelian categories:**

**(1) Commutative *-algebras and *homomorphisms.**

**(2) Banach *-algebras and continuous *-homomorphisms.**

**(3) Banach *-algebras and contractive *-homomorphisms.**

The concept of the reducing ideal was introduced by Gelfand and Naimark in their classic paper [2, p. 463]. It has subsequently been studied by Kelley and Vaught [5, p. 51] and the present author [7, p. 63] and [8, p. 930]. The concept is discussed in [10, pp. 210, 226] and [6, p. 259]. In [11, 1479] Yood gave a definition of the *-radical which agrees with our definition for Banach *-algebras but differs for certain other types of *-algebras.

Our main extension theorem (3.1, below) was previously known for $B^*$-algebras [1, Proposition 2.10.4]. It has a number of applications besides the one discussed here. For example it immediately implies the conclusion of [4, Theorem 23] with hypotheses weaker than those of [4, Theorem 22].

In § 1 we give necessary background information. The case of commutative *-algebras is considered in § 2 and of $U^*$-algebras in § 3. The category theory results are described in § 4 where we use the terminology of M. Gray [3] for the general theory of radicals.

In general we follow the terminology of Rickart's book [10]. Further details and related results will be found in the author's forthcoming monograph [9].

**1. Definitions and preliminary results.** We review some basic definitions and results for the convenience of the reader and in order to fix notation. Throughout this paper all algebras and linear spaces will have the complex field $\boldsymbol{C}$ as scalar field unless the real field is explicitly specified. No other scalar field is considered. The complex conjugate of $\lambda \in \boldsymbol{C}$ will be denoted by $\lambda^*$.

An involution on an algebra $\mathfrak{A}$ is a conjugate linear, anti-multiplicative, involutive map of $\mathfrak{A}$ onto itself. A $*$-algebra is an algebra together with a fixed involution which will always be denoted by $(*)$. A subset of a $*$-algebra is called a $*$-subset iff it closed under the involution. A map between $*$-algebras is called a $*$-map iff it preserves their involutions (i.e. $\varphi(a^*) = \varphi(a)^*$). A $*$-representation $T$ of a $*$-algebra is a $*$-homomorphism (i.e. an algebra homomorphism which is also a $*$-map) into the $*$-algebra $[\mathfrak{H}_T]$ of all bounded linear operators on some Hilbert space $\mathfrak{H}_T$. The meaning of each more specific term with a $*$-prefix (e.g. $*$-subalgebra, $*$-isomorphism) follows from these definitions. In particular a Banach $*$-algebra is simply a $*$-algebra with a norm relative to which it is a Banach algebra. No relationship between the involution and norm is postulated.

We review briefly the standard Gelfand-Naimark construction of $*$-representations from positive linear functionals since later proofs depend intimately on this material (cf. [2], [6], [9] or [10]). A linear functional $\omega$ on a $*$-algebra $\mathfrak{A}$ is called positive iff

$$\omega(a^*a) \geqq 0 \qquad \forall a \in \mathfrak{A}\,. \tag{1.1}$$

For any positive linear functional $\omega$ denote the left ideal

$$\{a \in \mathfrak{A}\colon \omega(a^*a) = 0\} = \{a \in \mathfrak{A}\colon \omega(b^*a) = 0,\ \forall b \in \mathfrak{A}\} \tag{1.2}$$

by $\mathfrak{A}_\omega$. Let

$$\mathfrak{A}^\omega = \mathfrak{A}/\mathfrak{A}_\omega\,. \tag{1.3}$$

For each $a \in \mathfrak{A}$ let $a^\omega$ be the image $a + \mathfrak{A}_\omega$ of $a$ in $\mathfrak{A}^\omega$. Then for all $a^\omega, b^\omega \in \mathfrak{A}^\omega$

$$(a^\omega, b^\omega) = \omega(b^*a) \tag{1.4}$$

is well defined and gives $\mathfrak{A}^\omega$ the structure of a pre-Hilbert space (i.e. a possibly incomplete inner-product space). The left regular representation of $\mathfrak{A}$ on itself induces a $*$-homomorphism $\tilde{T}^\omega$ of $\mathfrak{A}$ into the $*$-algebra of all (not necessarily bounded) linear operators on $\mathfrak{A}^\omega$ which have adjoints on $\mathfrak{A}^\omega$. The positive linear functional $\omega$ is called *admissible* iff the range of $\tilde{T}^\omega$ consists of bounded operators so that $\tilde{T}^\omega$ induces a $*$-representation $T^\omega$ of $\mathfrak{A}$ on the Hilbert space completion $\mathfrak{A}^{\omega-}$ of $\mathfrak{A}^\omega$.

An admissible positive linear functional $\omega$ is called representable iff there is some *-representation $T$ and some topologically cyclic vector $x \in \mathfrak{H}_T$ for $T$ such that

$$\omega(a) = (T_a x, x) \qquad \forall a \in \mathfrak{A}. \tag{1.5}$$

The set of representable positive linear functionals on a *-algera $\mathfrak{A}$ will be denoted by $R(\mathfrak{A})$. For each nonzero $\omega$ in $R(\mathfrak{A})$

$$|\omega| = \sup\{\omega(a)^2/\omega(a^*a) : a \in \mathfrak{A} \sim \mathfrak{A}_\omega\} \tag{1.6}$$

is finite. For the zero linear functional, which always belongs to $R(\mathfrak{A})$, we set $|0| = 0$. For each $\omega \in R(\mathfrak{A})$ there is a unique vector $x_\omega$ in $\mathfrak{A}^{\omega-}$ such that

$$T^\omega_a x_\omega = a^\omega \qquad \forall a \in \mathfrak{A}. \tag{1.7}$$

[9, Theorem 1.4.8]. This vector is a topologically cyclic vector for $T^\omega$ which also satisfies

$$\|x_\omega\|^2 = |\omega| \text{ and } \omega(a) = (T^\omega_a x_\omega, x_\omega) \qquad \forall a \in \mathfrak{A}. \tag{1.8}$$

For a *-algebra $\mathfrak{A}$ let

$$R_1(\mathfrak{A}) = \{\omega \in R(\mathfrak{A}) : |\omega| \leqq 1\}. \tag{1.9}$$

A linear functional $\omega$ on $\mathfrak{A}$ is called a *state* iff $\omega \in R(\mathfrak{A})$ and $|\omega| = 1$. A linear functional $\omega \in R(\mathfrak{A})$ is called *pure* iff $\omega = \omega_1 + \omega_2$ with $\omega_1, \omega_2 \in R(\mathfrak{A})$ implies that $\omega_1$ and $\omega_2$ are (nonnegative real) multiples of $\omega$. Let $P(\mathfrak{A})$ denote the set of pure states of $\mathfrak{A}$. Then $P(\mathfrak{A}) \cup \{0\}$ is the set of extreme points of the convex set $R_1(\mathfrak{A})$.

If $\mathfrak{A}$ is a Banach *-algebra it is well known that $R_1(\mathfrak{A})$ is compact in the $\mathfrak{A}$-topology. Thus $R_1(\mathfrak{A})$ is the closed convex hull of $P(\mathfrak{A}) \cup \{0\}$ by the Krein-Milman theorem. If $\mathfrak{A}$ is an arbitrary *-algebra (e.g. {complex polynomials} with conjugation of coefficients as the involution) then $R_1(\mathfrak{A})$ need not be compact.

LEMMA 1.1. *If $\mathfrak{A}$ is any *-algebra, $R_1(\mathfrak{A})$ is the closed convex hull of $P(\mathfrak{A}) \cup \{0\}$.*

*Proof.* For any $\omega \in R_1(\mathfrak{A})$ let

$$\mathfrak{S}_\omega = \{\omega' \in R_1(\mathfrak{A}) : \|T^{\omega'}_a\| \leqq \|T^\omega_a\| \text{ for all } a \in \mathfrak{A}\}.$$

A slight adaptation of a well known proof [10, p. 222] shows that $\mathfrak{S}_\omega$ is compact and convex [9, Proposition 1.5.6]. Similarly one can adapt another well known proof [10, p. 225] to show that the set of extreme points of $\mathfrak{S}_\omega$ is $\{0\} \cup (\mathfrak{S}_\omega \cap P(\mathfrak{A}))$ [9, Proposition 1.6.6]. Thus $\mathfrak{S}_\omega = \overline{\text{co}}\,(\{0\} \cup (\mathfrak{S}_\omega \cap P(\mathfrak{A})))$. Therefore

$$R_1(\mathfrak{A}) = \bigcup_{\omega \in R_1(\mathfrak{A})} \mathfrak{S}_\omega = \bigcup_{\omega \in R_1(\mathfrak{A})} \overline{\text{co}}\,(\{0\} \cup P(\mathfrak{A}))$$
$$\subseteqq \overline{\text{co}}\,(\{0\} \cup P(\mathfrak{A})) \subseteqq R_1(\mathfrak{A}) \,.$$

LEMMA 1.2. *Let $\mathfrak{A}$ be a *-algebra and let $\omega \in R(\mathfrak{A})$. The following are equivalent.*

(a) *$\omega$ is pure.*

(b) *$T^\omega$ is topologically irreducible.*

(c) *The set $(T^\omega)'$ of operators in $[\mathfrak{A}^{\omega-}]$ which commute with $T^\omega_a$ for each $a \in \mathfrak{A}$ is the set of complex multiples of the identity.*

*Proof.* [10, p. 211 and 223], [9, Theorems 1.6.1 and 1.6.5].

DEFINITION 1.3. For any *-algebra $\mathfrak{A}$ the reducing ideal of $\mathfrak{A}$ is denoted by $\mathfrak{A}_R$ and defined by

$$\mathfrak{A}_R = \bigcap \{\text{Ker}\,(T)\colon T \text{ is a *-representation of } \mathfrak{A}\}.$$

If $\mathfrak{J}$ is a *-ideal of a *-algebra $\mathfrak{A}$ then $\mathfrak{J}$ is a two-sided ideal and $\mathfrak{A}/\mathfrak{J}$ is a *-algebra in an obvious sense.

PROPOSITION 1.4. *Let $\mathfrak{A}$ be a *-algebra. Then the reducing ideal $\mathfrak{A}_R$ of $\mathfrak{A}$ is a *-ideal which equals:*

$$\bigcap \{\text{Ker}\,(T)\colon T \textit{ is a topologically irreducible *-representation of } \mathfrak{A}\}$$
$$= \bigcap \{\text{Ker}\,(T^\omega)\colon \omega \in R(\mathfrak{A})\} = \bigcap \{\text{Ker}\,(T^\omega)\colon \omega \in P(\mathfrak{A})\}$$
$$= \bigcap \{\mathfrak{L}_\omega\colon \omega \in R(\mathfrak{A})\} = \bigcap \{\mathfrak{L}_\omega\colon \omega \in P(\mathfrak{A})\}$$
$$= \{a \in \mathfrak{A}\colon \omega(a) = 0,\ \forall \omega \in R(\mathfrak{A})\} = \{a \in \mathfrak{A}\colon \omega(a) = ,\ \forall \omega \in P(\mathfrak{A})\}\,.$$

*Furthermore $(\mathfrak{A}/\mathfrak{A}_R)_R = \{0\}$. If $\mathfrak{A}$ is a Banach *-algebra then $\mathfrak{A}_R$ is closed so that $\mathfrak{A}/\mathfrak{A}_R$ is a Banach *-algebra.*

*Proof.* Use Lemma 1.1 to adapt the proof of [10, Theorem 4.4.10]. For details and further results see [9, Theorem 1.7.2 and 1.7.5].

Lemma 1.1 and this proposition do not seem to have been noted previously in this degree of generality. However they were essentially known.

We now turn to the theory of $U^*$-algebras. For additional information see [7], [8], or [9].

If $\mathfrak{A}$ is a *-algebra without an identity let $\mathfrak{A}^1$ denote the *-algebra with identity which has $\boldsymbol{C} \oplus \mathfrak{A}$ as underlying linear space and in which the multiplication and involution are defined by $(\lambda \oplus a)(\mu \oplus b) = \lambda\mu \oplus (\lambda b + \mu a + ab)$ and $(\lambda \oplus a)^* = \lambda^* \oplus a^*$ for all $\lambda, \mu \in \boldsymbol{C}$ and all $a, b \in \mathfrak{A}$. We regard $\mathfrak{A}$ as embedded in $\mathfrak{A}^1$ by the map $a \to 0 \oplus a$. If $\mathfrak{A}$ already has an identity let $\mathfrak{A}^1 = \mathfrak{A}$. In either case we write $\lambda + a$

for $\lambda 1 + a$ where 1 is the identity of $\mathfrak{A}^1$. Then, for instance, the spectrum of an element in $\mathfrak{A}$ is the same with relation to $\mathfrak{A}$ or $\mathfrak{A}^1$. Furthermore the Jacobson radical of $\mathfrak{A}$ and $\mathfrak{A}^1$ agree and the reducing ideal of $\mathfrak{A}$ and $\mathfrak{A}^1$ agree.

DEFINITION 1.5. A *-algebra $\mathfrak{A}$ is called a $U^*$-algebra iff $\mathfrak{A}$ is contained in the linear span of the set $\mathfrak{A}^1_U$ of unitary elements in $\mathfrak{A}^1$. If $\mathfrak{A}$ is a $U^*$-algebra and $a \in \mathfrak{A}$ then

$$\nu_{\mathfrak{A}}(a) = \inf\left\{\sum_{j=1}^{n} \lambda_j \colon a = \sum_{j=1}^{n} \lambda_j u_j \text{ where } n \in \boldsymbol{N}, \lambda_j \in \boldsymbol{C}, \text{ and } u_j \in \mathfrak{A}^1_U\right\}.$$

LEMMA 1.6. *Let $\mathfrak{A}$ be a $U^*$-algebra. Then $\nu_{\mathfrak{A}}$ is an algebra pseudo-norm, (i.e. $\nu_{\mathfrak{A}}(\lambda a) = |\lambda| \nu_{\mathfrak{A}}(a)$, $\nu_{\mathfrak{A}}(a + b) \leqq \nu_{\mathfrak{A}}(a) + \nu_{\mathfrak{A}}(b)$, $\nu_{\mathfrak{A}}(ab) \leqq \nu_{\mathfrak{A}}(a)\nu_{\mathfrak{A}}(b)$ for all $a, b \in \alpha$).*

*Proof.* Obvious.

For any *-algebra $\mathfrak{A}$ let

$$\mathfrak{A}_{qU} = \{v \in \mathfrak{A} \colon v^*v = vv^* = v + v^*\} \tag{1.10}$$

be the set of quasi-unitary elements in $\mathfrak{A}$. For any subset $\mathfrak{S}$ of $\mathfrak{A}$ let $\mathfrak{S}^U$ be the linear span of $\mathfrak{S} \cap \mathfrak{A}_{qU}$.

LEMMA 1.7. *Let $\mathfrak{A}$ be a *-algebra. Then $\mathfrak{A}^U$ is a *-subalgebra of $\mathfrak{A}$ which is a $U^*$-algebra. Furthermore $\mathfrak{A}^U$ contains every *-subalgebra of $\mathfrak{A}$ which is a $U^*$-algebra. In particular $\mathfrak{A}$ is a $U^*$-algebra iff $\mathfrak{A} = \mathfrak{A}^U$. In this case*

$$\nu_{\mathfrak{A}}(a) = \inf\left\{\sum_{j=1}^{n} \lambda_j \colon a = \sum_{j=1}^{n} \lambda_j v_j,\ 0 = \sum_{j=1}^{n} \lambda_j \text{ where } n \in \boldsymbol{N}, \lambda_j \in \boldsymbol{C} \text{ and } v_j \in \mathfrak{A}_{qU}\right\}.$$

*Finally if $\mathfrak{J}$ is a one- or two-sided ideal in $\mathfrak{A}$ then $\mathfrak{J}^U$ is a *-ideal in $\mathfrak{A}$.*

*Proof.* Straightforward or see [8] or [9].

LEMMA 1.8. *Let $\mathfrak{A}$ be a $U^*$-algebra and let $\mathfrak{B}$ be a *-algebra. Let $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ be a *-homomorphism. Then $\varphi(\mathfrak{A})$ is a $U^*$-algebra and $\nu_{\varphi(\mathfrak{A})}(\varphi(a)) \leqq \nu_{\mathfrak{A}}(a)$ for all $a \in \mathfrak{A}$. Furthermore if $\mathfrak{B}$ is the algebra of all (not necessarily bounded) linear operators with adjoints on a pre-Hilbert space, then $\varphi(\mathfrak{A})$ is contained in the set of bounded operators and $\|\varphi(a)\| \leqq \nu_{\mathfrak{A}}(a)$ for all $a \in \mathfrak{A}$.*

*Proof.* This follows directly from Lemma 1.7 or see [7], [8] or

[9].

By slight abuse of language we call a *-homomorphism into the type of *-algebra described in the last sentence of Lemma 1.8 a *-representation on a pre-Hilbert space. When the range of such a map consists of bounded operators we call it a normed *-representation on a pre-Hilbert space. (Of course any *-representation of any *-algebra (by definition, on a Hilbert space) is automatically normed [10, p. 205] or [9, Corollary 1.2.4].)

COROLLARY 1.9. *Every *-representation of a $U^*$-algebra on a pre-Hilbert space is normed. Every positive linear functional on a $U^*$-algebra is admissible. A positive linear functional on a $U^*$-algebra $\mathfrak{A}$ is representable iff it is the restriction of some positive linear functional on $\mathfrak{A}^1$.*

*Proof.* For the last sentence see [10, p. 218] or [9, Theorem 1.4.8].

DEFINITION 1.10. Let $\mathfrak{A}$ be a *-algebra. For any $a \in \mathfrak{A}$ let

$\gamma_{\mathfrak{A}}(a) = \sup\{\|T_a\|: T$ is a *-representation of $\mathfrak{A}$ on a Hilbert space$\}$ .

It is not hard to show [9, Theorem 2.1.2] that $\gamma_{\mathfrak{A}}(a) = \sup\{\|T_a\|: T$ is a topologically irreducible *-representation of $\mathfrak{A}$ on a Hilbert space$\} = \sup\{\omega(a^*a)^{1/2}: \omega \in R_1(\mathfrak{A})\} = \sup\{\omega(a^*a)^{1/2}: \omega \in P(\mathfrak{A})\}$. In a perfectly general *-algebra $\gamma_{\mathfrak{A}}(a) = \infty$ is possible. However if $\gamma_{\mathfrak{A}}$ is finite valued then it is the largest algebra pseudonorm on $\mathfrak{A}$ which satisfies the $B^*$-condition: $\gamma_{\mathfrak{A}}(a^*a) = \gamma_{\mathfrak{A}}(a)^2$ for all $a \in \mathfrak{A}$. We call $\gamma_{\mathfrak{A}}$ the Gelfand-Naimark pseudo-norm on $\mathfrak{A}$. Note that $\mathfrak{A}_R = \{a \in \mathfrak{A}: \gamma_{\mathfrak{A}}(a) = 0\}$.

COROLLARY 1.11. *If $\mathfrak{A}$ is a $U^*$-algebra then*

$$\gamma_{\mathfrak{A}}(a) \leqq \nu_{\mathfrak{A}}(a)$$

*for all $a \in \mathfrak{A}$.*

*Proof.* Obvious from Lemma 1.8.

THEOREM 1.12. *Let $\mathfrak{A}$ be a Banach *-algebra. Then $\mathfrak{A}$ is a $U^*$-algebra and $\gamma_{\mathfrak{A}} = \nu_{\mathfrak{A}}$.*

*Proof.* [7, Theorem 4] or [9, Theorem 3.1.12].

**2. Commutative *-algebras.** We are now in a position to treat this case easily. Several of our results are essentially known but are usually stated in less generality.

THEOREM 2.1. *Let $\mathfrak{A}$ be a commutative *-algebra. Then $P(\mathfrak{A})$ is the set of *-homomorphisms of $\mathfrak{A}$ onto $\boldsymbol{C}$.*

*Proof.* Suppose $\omega$ is a pure state. Then $(T^{\omega})' = \boldsymbol{C}I$ by Lemma 1.2 where $I$ is the identity operator in $[\mathfrak{A}^{\omega^-}]$. Since $\mathfrak{A}$ is commutative $T^{\omega}_{\mathfrak{A}} \subseteqq (T^{\omega}_{\mathfrak{A}})'$. Since $\omega \neq 0$, $T^{\omega} \neq 0$ so $T^{\omega}_{\mathfrak{A}} = \boldsymbol{C}I$. Let $T^{\omega}_a = \varphi(a)I$ for all $a \in \mathfrak{A}$. Then $\varphi$ is a *-homomorphism of $\mathfrak{A}$ onto $\boldsymbol{C}$ and $\omega(a) = (T^{\omega}_a x_{\omega}, x_{\omega}) = (\varphi(a)x_{\omega}, x_{\omega}) = \varphi(a) \,|\, \omega \,|\, = \varphi(a)$ for all $a \in \mathfrak{A}$. Thus $\omega = \varphi$ is a *-homomorphism of $\mathfrak{A}$ onto $\boldsymbol{C}$.

Conversely suppose $\omega$ is a *-homomorphism of $\mathfrak{A}$ onto $\boldsymbol{C}$. Then $\omega(a^*a) = \omega(a^*)\omega(a) = |\,\omega(a)\,|^2$ for all $a \in \mathfrak{A}$ so that $\omega$ is a state. The map $a^{\omega} \rightarrow \omega(a)$ for all $a \in \mathfrak{A}$ is a linear isometry of $\mathfrak{A}^{\omega}$ onto $\boldsymbol{C}$. Thus $\mathfrak{A}^{\omega} = \mathfrak{A}^{\omega^-}$ is linearly isometric to $\boldsymbol{C}$ so that $T^{\omega} \neq 0$ is irreducible. Therefore $\omega$ is a pure state by Lemma 1.2.

COROLLARY 2.2. *Let $\mathfrak{A}$ be a commutative *-algebra. For each $a \in \mathfrak{A}$ let $\check{a}$: $P(\mathfrak{A}) \rightarrow \boldsymbol{C}$ be defined by $\check{a}(\omega) = \omega(a)$ for all $\omega \in P(\mathfrak{A})$. Let $P(\mathfrak{A})$ carry the weakest topology which makes each $\check{a}$ continuous. Let $C_{\infty}(P(\mathfrak{A}))$ be the set of continuous but not necessarily bounded complex valued functions on $P(\mathfrak{A})$. Then $P(\mathfrak{A})$ is Tychonoff space and*

$$(2.1) \qquad (\check{\ }): \mathfrak{A} \longrightarrow C_{\infty}(P(\mathfrak{A}))$$

*is a *-homomorphism with kernel $\mathfrak{A}_R$.*

*Proof.* Immediate from Theorem 2.1 and Proposition 1.4.

THEOREM 2.3. *Let $\mathfrak{A}$ be a commutative *-algebra. Let $\mathfrak{B}$ be a *-ideal of $\mathfrak{A}$ and let $\mathfrak{J}$ be a *-ideal of $\mathfrak{B}$. For each $\omega \in P(\mathfrak{J})$ there is an $\bar{\omega} \in P(\mathfrak{A})$ such that $\omega$ is the restriction of $\bar{\omega}$.*

*Proof.* Theorem 2.1 shows that $\omega$ is a *-homomorphism of $\mathfrak{J}$ onto $\boldsymbol{C}$. Let $e \in \mathfrak{J}$ satisfy $\omega(e) = 1$. We may assume $e = e^*$ since $\omega$ is a *-map. For any $a \in \mathfrak{A}$, $ea \in \mathfrak{B}$ so $e^2a \in \mathfrak{J}$. Define $\bar{\omega}(a) = \omega(e^2a)$ for all $a \in \mathfrak{A}$. Then $\bar{\omega}$ is clearly linear and if a, $b \in \mathfrak{A}$ then $\bar{\omega}(ab) = \omega(e^2a) = \omega(e)^2\omega(ae^2b) = \omega(e^2ae^2b) = \omega(e^2a)\omega(e^2b) = \bar{\omega}(a)\bar{\omega}(b)$, and $\bar{\omega}(a^*) = \omega(e^2a^*b) = \omega(e^2a)^* = \bar{\omega}(a)^*$. Thus $\bar{\omega}$ is a *-homomorphism of $\mathfrak{A}$ onto $\boldsymbol{C}$ and thus by Theorem 2.1 $\bar{\omega} \in P(\mathfrak{A})$. If $a \in \mathfrak{J}$ then $\bar{\omega}(a) = \omega(e^2a) = \omega(e)^2\omega(a) = \omega(a)$. Thus $\bar{\omega}$ satisfies the theorem.

COROLLARY 2.4. *Let $\mathfrak{A}$ be a commutative *-algebra. Let $\mathfrak{J}$ be a *-ideal of $\mathfrak{A}_R$ (e.g. a *-ideal of $\mathfrak{A}$ included in $\mathfrak{A}_R$). Then $\mathfrak{J}_R = \mathfrak{J}$. In* *particular* $(\mathfrak{A}_R)_R = \mathfrak{A}_R$.

*Proof*. If $\mathfrak{I}_R \neq \mathfrak{I}$ then there is some nonzero pure state on $\mathfrak{I}$ by Proposition 1.4. Thus Theorem 2.3 shows that there is a pure state on $\mathfrak{A}$ which does not vanish on $\mathfrak{I}$. This contradicts Proposition 1.4.

3. $U^*$**-algebras.** Although our primary interest is in Banach $^*$-algebras it seems difficult to give the following proof in that setting without using the (more general) structure of $U^*$-algebras.

THEOREM 3.1. *Let $\mathfrak{A}$ be a $U^*$-algebras. Let $\mathfrak{I}$ be a $^*$-ideal of $\mathfrak{A}$. Let $T$ be a $^*$-representation of $\mathfrak{I}$. Then there is a $^*$-representation $\bar{T}$ of $\mathfrak{A}$ on $[\mathfrak{H}_T]$ which extends $T$. If $T$ is essential then $\bar{T}$ is unique, and the set of topologically cyclic vectors for $T$ equals the set of topologically cyclic vectors for $\bar{T}$. Thus when $T$ is essential it is topologically cyclic or topologically irreducible iff $\bar{T}$ has the corresponding property.*

*Proof*. If $T$ is not essential it is the direct sum of a zero sub-$^*$-representation $T^0$ on $\mathfrak{H}_0$ and an essential sub-$^*$-representation $T^1$ on $\mathfrak{H}_1$. We can extend $T^0$ as a zero $^*$-representation $\bar{T}^0: \mathfrak{A} \to [\mathfrak{H}_0]$. Thus if we can extend $T^1$ to $\bar{T}^1: \mathfrak{A} \to [\mathfrak{H}_1]$ then $T^0 \oplus T^1$ extends $T$. Therefore we need only consider the case of essential $^*$-representations.

Suppose $T$ is essential and let $\mathfrak{X}$ be the subset of $\mathfrak{H}$, $T_{\mathfrak{I}}\mathfrak{H}_T = \{T_b x: b \in \mathfrak{I}, x \in \mathfrak{H}_T\}$. Then $\mathfrak{X}$ is dense in $\mathfrak{H}_T$. Let $\bar{T}: \mathfrak{A} \to [\mathfrak{H}_T]$ by any $^*$-representation which extends $T$. Let $a \in \mathfrak{A}$ and $x \in \mathfrak{X}$. Then $x = T_b y$ for $b \in \mathfrak{I}$ and $y \in \mathfrak{H}_T$. Thus

$$\bar{T}_a x = \bar{T}_a T_b y = \bar{T}_a \bar{T}_b y = \bar{T}_{ab} y = T_{ab} y \,.$$

Since $T$ is normed (Corollary 1.9) and $\mathfrak{X}$ is dense this shows that there is at most one extension $\bar{T}: \mathfrak{A} \to [\mathfrak{H}_T]$ of $T$.

Suppose $z$ is a topologically cyclic vector for $T$. Let $\mathfrak{X} = T_{\mathfrak{I}} z$. Then $\mathfrak{X}$ is dense again. For $a \in \mathfrak{A}$ and $x \in \mathfrak{X}$ define $\bar{T}^1_a x = T_{ab} z$ where $x = T_b z$ with $b \in \mathfrak{I}$. We must first show that this is well defined. Suppose $x = T_d z$ with $d \in \mathfrak{I}$ also. Let $a = \sum_{n=1}^N \lambda_n v_n$ where $\lambda_n \in C$, $v_n \in \mathfrak{A}_{qU}$, and $\sum_{n=1}^N \lambda_n = 0$. Then $T_{ab} z - T_{ad} z = \sum_{n=1}^N \lambda_m (T_{v_n b - b} z - T_{v_n d - d} z)$. However for each $n$

$$\begin{aligned} \| T_{v_n b - b} z - T_{v_n d - d} z \|^2 &= \| T_{v_n (b-d) - (b-d)} z \|^2 \\ &= ((T_{(b-d)^*(v_n^* v_n - v_n - v_n^*)(b-d)} + T_{(b-d)^*(b-d)}) z, z) \\ &= \| T_{b-d} z \|^2 = 0 \,. \end{aligned}$$

Thus $T_{ab} z = T_{ad} z$ and $\bar{T}^1_a x$ is well defined for each $x \in \mathfrak{X}$. For $a \in \mathfrak{I}$

and $x = T_b z \in \mathfrak{X}$, $\bar{T}^1_a x = T_{ab} z = T_a T_b z = T_a x$. It is easy to check that $\bar{T}^1$ is a *-representation of $\mathfrak{A}$ on the pre-Hilbert space $\mathfrak{X}$. Corollary 1.9 shows that $\bar{T}^1$ is normed and hence can be extended (in the sense of extensions of *-representations) to a unique *-representation $\bar{T}: \mathfrak{A} \to [\mathfrak{H}_T]$ which extends $T: \mathfrak{J} \to [\mathfrak{H}_T]$. Clearly $z$ is a topologically cyclic vector for $\bar{T}$ since $\bar{T}_{\mathfrak{A}} z \supseteq T_{\mathfrak{J}} z$. This concludes the proof of the theorem when $T$ is topologically cyclic.

Suppose $T$ is essential but not necessarily topologically cyclic. Then $T = \bigoplus_{\alpha \in A} T^\alpha$ is the internal direct sum of a family $\{T^\alpha : \alpha \in A\}$ of topologically cyclic sub-*-representations on $T$-invariant subspaces $\{\mathfrak{H}_\alpha : \alpha \in A\}$. For each $\alpha \in A$ we have shown how to construct a *-representation $\bar{T}^\alpha: \mathfrak{A} \to [\mathfrak{H}_\alpha]$ which extends $T^\alpha: \mathfrak{J} \to [\mathfrak{H}_\alpha]$. The direct sum $\bigoplus_{\alpha \in A} \bar{T}_\alpha: \mathfrak{A} \to [\mathfrak{H}_T]$ is defined since $\gamma_{\mathfrak{A}} \leqq \nu_{\mathfrak{A}}$ by Corollary 1.11. It extends $T: \mathfrak{J} \to [\mathfrak{H}_T]$. We have already shown that only one such extension is possible. Thus any essential *-representation of $\mathfrak{J}$ has a unique extension to $\mathfrak{A}$.

Suppose $z$ is a topologically cyclic vector for $\bar{T}$ and $T$ is essential then $\bar{T}_a T_b z = T_{ab} z$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{J}$ so that $T_{\mathfrak{J}} z^-$ is a closed $\bar{T}$-invariant subspace of $\mathfrak{H}_T$ containing $z$ by [10, p. 206] or [9, 1.2.10]. The topological cyclicity of $z$ for $\bar{T}$ shows that $T_{\mathfrak{J}} z^- = \mathfrak{H}_T$ so that $z$ is a topologically cyclic vector for $T$.

When $T$ is essential we have shown that the set of topologically cyclic vectors for $\bar{T}$ equals the set of topologically cyclic vectors for $T$. Since a *-representation is topologically cyclic iff its set of topologically cyclic vectors is nonempty and is topologically irreducible iff every nonzero vector is topologically cyclic this establishes the last sentence of the theorem.

COROLLARY 3.2. *If $\mathfrak{A}$ is a $U^*$-algebra and $\mathfrak{J}$ is a *-ideal of $\mathfrak{A}$ included in $\mathfrak{A}_R$ then $\mathfrak{J}_R = \mathfrak{J}$. In particular $(\mathfrak{A}_R)_R = \mathfrak{A}_R$.*

*Proof.* If $\mathfrak{J}_R \neq \mathfrak{J}$ there is a nonzero *-representation $T$ of $\mathfrak{J}$. Then Theorem 3.1 shows that there is a *-representation $\bar{T}$ of $\mathfrak{A}$ which does not vanish on $\mathfrak{J} \subseteq \mathfrak{A}_R$. This contradicts the definition of $\mathfrak{A}_R$.

COROLLARY 3.3. *If $\mathfrak{A}$ is a $U^*$-algebra and $\mathfrak{J}$ is any *-ideal of $(\mathfrak{A}_R)^U$ then $\mathfrak{J}_R = \mathfrak{J}$. In particular $((\mathfrak{A}_R)^U)_R = (((\mathfrak{A}_R)^U)_R)^U = (\mathfrak{A}_R)^U$.*

*Proof.* The last sentence of Lemma 1.7 and Corollary 3.2 together show that $((\mathfrak{A}_R)^U)_R = (\mathfrak{A}_R)^U$. Thus these sets clearly equal $(((\mathfrak{A}_R)^U)_R)^U$. Thus this corollary follows from Corollary 3.2 applied to $(\mathfrak{A}_R)^U$ in place of $\mathfrak{A}$.

COROLLARY 3.4. *If $\mathfrak{A}$ is a Banach *-algebra and $\mathfrak{J}$ is a *-ideal* *of* $\mathfrak{A}_R$ *then* $\mathfrak{J}_R = \mathfrak{J}$. *In particular* $(\mathfrak{A}_R)_R = \mathfrak{A}_R$.

*Proof.* Theorem 1.12 and Proposition 1.4 together show that $\mathfrak{A}$ and $\mathfrak{A}_R$ are $U^*$-algebras so that $\mathfrak{A}_R = (\mathfrak{A}_R)^U$. Thus this corollary follows from Corollary 3.3.

**4. Remarks on categorical consequences.** In this section we wish to indicate the consequences of our results in the language of categories. In reference [3] we find a strong notion of radical subcategory which we will use. In fact what is called a radical in [3] is sometimes called a hereditary radical (cf. p. 125 of N. J. Divinsky, *Rings and Radicals,* University of Toronto Press, 1965). From one viewpoint our results may be considered as a quite different example of this theory.

We will show first that each of the three categories listed in the introduction is both semi-abelian and co-semi-abelian. The trivial *-algebra $\{0\}$ is a zero-object in each of these categories and also in each of the other categories which we will consider. We examine the categorically defined kernels, cokernels, images, and co-images in these categories.

In all three of the categories listed in the introduction the kernel of $f \in \operatorname{Hom}(\mathfrak{A}, \mathfrak{B})$ is simply (the subobject represented by the injection into $\mathfrak{A}$ of) the set theoretic kernel $\operatorname{Ker}(f)$ of $f$.

Consider the following categories.

(4) $U^*$-algebras and *-homomorphisms.

(5) Banach *-algebras and *-homomorphisms.

Since the image of any $U^*$-algebra is a $U^*$-algebra it is easy to see that the kernel of $f \in \operatorname{Hom}(\mathfrak{A}, \mathfrak{B})$ in category (4) is (the subobject represented by the injection into $\mathfrak{A}$ of) $(\ker(f))^U$ where again $\operatorname{Ker}(f)$ is the set theoretic kernel of $f$. In category (5) morphisms do not always have kernels, since there is not in general any maximal subobject of $\operatorname{Ker}(f)$ on which a Banach *-algebra norm can be defined. Notice that when such a maximal subobject does exist it must be included in $(\operatorname{Ker}(f))^U$.

In the category

(6) *-algebras and *-homomorphisms

the set theoretic kernel "is" the categorical kernel.

In categories (1), (4) and (6) the cokernel of $f \in \operatorname{Hom}(\mathfrak{A}, \mathfrak{B})$ is represented by

$$\mathfrak{B} \longrightarrow \mathfrak{B}/(\text{*-ideal generated by } f(\mathfrak{A}))\ .$$

In categories (2) and (3) the cokernel of $f \in \operatorname{Hom}(\mathfrak{A}, \mathfrak{B})$ is represented by

$$\mathfrak{B} \longrightarrow (\mathfrak{B}/(\text{closed *-ideal generated by } f(\mathfrak{A})))\ .$$

Morphisms in category (5) do not always have cokernels, since there is not always a smallest *-ideal containing $f(\mathfrak{A})$ such that the quotient may be embedded in a Banach *-algebra.

In categories (1), (2), (3), (4), (6) the image of $f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ is represented by the map

$$\mathfrak{A}/\mathrm{Ker}(f) \longrightarrow \mathfrak{B}$$

induced by $f$. Morphisms in category (5) do not always have images.

The co-image of $f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ in categories (1), (2), (3), (4), and (6) is represented by the natural morphism

$$\mathfrak{A} \longrightarrow \mathfrak{A}/\mathrm{Ker}(f) .$$

Morphisms in category (5) do not always have co-images.

DEFINITION 4.1. A category with a zero object is called semi-abelian if:

(a) Every morphism may be factored into a representative of its co-image followed by a representative of its image, and

(b) Every morphism has a cokernel.

A category with a zero object is called co-semi-abelian iff it satisfies (a) and

(c) Every morphism has a kernel.

PROPOSITION 4.2. *Categories (1), (2), (3), (4), and (6) are each both semi-abelian and co-semi-abelian.*

*Proof.* This follows from the remarks above.

DEFINITION 4.3. Let $\mathscr{C}$ be a semi-abelian category. A radical subcategory of $\mathscr{C}$ is a full subcategory $\mathscr{R}$ such that

(a) If $\mathfrak{A} \in \mathscr{R}, f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ and $i \in \mathrm{Hom}(\mathfrak{J}, \mathfrak{B})$ represents the image of $f$ then $\mathfrak{J} \in \mathscr{R}$.

(b) If $\mathfrak{A} \in \mathscr{R}, f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ and $k \in \mathrm{Hom}(\mathfrak{K}, \mathfrak{A})$ represents the kernel of $f$ then $\mathfrak{K} \in \mathscr{R}$.

(c) For each $\mathfrak{A} \in \mathscr{C}$ there is a unique subobject $\mathfrak{A}_{\mathscr{R}}$ or $\mathfrak{A}$ which satisfies

($c_1$) $\mathfrak{A}_{\mathscr{R}}$ is a kernel.

($c_2$) $\mathfrak{A}_{\mathscr{R}}$ is represented by a monomorphism with an object of $\mathscr{R}$ as domain.

($c_3$) $\mathfrak{A}_{\mathscr{R}}$ includes any subobject of $\mathfrak{A}$ which is a kernel and is also represented by a monomorphism with an object of $\mathscr{R}$ as domain.

(d) If $u \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ is a representative of the cokernel of a representative $v \in \mathrm{Hom}(\mathfrak{H}, \mathfrak{A})$ of $\mathfrak{A}_{\mathscr{R}}$ then the subobject $\mathfrak{B}_{\mathscr{R}}$ is the zero-subobject of $\mathfrak{B}$.

THEOREM 4.4. *In each of the categories* (1), (2), *and* (3) *the full subcategory defined by the class of objects* $\mathfrak{A}$ *such that* $\mathfrak{A} = \mathfrak{A}_R$ *is a radical subcategory.*

*Proof.* Proposition 1.4 and Corollary 2.4 and 3.4, together with the identification of the kernels, cokernels, images and co-images in these categories, establish this result.

This theorem justifies the term *-radical as a name for the reducing ideal in these three categories.

In the semi-abelian category (4) of $U^*$-algebras we do not know whether the reducing ideal is always a $U^*$-algebra, i.e.

$$\mathfrak{A}_R = (\mathfrak{A}_R)^U . \tag{4.1}$$

In fact we do not know whether every closed *-ideal is always a $U^*$-algebra. If $\mathfrak{A}_R$ is always a $U^*$-algebra then Theorem 4.4 is true for category (4) also. Otherwise one might consider the full subcategory $\mathscr{R}$ defined by the class of objects $\mathfrak{A}$ such that $\mathfrak{A} = (\mathfrak{A}_R)^U$. This subcategory satisfies (a), (b), and (c) of Definition 4.3 with $\mathfrak{A}_{\mathscr{R}} = (\mathfrak{A}_R)^U$. However it will not satisfy (d) unless

$$((\mathfrak{A}/(\mathfrak{A}_R)^U)_R)^U = \{0\} . \tag{4.2}$$

It is possible that condition (4.2) is true for all $U^*$-algebras. If it is not true for all $U^*$-algebras perhaps there is a full subcategory of category (4) in which either condition (4.1) or (4.2) holds. This subcategory might have a radical subcategory associated with the reducing ideal. Notice that categories (2) and (3) are nonfull subcategories of category (4) in which (4.1) holds.

It seems unlikely that the semi-abelian category (6) has a radical subcategory defined by the reducing ideal. However a counterexample is probably quite weird. (Note added in proof: I have found a counterexample which is not particularly weird.)

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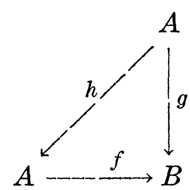
# QUASI PROJECTIVES IN ABELIAN AND MODULE CATEGORIES

K. M. RANGASWAMY AND N. VANAJA

**If $R$ is a ring without zero divisors then it is shown that any torsion-free quasi-projective left $R$-module $A$ is projective provided $A$ is finitely generated or $A$ is "big". It is proved that the universal existence of quasi-projective covers in an abelian category with enough projectives always implies that of the projective covers. Quasi-projective modules over Dedekind domains are described and as a biproduct we obtain an infinite family of quasi-projective modules $Q$ such that no direct sum of infinite number of carbon copies of $Q$ is quasi projective. Perfect rings are characterised by means of quasi-projectives. Finally the notion of weak quasi-projectives is introduced and weak quasi-projective modules over a Dedekind domain are investigated.**

**1. Introduction.** An object $A$ in a category $\mathscr{A}$ is called quasi-projective [14] if given an epimorphism $A \xrightarrow{f} B$ and a morphism $g: A \rightarrow B$, there is $h: A \rightarrow A$ making the following diagram




commutative. This paper starts with the investigation of the quasi-projectives in an abelian category. Utilising a few basic lemmas, it is shown that the universal existence of the quasi-projective covers in an abelian category $\mathscr{A}$ implies that of the projective covers, provided $\mathscr{A}$ possesses enough projectives and this answers affirmatively a question of Faith [4] in a general form. Next we consider quasi-projectives in the category of modules. It turns out that "big" torsion-free quasi-projectives over rings without zero divisors are always projective. Artin semi-simple rings are characterised as those rings over which quasi-projectives and projectives coincide. In § 5, quasi-projectives over a Dedekind domain $R$ are investigated: A quasi-projective $R$-module is either torsion or torsion-free. A torsion $R$-module is quasi-projective if and only if it is quasi-injective but not injective. If $R$ is a complete discrete valuation ring, then the torsion-free quasi-projective $R$-modules are just the free $R$-modules and the torsion-free modules of finite rank. Suppose $R$ is a Dedekind domain which is not a complete discrete valuation ring and $\sigma$ is the number of distinct prime ideals of $R$. If $\sigma \leqq 2^{\aleph_0}$, then all the torsion-free $R$-quasi-projectives are projective. If $\sigma > 2^{\aleph_0}$, then a torsion-free quasi-projective $R$-module $A$ is projective if either (i) rank $A \leqq \aleph_0$ or (ii) rank $A > \sigma$. In the case when $\aleph_0 <$ rank $A < \sigma$, $A$ is torsion-less, $\aleph_1$-projective and contains a free summand $F$ having the same rank as $A$. As a biproduct we at once get an infinite family of quasi-projective modules $A$ such that no direct sum of infinite number of copies of $A$ is quasi-projective. In § 6, Perfect rings are characterised as those rings $R$ such that $R$-quasi-projectivity survives under direct limits. A weakened form of quasi-projectivity — called weak quasi-projectivity — is considered in the last section and weak quasi-projectives over a Dedekind domain are completely characterised.

**2. Preliminaries.** All the rings that we consider are associative and are assumed to possess an identity and all the modules unitary left modules. A sub-module $S$ of an $R$-module $M$ is called *fully invariant* if $S$ is stable under every $R$-endomorphism of $M$. $S$ is called a *small* submodule, if $S + T = M$ implies $T = M$ for any submodule $T$ of $M$. A projective module $P$ is called a *projective cover* of $M$ if there is an epimorphism $P \to M$ whose kernel is small. A module $M$ over an integral domain is called *reduced* if 0 is the only divisible submodule of $M$. By the *rank* of a torsion-free module $M$ over a Dedekind domain $R$ we shall mean the cardinality of a maximal $R$-independent subset of $M$. An $R$-module $M$ is called *quasi-injective* if for any exact sequence $0 \to S \xrightarrow{i} M$, the induced sequence

$$\mathrm{Hom}_R(M, M) \xrightarrow{i^*} \mathrm{Hom}_R(S, M) \longrightarrow 0$$

is exact, where $i^*(f) = i \circ f$ for all $f$ in $\mathrm{Hom}_R(M, M)$. For the basic results in category theory, modules and abelian groups, the reader is referred to [5], [6], [10] and [11].

**3. Quasi-projectivity in abelian categories.** In this section, we examine the properties of quasi-projective objects in an abelian category. The main result shows that the universal existence of quasi-projective covers in an abelian category $\mathscr{A}$ implies that of projective covers, provided $\mathscr{A}$ possesses enough projectives.

NOTE. In conformity with our notation in the subsequent sections, *a composite $f \circ g$ of two morphisms is obtained by applying $f$*

*first and then g.*

LEMMA 3.1 [14]. *In an abelian category, any retract of a quasi-projective is quasi-projective.*

The following lemma gives a condition under which an object becomes projective.

LEMMA 3.2. *An object $A$ in an abelian category is projective if and only if there exists an epimorphism $P \dashrightarrow A$ with $P$ projective and $A \oplus P$ is quasi-projective.*

*Proof.* We prove only the "if" part. Let $f: P \rightarrow A$ be the given epimorphism, $A \xrightarrow{i} A \oplus P \xrightarrow{j} A = 1_A$ and $P \xrightarrow{i'} A \oplus P \xrightarrow{j'} P = 1_P$. By the quasi-projectivity of $A \oplus P$, there exists $g: A \oplus P \rightarrow A \oplus P$ such that $A \oplus P \xrightarrow{j} A = A \oplus P \xrightarrow{g} A \oplus P \xrightarrow{j'} P \xrightarrow{f} A$. Then

$$1_A = ij = i(g \circ j' \circ f) = (i \circ g \circ j')f .$$

Thus $A$ is a retract of $P$ and hence projective.

Dualizing 3.2, we obtain

LEMMA 3.2′. *An object $A$ is an abelian category is injective if and only if there is a monomorphism $A \rightarrow I$ with $I$ injective and $A \oplus I$ is quasi-injective.*

Next we examine the universal existence of quasi-projective covers.

DEFINITION 3.3. ( i ) An epimorphism $f$ in a category is called a *minimal epimorphism* if, whenever $g \circ f$ is an epimorphism, $g$ itself is an epimorphism.

(ii) $A \rightarrow X$ is called a *projective* (*quasi-projective*) *cover* in a category, if $A$ is projective (quasi-projective) and $f$ is a minimal epimorphism.

(iii) A category $\mathscr{A}$ is called *perfect* (*quasi-perfect*) if every object in $\mathscr{A}$ possesses a projective (quasi-projective) cover.

(iv) A category is said to possess *enough projectives,* if, to every object $A$, there is an epimorphism $P \rightarrow A$ with $P$ projective.

REMARK. ( i ) For an axiomatic treatment of minimal epimorphisms see [1]. Observe that in the category of $R$-modules, an epimorphism $f: A \rightarrow B$ is minimal if and only if Ker $f$ is small in $A$.

(ii) The notion of a perfect category has been considered in [2], [3].

(iii) Our definition of a quasi-projective cover is slightly different from the one defined in [14] for modules. However, it is easy to see that for the category of modules over a ring $R$, the universal existence of quasi-projective covers according to the new definition is equivalent to the universal existence of quasi-projective covers according to the definition given in [14].

It is clear that a perfect abelian category is quasi-perfect. Conversely, is a quasi-perfect abelian category perfect? This is the category-theoretical formulation of a question raised by C. Faith [4][1]. The following theorem answers this:

THEOREM 3.4. *An abelian category $\mathscr{A}$ is perfect if and only if it is quasi-perfect and possesses enough projectives.*

*Proof.* *IF part*: Let $A \in \mathscr{A}$ and $P \xrightarrow{u} A$ an epimorphism with $P$ projective. Let $g: Q' \to A \oplus P$ be a quasi-projective cover of $A \oplus P$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} & & Q & \xrightarrow{i''} & Q' & & & & \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus P & \xrightarrow{j'} & P & \longrightarrow & 0 \end{array}$$

where the square is a pull-back and

$$A \xrightarrow{i} A \oplus P \xrightarrow{j} A = 1_A, \; P \xrightarrow{i'} A \oplus P \xrightarrow{j'} P = 1_P\,.$$

By Lemma 2.61 of [5],

$$0 \longrightarrow Q \xrightarrow{i''} Q' \xrightarrow{g \circ j'} P \longrightarrow 0$$

is an exact sequence which splits since $P$ is projective. Let $f: P \to Q'$ be such that $f \circ g \circ j' = 1_P$. Since $g$ is epic and the square is a pull-back, $g'$ is also epic. We claim $g'$ is minimal. Let $h': C \to Q$ be such that $h' \circ g'$ is epic. Let $h = (h' \circ i'') \oplus f$. Consider the following commutative diagram

[1] While this paper was being written we found out that this question has been recently answered independently by A. Koehler [12], K. R. Fuller, D. A. Hill and J. Golan for the category of $R$-modules.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & C \oplus P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_P} & & \\
0 & \longrightarrow & Q & \longrightarrow & Q' & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1_P} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & A \oplus P & \xrightarrow{j} & P & \longrightarrow & 0
\end{array}$$

where the top row is split exact with the obvious maps. By the 5-lemma, $h \circ g$ is epic and since $g$ is minimal, $h$ is epic. Since

$$C \longrightarrow C \oplus P \longrightarrow P \longrightarrow 0$$

is exact, again by Lemma 2.61 of [5], the left top square is a pull-back. Since $h$ is epic, $h'$ is also epic. Thus $g'$ is minimal epic. Since $P$ is projective and $u: P \rightarrow A$, there exists $v: P \rightarrow Q$ such that $v \circ g' = u$. By the minimality of $g'$, $v$ is an epimorphism. Then the quasi-projectivity of $Q \oplus P$ and the Lemma 3.2 imply that $Q$ is projective. Thus $g': Q \rightarrow A$ is a projective cover of $A$ and we conclude that the category is perfect.

REMARK 1. Theorem 3.4 is best possible in the sense that it fails to be true if $\mathscr{A}$ is not an abelian category. To see this, let $\mathscr{A}_b$ be the category of all the abelian groups and $\mathscr{A}$ the full subcategory of $\mathscr{A}_b$ consisting of all the cyclic groups. Then $\mathscr{A}$ is not abelian. $\mathscr{A}$ has enough projectives and is clearly quasi-perfect (every object in $\mathscr{A}$ is quasi-projective). But $\mathscr{A}$ is not perfect since the prime cyclic group $Z(p)$ possesses no projective cover in $\mathscr{A}$.

REMARK 2. A quasi-perfect abelian category need not possess enough projectives. The category $\mathscr{F}_p$ of all finite abelian $p$-groups is one such. The quasi-projectives in $\mathscr{F}_p$ are the direct sums of isomorphic cyclic $p$-groups [7]. $\mathscr{F}_p$ is abelian and is readily seen to be quasi-perfect. But it possesses no non-trivial projectives.

**4. Quasi-projectives in the category of modules.** In this section we indicate some of the simple properties of quasi-projective modules over a ring. We also investigate when a quasi-projective module over a ring $R$ without zero-divisors becomes projective. It turns out in a surprisingly simple way that the "big" torsion-free quasi-projectives over such $R$ are projective. Some of the preliminary lemmas in this section hold in any abelian category but, for the sake of convenience, we will consider only the module case. Lemmas 4.3 and 4.4 occur in [7], but are proved here for the same of completeness.

LEMMA 4.1. [14]. *If $A$ is a quasi-projective $R$-module and $S$ is fully invariant in $A$, then $A/S$ is quasi-projective.*

COROLLARY. *Let $I$ be a two sided ideal of a ring $R$. Then $R/I$ is quasi-projective as an $R$-module.*

The converse of Lemma 4.1 is not always true. It holds, however, under some restriction on $S$, as indicated below.

LEMMA 4.2. *Let $S$ be a small submodule of a quasi-projective module $A$. Then $A/S$ is quasi-projective if and only if $S$ is fully invariant in $A$.*

To prove this, replace the word, "projective" in the proof of proposition 2.2 of [14] by "quasi-projective".

The following lemma gives a condition when a submodule of a quasi-projective module becomes a summand.

LEMMA 4.3. *Let $S$ be a submodule of a quasi-projective module $A$. Then $S$ is a summand if and only if $A/S$ is isomorphic to a summand of $A$.*

*Proof.* Let $A = B \oplus C$ and $f\colon B \to A/S$ be an isomorphism. Define $g\colon A \to A/S$ by $g \mid B = f$ and $g \mid C = 0$[2]. By the quasi-projectivity of $A$, $g$ lifts to an endomorphism $h$ of $A$ such that $h \circ p = g$, where $p\colon A \to A/S$ is the natural map. Set $p' = f^{-1} \circ h$. Since $p' \circ p = 1_{A/S}$, the sequence $0 \to S \to A \to A/S \to 0$ splits and thus $S$ is a summand of $A$.

Dualising 4.3, we obtain a corresponding statement for quasi-injectives.

LEMMA 4.3′. *Let $S$ be a submodule of a quasi-injective module $A$. Then $S$ will be a summand if and only if $S$ is isomorphic to a summand of $A$.*

REMARK. Lemma 3.2 and 3.2′ can also be easily deduced from 4.3 and 4.3′ respectively.

LEMMA 4.4. *Let $A$ be a quasi-projective module. Then the exact*

[2] $g \mid B$ denotes the restriction of the map $g$ to $B$.

*sequence* $0 \to T \to S \xrightarrow{f} A \to 0$ *splits, whenever* $S$ *is a submodule of* $A$.

*Proof.* Let $g\colon A \to A/T$ be an epimorphism such that $g \mid S = f$. Let $h\colon A \to A/T$ be monic with $\operatorname{Im} h = S/T$. Then there exists an endomorphism $h'$ of $A$ satisfying $h' \circ g = h$. Since $\operatorname{Im} h' = S$, it is readily seen that $h'$ is a split map of the sequence $0 \to T \to S \xrightarrow{f} A \to 0$. Hence the Lemma.

Dualising 3.4, we obtain an analogous property of quasi-injectives.

LEMMA 4.4′. *If* $A$ *is quasi-injective, then the exact sequence* $0 \to A \xrightarrow{i} X \to Y \to 0$ *splits whenever* $X$ *is a quotient of* $A$.

As an easy application of Lemma 4.4 we show that big torison-free quasi-projectives over an integral domain are projective.

THEOREM 4.5. *Let* $R$ *be a ring without zero divisors. Then any torsion-free quasi-projective* $R$*-module containing an* $R$*-independent subset of cardinality exceeding the cardinality of* $R$ *is projective.*

We may assume, without loss in generality, that $R$ is infinite (since otherwise $R$ becomes a field). Let $A$ be a quasi-projective torsion-free $R$-module and $S$ a maximal $R$-independent subset with $|S| \geqq |R|$. Let $F$ be the (free) submodule generated by $S$. Then $|A| = |S| \cdot |R| = |S|$ and so $A$ can be obtained as an epimorphic image of $F$. Since $F$ is free, $A$ is projective by Lemma 4.4.

REMARK. (i) From the proof of 4.5 it is clear that, if $R$ has no zero divisors, then a torsion-free quasi-projective $R$-module $A$ is projective exactly when $\bigoplus_m A$ is quasi-projective for every cardinal $m$.

(ii) K. H. Fuller and D. A. Hill (Notices, Amer. Math. Soc., 16 (1969) 961) show that if $A$ is finitely generated quasi-projective, then $\bigoplus_m A$ is quasi-projective for any $m$. An immediate deduction from (i) above: *If* $R$ *has no zero divisors, then a finitely generated torsion-free quasi-projective* $R$*-module is projective.*

COROLLARY 4.6. *A quasi-projective module over a ring without zero divisors is projective if and only if it is torsion-free and possesses a projective cover.*

We need only to prove the "if" part. Let $A$ be torsion-free quasi-projective and $A \cong P/S$, $P$ projective and $S$ small. By Lemma 4.2, $S$ is fully invariant in $P$. If $m$ denotes the cardinality of $R$, then $\bigoplus_m A \cong (\bigoplus_m P)/(\bigoplus_m S)$ is quasi-projective, since $\bigoplus_m S$ is fully invariant in $\bigoplus_m P$. The projectivity of $A$ then follows from Thorem 4.5.

REMARK. One can deduce that over a ring without zero divisors a quasi-projective module with a projective cover is either torsion or torsion-free. For, suppose $A$, $P$ and $S$ are as in the preceeding proof and $A$ contains a torsion-free element $a \neq 0$. *If* $m > \aleph_0 \cdot |R| \cdot |A|$, then $\bigoplus_m A$ is quasi-projective, has cardinality $m$ and contains a free submodule $F$ of rank $m$. By Lemma 4.4, $\bigoplus_m A$ and hence $A$ is projective (and torsion-free).

The following theorem characterises Artin Semisimple rings by means of quasi-projectives.

THEOREM 4.7. *The following properties are equivalent for any ring $R$:*

(i) *$R$ is Artin Semi-simple.*

(ii) *The $R$-modules with a projective cover are precisely the quasi-projectives.*

(iii) *Every quasi-projective $R$-module is projective.*

*Proof.* Trivially (i) implies (ii).

Assume (ii). Let $Q$ be quasi-projective. By assumption $Q$ possesses a projective cover $P$. Then $P \oplus Q$ will have a projective cover and hence is quasi-projective by hypothesis. Lemma 3.2 then implies that $Q$ is projective.

Assume (iii). Since any simple $R$-module is quasi-projective, it becomes projective by assumption. Then all the maximal left ideals of $R$ are direct summands of the left $R$-module $R$ and since $R$ has 1, we conclude that $R$ is Artinian Semi-simple. This completes the proof.

REMARK 1. Observe that if every $R$-module is quasi-projective then, by Lemma 3.2, $R$ satisfies the condition (iii) above and hence $R$ is Artinian Semi-simple.

REMARK 2. Johnson and Wong [9] showed that the quasi-injective modules over any ring $R$ are exactly the fully invariant submodules of injective $R$-modules. A natural question is whether this can be dualised to quasi-projectives. Precisely, *must every quasi-projective $R$-module $A$ be of the form $P/S$ with $P$ projective and $S$ fully invariant in $P$*? Jans and Wu [14] answered this in the affirmative under the assumption that $A$ has a projective cover. In the general case, the answer turns out to be in the negative. To see this, consider $M = \bigoplus (Z/pZ)$, where $Z$ is the ring of integers, $\bigoplus$ is a $Z$-module direct sum and $p$ runs over the set of all primes in $Z$. Clearly $M$ is a quasi-projective $Z$-module [7]. But $M$ cannot be written as $P/S$, where $P$ is a projective (hence free) abelian group and $S$ fully invariant in $P$, since the only fully invariant subgroups of a free abelian group $F$ are of the form $nF$, $n = 1, 2, \cdots$.

REMARK 3. In the statement of the Theorem 4.7 (ii), if we replace "precisely" by "necessarily", we obtain a characterisation of Jacobson semi-simple rings: *A ring $R$ is Jacobson semi-simple if and only if the $R$-modules possessing projective covers are necessarily quasi-projective.* To see this, assume the "if" part. Then, by Lemma 4.2, the small submodules of any projective $R$-module $P$ are fully invariant in $P$. In particular, let $P = R_1 \bigoplus R_2$ with $R_i = R$ and let $J_i = J$, the Jacobson radical of $R$, for $i = 1, 2$. Now $J_1$ is small in $R_1$ and hence in $P$. But then $J_1$ would be fully invariant in $P$, an impossibility since $J_1$ can be mapped onto $J_2$ by an endomorphism of $P$. Thus $J_1 = 0$ and $R$ is Jacobson Semi-simple. The converse follows on noting that if $R$ is Jacobson Semi-simple, then 0 is the only small submodule of any projective $R$-module.

5. **Quasi-projectives over Dedekind domains.** In this section we propose to describe the quasi-projective modules over an arbitrary Dedekind domain $R$. First, observe that if $A$ is any quasi-projective $R$-module, then any exact sequence $0 \to S \xrightarrow{i} A \xrightarrow{j} A/S \to 0$ yields the following two exact sequences.

$$0 \longrightarrow \operatorname{Hom}_R(A, S) \xrightarrow{i'} \operatorname{Hom}_R(A, A) \xrightarrow{j'} \operatorname{Hom}_R(A, A/S) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Ext}^1_R(A, S) \xrightarrow{i''} \operatorname{Ext}^1_R(A, A) \xrightarrow{j''} \operatorname{Ext}_R(A, A/S) \longrightarrow 0 .$$

We first consider the torsion free quasi-projective modules. To avoid the trivial situations, the integral domains that we consider are not fields, unless explicitly stated.

LEMMA 5.1. *Let $R$ be a Dedekind domain. Then the quotient field $K$ of $R$ is a quasi-projective $R$-module if and only if $R$ is a complete discrete valuation ring.*

*Proof.* Suppose $K$ is quasi-projective. Given any $f \in \operatorname{Hom}_R(K/R, K/R)$, there exists a $f' \in \operatorname{Hom}_R(K, K)$ such that $f' \circ j = j \circ f$ where $j$ is the natural map from $K$ onto $K/R$. Let $f'' = f' \mid R$. Since $Rf' \subseteqq R$, $f''$ is given by a multiplication by an element of $R$. It is readily seen that the association $f \mapsto f''$ gives an isomorphism of $\mathrm{Hom}_R(K/R, K/R)$ onto $R$. Now the exact sequence $0 \to R \to K \to K/R \to 0$ yields an exact sequence

$$\mathrm{Hom}_R(K/R, K) = 0 \longrightarrow \mathrm{Hom}_R(K/R, K/R) \longrightarrow \mathrm{Ext}^1_R(K/R, R) \longrightarrow \mathrm{Ext}^1_R(K/R, K) = 0$$

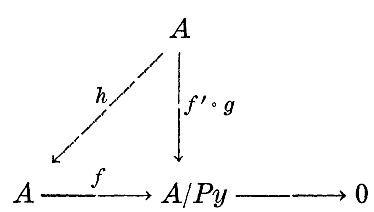
(the first term is zero since $K/R$ is torsion and $K$ is torsion-free). Thus $R \cong \mathrm{Hom}_R(K/R, K/R) \cong \mathrm{Ext}^1_R(K/R, R)$ and the Corollary 7.9 of [13] implies that $R$ is a complete discrete valuation ring.

Conversely, suppose $R$ is a complete discrete valuation ring. Then any $R$-submodule $S$ of $K$ is isomorphic to $R$ or $K$ and hence, by Theorem 7.9 of [13], $\mathrm{Ext}^1_R(K, S) = 0$. $K$ is then clearly quasi-projective.

We shall first describe the torsion-free quasi-projectives over Dedekind domains which are not complete discrete valuation rings.

LEMMA 5.2. *Suppose $R$ is a Dedekind domain which is not a complete discrete valuation ring. Then any torsion-free quasi-projective $R$-module $A$ is torsionless.*

*Proof.* Let $0 \neq x \in A$ and $S$ the pure submodule generated by $x$. Since $R$ is not a complete discrete valuation ring, $A$ (and therefore $S$) is reduced, by Lemma 3.1. Thus $S \neq PS$ for some prime ideal $P$ of $R$. Then $S/PS$, being bounded and pure, is a summand of $A/PS$ (Theorem 5 [11]). A nonzero cyclic summand of $S/PS$ will be isomorphic to $R/P$ and can be written as $Ry/Py$, for some $y \in S$. Let $g$: $S/PS \to Ry/Py$ be a nonzero map. Consider the following diagram




where $f'$: $A \to S/PS$ is obtained via the projection $A/PS \to S/PS$ and $f$ is the natural map. By the quasi-projectivity of $A$, there exists $h$: $A \to A$ making the diagram commutative. Now $A(h \circ f) = A(f' \circ g) \subseteq Ry/Py$, so that $Ah \subseteq Ry$. Thus $h$: $A \to Ry \cong R$ and $xh \neq 0$ since $h$ does not vanish on the rank 1 submodule $S$. It follows that $A$ is torsionless.

COROLLARY 5.3. *Let $R$ be a Dedekind domain which is not a complete discrete valuation ring. Then any torsion-free $R$-module $A$ is $\aleph_1$-projective. Hence any torsion-free $R$-module of atmost countable rank is projective.*

*Proof.* Let $S$ be a submodule of $A$ of rank 1. By Lemma 5.2, $A$ is torsionless so that for each $a \neq 0$ in $S$, there exists $f: A \to R$ such that $af \neq 0$. Since $S$ has rank 1 and im $f$ is torsion-free, $f \mid S$ is mono. As $R$ is hereditary, $S$ is projective. By finite induction, it is clear that any submodule of $A$ of finite rank is projective. Then a well-known step-wise argument (see for example Lemma 8.3.1 [13]) yields that any submodule of countable rank of $A$ is projective.

In the following $\sigma$ denotes cardinality of the set of all distinct prime ideals of $R$.

PROPOSITION 5.4. *Let $R$ be a Dedekind domain. Then any torsion-free quasi-projective of rank $m \geqq \sigma\aleph_0$ is projective.*

*Proof.* Let $A$ be a torsion-free $R$-module of rank $m \geqq \sigma\aleph_0$ and $K$ be the quotient field of $R$. It is easy to see that $R(P^\infty)$ is countably generated. Now $K/R$ is $\bigoplus_P R(P^\infty)$, where $P$ runs over the set of distinct non-zero prime ideals of $R$ and hence $K$ has a generating set of cardinality $\sigma\aleph_0$. If $D$ is an injective hull of $A$, then $D \cong \bigoplus_m K$ has a generating set of cardinality $m$. It is then readily seen that $A$ itself is generated by $m$ elements. Let $F$ be a free submodule of $A$ of rank $m$ (for example $F$ may be the submodule generated by a maximal $R$-independent subset of $A$). $A$ can be got as an epimorphic image of $F$ and hence by Lemma 4.4, $A$ is a direct summand of $F$ and hence projective.

Combining 5.3 and 5.4, we get the following.

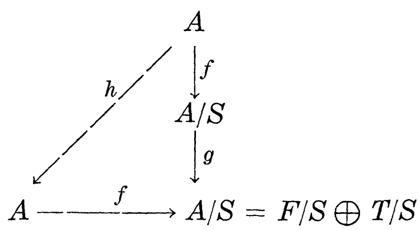
THEOREM 5.5. *Let $R$ be a Dedekind domain which is not a complete discrete valuation ring and $\sigma \leqq \aleph_0$. Then a torsion-free $R$-module is quasi-projective if and only if it is projective.*

REMARK. If we assume the continuum hypothesis and use 5.3 and 5.4, then we can sharpen 5.5 to the following: *Let $R$ be a Dedekind domain wich is not a complete discrete valuation ring and $\sigma \leqq 2^{\aleph_0}$. Then any torsion-free quasi-projective $R$-module is projective.*

Next we consider the case when $\sigma > 2^{\aleph_0}$.

PROPOSITION 5.6. *Let $R$ be a Dedekind domain and $A$ be a* *torsion-free quasi-projective $R$-module of infinite rank $m$. Then $A$ contains a free summand of rank $m$.*

*Proof.* Let $P$ be any non-zero prime ideal of $R$. $R(P^{\infty})$ is a countably generated injective $R$-module. If $Q = \bigoplus_m R(P^{\infty})$, then, as $R$ is Noetherian, $Q$ is an injective $R$-module. Clearly $Q$ has a generating set of cardinality $m$. Let $F$ be the free-submodule generated by a maximal $R$-independent subset of $A$. Then $Q$ can be obtained as a quotient of $F$, $Q \cong F/S$ for some submodule $S$. Consider the following diagram,




where $g\colon A/S \to F/S$ is a projection of $A/S$ onto the injective summand $F/S$ and $f$ is the natural map. By the quasi-projectivity of $A$, there exists $h\colon A \to A$ such that $h \circ f = f \circ g$. It is clear that $Ah \subseteqq F$ and since $R$ is hereditary $Ah$ is projective. As $F/S$ is a direct sum of $m$ copies of $R(P^{\infty})$, it is clear that the rank of $Ah = m$. Thus $A = F' \oplus K$, where $K$ is the kernel of $h$ and $F'$ is a projective module of infinite rank $m$ and hence is free **[11]**.

Combining 5.3, 5.4 and 5.6 we get,

THEOREM 5.7. *Let $R$ be a Dedekind domain with $\sigma > 2^{\aleph_0}$. Then any torsion-free quasi-projective $R$-module $A$ is projective if either* (i) *rank $A \leqq \aleph_0$ or* (ii) *rank $A \geqq \sigma$. In the case when $\aleph_0 <$ rank $A < \sigma$, $A$ is torsionless, $\aleph_1$-projective and contains a free summand $F$ having the same rank as $A$.*

The following theorem characterises torsion-free quasi-projectives over a complete discrete valuation ring.

THEOREM 5.8. *Suppose $R$ is a complete discrete valuation ring. Then the torsion-free quasi-projective $R$-modules are just the free $R$-modules and the torsion-free $R$-modules of finite rank.*

*Proof.* By Kaplansky [10], any torsion-free $R$-module of finite rank is of the form $(\bigoplus_{i=1}^{n} K_i) \oplus (\bigoplus_{j=1}^{m} R_j)$ where each $R_j \cong R$ and each $K_i \cong K$, the quotient field of $R$. Thus if $A$ is any finite rank torsion-free $R$-module and $S$ is any submodule, then both are direct sums of finite number of copies of $K$ and $R$, so that

$$\mathrm{Ext}^1_R(A, S) \cong \bigoplus_r \mathrm{Ext}^1_R(K, R) ,$$

where $r$ is finite. By Lemma 5.1, $K$ is quasi-projective so that

$$\mathrm{Ext}^1_R(K, R) = 0 .$$

Thus $\mathrm{Ext}^1_R(A, S) = 0$, whence $\mathrm{Hom}_R(A, A) \xrightarrow{f'} \mathrm{Hom}_R(A, A/S) \to 0$ is exact for every submodule $S$ of $A$, where $f'$ is induced by the natural map $f\colon A \to A/S$. The quasi-projectivity of $A$ then follows. On the other hand if $A$ is a torsion-free quasi-projective $R$-module of infinite rank, then by Proposition 5.4, $A$ is projective and hence free.

COROLLARY 5.9. *If $A$ is quasi-projective, then a direct sum $\bigoplus A$ of copies of $A$ need not be quasi-projective.*

EXAMPLE. Suppose $A$ is any torsion-free module of finite rank over a complete discrete valuation ring $R$ such that $A$ is not projective (for example $A = K$, the quotient field of $R$). Then any finite direct sum of copies $A$ is quasi-projective but, by 5.8, no direct sum of infinite number of copies of $A$ can be quasi-projective.

We shall now describe the torsion quasi-projectives over $R$.

THEOREM 5.10. *A torsion module $A$ over a Dedekind domain $R$ is quasi-projective if and only if each $P$-primary component $A_P$ is a direct sum copies of the same cyclic module $R/P^k$ for some fixed positive integer $k$ depending on $P$.*

*Proof.* Since a $P$-primary module over $R$ can be viewed as a module over the principal ideal domain $R_P$, and quasi-projectivity survives under this transition, we may assume that $R$ itself is a principal ideal domain. Our proof would be sketchy since it is similar to the one given in [7]. Now $R(P^\infty)$ is not quasi-projective since otherwise, by Lemma 4.3, every submodule of $R(P^\infty)$ would be a summand. Thus a torsion quasi-projective $R$-module $A$ is necessarily reduced. Again, by Lemma 4.3, $A$ cannot contain a summand of the form $(R/P^{k_1}) \oplus (R/(P^{k_2}))$ with $k_1 > k_2$, since there is an epimorphism $R/(P^{k_2}) \to R/(P^{k_1})$ whose kernel is not a summand. Thus the basic submodules $B_P$ (see [6]) of each $P$-primary component $A_P$ are bounded and since the $A_P$ are reduced, each $A_P$ coincides with $B_P$ which is clearly a direct sum of isomorphic cyclic modules. The "only if" part follows.

Conversely, if $A$ is a direct sum $\bigoplus_m R/(P^k)$ of isomorphic cyclic modules, then $A \cong F/P^k F$, where $F$ is free, say, $F = \bigoplus_m R$. Since

$P^kF$ is fully invariant in $F$, $A$ is quasi-projective, by 4.1.

COROLLARY 5.11. *A torsion module $A$ over a Dedekind domain $R$ is quasi-projective if and only if $A$ is quasi-injective but not injective.*

*Proof.* By Johnson and Wong [9], the quasi-injectives are precisely the fully invariant submodules of injective modules. The corollary then follows on noting that $P$-primary injective $R$-modules are direct sums of copies of $R(P^\infty)$ and their proper fully invariant submodules are direct sums of isomorphic cyclic $P$-primary modules.

The following theorem concludes our investigation of quasi-projectives over Dedekind domains.

THEOREM 5.12. *A quasi-projective module over a Dedekind domain is either torsion or torsion-free.*

*Proof.* Suppose $A$ is a quasi-projective $R$-module with its maximal torsion submodule $A_t \neq 0$. Since $R(P^\infty)$ is not quasi-projective for any prime ideal $P$, $A_t$ is reduced and thus $A$ has torsion cyclic summands [11]. Let $A = (R/P^k) \oplus B$. Now if $R$ is not a complete discrete valuation ring, $B/B_t$ is torsion-free quasi-projective and hence is torsionless (5.2) so that $B$ has a projective summand $I$ of rank 1. If $R$ is a complete discrete valuation ring, then as in the proof of 5.10, one can then show that $B_t = B_P$ is a bounded direct sum of isomorphic cyclic modules, where $P$ is the unique nonzero prime ideal of $R$. Hence $B = B_P \oplus B/B_P$, so $B/B_P$ is a torsion-free quasi-projective $R$-module and hence contains a summand isomorphic to $R$ or $K$, the quotient field of $R$ (5.8). Thus, in either case, $A$ has a summand of the form $(R/P^k) \oplus C$, where $C \cong K$, the quotient field of $R$ or $C \cong I$, an ideal of $R$. Choose a submodule $S$ of $C$ such that $S \cong R$ or $S \cong IP^k$ according as $C \cong K$ or $C \cong I$. Then there exists a nonzero morphism $g\colon R/P^k \to C/S$. Consider the following diagram.

$$\begin{array}{ccc} & & (R/P^k) \oplus C \\ & & \big\downarrow {\scriptstyle g'} \\ (R/P^k) \oplus C & \xrightarrow{\ f'\ } & (R/P^k) \oplus (C/S) \end{array}$$

where $f' = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}$, $f$ being the natural map and $g' = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}$, where $g$ is any nonzero homomorphism $R/P^k \to C/S$. This $g'$ cannot be lifted to an endomorphism $h$ of $(R/P^k) \oplus C$ satisfying $h \circ f = g'$, a contradiction. We thus conclude that $A$ is either torsion or torsion-free.

**6. Perfect rings.** In this section perfect rings are characterised by means of quasi-projective $R$-modules.

THEOREM 6.1. *Let $R$ be any ring. Then the following properties are equivalent.*

( i ) *$R$ is left perfect.*

(ii) *A direct limit of quasi-projective left $R$-modules is quasi-projective.*

(iii) *A direct limit of finitely generated quasi-projectives over $R$ is quasi-projective.*

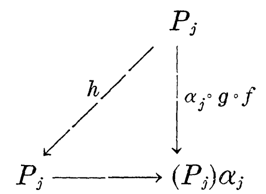
(iv) *Any flat left $R$-module is quasi-projective.*[3]

*Proof.* Let $Q = \varinjlim Q_i$, $i \in I$ where $I$ is a directed set and the $Q_i$'s are quasi-projective $R$-modules. To each $i \in I$, there exists, by hypothesis, an exact sequence $0 \to K_i \xrightarrow{u_i} P_i \xrightarrow{v_i} Q_i \to 0$ where $P_i$ is projective and $K_i$ is small in $P_i$. Now $\{P_i\}_{i \in I}$ and $\{K_i\}_{i \in I}$ can be made into directed systems in a natural way so that we get a directed system of exact sequences. Let $K = \varinjlim K_i$ and $P = \varinjlim P_i$. Suppose for each $i \in I$ $\alpha_i$: $P_i \to P$ and $\beta_i$: $K_i \to K$ are the natural maps associated with the direct limits. Since the direct limit commutes with exact sequences, $0 \to K \xrightarrow{u} P \xrightarrow{v} Q \to 0$ is exact. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \xrightarrow{u_i} & P_i & \xrightarrow{v_i} & Q_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \alpha_i} & & \downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow{u} & P & \xrightarrow{v} & Q & \longrightarrow & 0 . \end{array}$$

We claim that $Ku$ is fully invariant in $P$. Let $f \in \mathrm{End}_R(P)$ and $k \in K$. As $R$ is perfect, $P$ is a direct sum of cyclic projective $R$-modules [12]. Let $P'$ be a finitely generated summand of $P$ containg $(k)u$ and let $P \xrightarrow{g} P'$ be the natural projection. As $(P')f$ is finitely generated, we can choose a $j \in I$ and a $k_j \in K_j$ such that $(P_j)\alpha_j \supset (P')f$ and $(k_j)\beta_j = k$. Consider the following diagram:

[3] In a private communication Dr. J. Golan has indicated that he has also proved the equivalence of (iv) and (i).




where $h$ exists by the projectivity of $P_j$. As $(K_j)u_j$ is fully invariant in $P_j$ (by 4.2), $(k_j)u_j h \in (K_j)u_j$. Now

$$(k)u \circ f = (k)u \circ g \circ f (\text{as } g \mid P' = 1_{P'}) = (k_j)\beta_j \circ u \circ g \circ f = (k_j)u_j \circ \alpha_j \circ g \circ f$$
$$= (k_j)u_j \circ h \circ \alpha_j \in (K_j)u_j \circ \alpha_j = (K_j)\beta_j \circ u \subseteqq (K)u .$$

Thus $(K)u$ is fully invariant in $P$ whence $Q \cong P/(K)u$ is quasi-projective.

Clearly (ii) $\Rightarrow$ (iii) and, since a flat module is a direct limit of finitely generated projectives, (iii) implies (iv).

Assume (iv). Let $A$ be flat and $P$ projective such that $A \cong P/S$. Since $A \oplus P$ is flat, it is quasi-projective, by hypothesis. Then Lemma 3.2 implies that $A$ is projective. Thus a direct limit of projective left $R$-modules is projective and so $R$ is left perfect, by theorem $P$ of [2]. This proves (i).

REMARK. If $R$ is left perfect and $A$ is a quasi-projective lelf $R$-module, then a direct sum of any number of carbon copies of $A$ is again quasi-projective. This property, however, does not characterize the perfect rings. Indeed, the investigations made in § 5 show that if $R$ is a countable Dedekind domain which is not a complete discrete valuation ring and $A$ is a quasi-projective $R$-module, then $\bigoplus_m A$ is quasi-projective for any cardinal number $m$.

## 7. Generalization.

In this section, we consider a weakened form of quasi-projectivity called $w$. quasi-projectives. The $w$. quasi-projective abelian groups were considered in [8]. We give a description of $w$. quasi-projectives over a Dedekind domain. It is also shown that $w$. quasi-perfect abelian categories with enough projectives are perfect.

DEFINITION. An object $A$ in a category $\mathscr{A}$ is called *weak quasi-projective* (for short, $w$. quasi-projective) if for any epimorphism $f: A \to B$ and any $g: A/B \to A/B$, there is a $g': A \to A$ making the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A/B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\quad f \quad} & A/B \end{array}$$

commutative.

It is clear that any quasi-projective is weak quasi-projective. But the converse is not true. The abelian group $Z(P^{\infty})$ is $w.$ quasi-projective, eventhough it is not a quasi-projective $Z$-module.

We start with the following lemma which gives a criterion for quasi-projectivity. The proof is straight forward and hence is omitted.

LEMMA 7.1. *An $R$-module $A$ is quasi-projective if and only if $A \oplus A$ is weak quasi-projective.*

REMARK. It is clear from 7.1 that, unlike the quasi-projective case, if $A$ is $w.$ quasi-projective then $A \oplus A$ need not be $w.$ quasi-projective.

The next lemma can be obtained by modifying the arguments of 3.2.

LEMMA 7.2. [8]. *If $A \oplus B$ is $w.$ quasi-projective and there is an epimorphism $f: A \to B$, then $B$ will be isomorphic to a summand of $A$.*

One can define a weak quasi-perfect category in the obvious manner. Using Lemma 7.1 and proceeding exactly as in the proof of Theorem 3.4, we obtain.

THEOREM 7.3. *A weak quasi-perfect abelian category with enough projectives is perfect.*

If we suitably modify the preceding investigation of the quasi-projectives over a Dedekind domain and make use of Lemma 7.2 we can obtain the following theorem whose proof is omitted.

THEOREM 7.4. *Let $R$ be a Dedekind domain.*

( i ) *A torsion $R$-module $A$ is weak quasi-projetive if and only if* *each P-primary component* $A_P$ *is either quasi-projective or* $A_P \cong R(P^\infty)$.

(ii) *If the number* $\sigma$ *of prime ideals of* $R$ *is* $\leqq 2^{\aleph_0}$ *then the torsion-free weak quasi-projectives are just the (torsion-free) quasi-projectives. If* $\sigma > 2^{\aleph_0}$, *then a torsionfree weak quasi-projective* $R$-*module* $A$ *is projective if either* $A$ *has rank* $\leqq \aleph_0$ *or* (ii) *rank* $A > \sigma$. *If* $\aleph_0 < \text{rank}\, A < \sigma$, $A$ *is* $\aleph_1$-*projective and contains a free summand* $F$ *whose rank is equal to rank* $A$.

(iii) *A properly mixed* $R$-*module* $A$ *is weak quasi-projective if and only if* $A \cong B \oplus C$ *where* $B$ *is reduced torsion-free quasi-projective of finite rank and* $C$ *is an injective submodule of* $K/R$, *where* $K$ *is the quotient field of* $R$.

The authors are indebted to the referee for pointing out a few inaccuracies and for offering many suggestions for improvement.

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MADURAI UNIVERSITY
AND
THE AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, A.C.T. 2600, AUSTRALIA
AND
THE UNIVERSITY OF BOMBAY
SANTA CRUZ (EAST), BOMBAY-28, INDIA

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# ON THE UNIVALENCE OF SOME ANALYTIC FUNCTIONS

G. M. SHAH

**Let**

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

**and**

$$g(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

**be analytic and satisfy**

(a) $$\mathbf{Re}\,(f(z)/[\lambda f(z) + (1-\lambda)\,g(z)]) > 0$$

**or**

(b) $$|\,f(z)/[\lambda f(z) + (1-\lambda)\,g(z)] - 1\,| < 1$$

**for** $|\,z\,| < 1,\ 0 \leqq \lambda < 1$ **.**

**We propose to determine the values of** $R$ **such that** $f(z)$ **is univalent and starlike for** $|\,z\,| < R$ **under the assumption (i)** $\mathbf{Re}\,(g(z)/z) > 0$**, or (ii)** $\mathbf{Re}\,(zg'(z)/g(z)) > \alpha,\ 0 \leqq \alpha < 1$**.**

**We also consider the case when** $n = 1$ **and** $\mathbf{Re}(g(z)/z) > 1/2$ **and show that under condition (a)** $f(z)$ **is univalent and starlike for** $|\,z\,| < (1-\lambda)/(3+\lambda)$**.**

2. LEMMA 1. *If* $p(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ *is analytic and satisfies* $\mathrm{Re}(p(z)) > \alpha$, $0 \leqq \alpha < 1$, *for* $|\,z\,| < 1$, *then*

(1) $$p(z) = [1 + (2\alpha - 1)z^n u(z)]/[1 + z^n u(z)]\ , \quad \textit{for}\ |\,z\,| < 1\ ,$$

*where* $u(z)$ *is analytic and* $|\,u(z)\,| \leqq 1$ *for* $|\,z\,| < 1$.

*Proof*. Let

(2) $$F(z) = [p(z) - \alpha]/(1 - \alpha) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots .$$

$F(z)$ is analytic and $\mathrm{Re}\,(F(z)) > 0$ for $|\,z\,| < 1$ and hence

(3) $$h(z) = [1 - F(z)]/[1 + F(z)] = d_n z^n + d_{n+1} z^{n+1} + \cdots ,$$

is analytic and $|\,h(z)\,| < 1$ for $|\,z\,| < 1$. Thus, by Schwarz's lemma

(4) $$h(z) = z^n u(z)\ ,$$

where $u(z)$ is analytic and $|\,u(z)\,| \leqq 1$ for $|\,z\,| < 1$. Now equations (2), (3) and (4) prove (1).

LEMMA 2. *Under the hypothesis of Lemma* 1 *we have for* $|\,z\,| < 1$

$| zp'(z)/p(z) | \leqq 2nz^n(1-\alpha)/\{(1-|z|^n)[1+(1-2\alpha)|z|^n]\}$.

*Proof.* Proceeding as in the proof of Lemma 1, we have in view of (3) and a result of Goluzin [1] that for $|z|<1$

$$(5) \qquad |h'(z)| \leqq n|z|^{n-1}(1-|h(z)|^2)/(1-|z|^{2n}) .$$

Using (3), the inequality (5) takes the form

$$|F'(z)| \leqq 2n|z|^{n-1}\operatorname{Re}(F(z))/(1-|z|^{2n}) .$$

Hence, in view of (2),

$$(6) \qquad |p'(z)| \leqq 2n|z|^{n-1}[\operatorname{Re}(p(z))-\alpha]/(1-|z|^{2n})$$

or,

$$(7) \qquad |zp'(z)/p(z)| \leqq 2n|z|^n(1-\alpha/(|p(z)|)/(1-|z|^{2n}) .$$

Equation (4) gives

$$(8) \qquad |h(z)| \leqq |z|^n \qquad \text{for } |z|<1 ,$$

and hence, by virtue of (3),

$$(9) \qquad |F(z)| \leqq (1+|z|^n)/(1-|z|^n) \qquad \text{for } |z|<1 .$$

From (2) and (9),

$$\begin{aligned} |p(z)| &= |\alpha+(1-\alpha)F(z)| \\ &\leqq \alpha+(1-\alpha)|F(z)| \\ &\leqq [1+(1-2\alpha)|z|^n]/(1-|z|^n) . \end{aligned}$$

The inequality (7), because of the last inequality, reduces to

$$|zp'(z)/p(z)| \leqq 2n|z|^n(1-\alpha)/\{(1-|z|^n)[1+(1-2\alpha)|z|^n]\} \text{ for } |z|<1$$

and this completes the proof.

We remark that in the case $\alpha=0$, the above lemma reduces to a result of MacGregor [2; Lemma 1] and the inequality (6) with $\alpha=0$, $n=1$, gives another result of MacGregor [2, Lemma 2].

LEMMA 3. *Under the hypothesis of Lemma* 1 *we have for* $|z|<1$ $\operatorname{Re}(p(z)) \geqq [1+(2\alpha-1)|z|^n]/(1+|z|^n)$.

*Proof.* We have from equation (3), $F(z)=[1-h(z)]/[1+h(z)]$ and also from (8), $|h(z)| \leqq |z|^n$ for $|z|<1$. Hence the image of $|z|<r$ $(0<r<1)$ under $F(z)$ lies in the interior of the circle with the line segment joining the points $(1-r^n)/(1+r^n)$ and $(1+r^n)/(1-r^n)$ as a diameter. Consequently $\operatorname{Re}(F(z)) \geqq (1-|z|^n)/(1+|z|^n)$ for

$|z| < 1$. The result now follows from the last inequality involving $F(z)$ and equation (2).

LEMMA 4. ([6]). *If* $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$ *is analytic and* $\operatorname{Re}(h(z)) > 0$ *for* $|z| < 1$, *then*

$$[1 - \lambda |h(z)|]^{-1} \leqq (1 - |z|^n)/[(1 - |z|^n) - \lambda(1 + |z|^n)]$$

*for* $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$, *where* $0 \leqq \lambda < 1$.

3. THEOREM 1. *Suppose that* $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots$, *and* $g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \cdots$ *are analytic and* $\operatorname{Re}(g(z)/z) > 0$ *for* $|z| < 1$. *If* $\operatorname{Re}(f(z)/[\lambda f(z) + (1 - \lambda) g(z)]) > 0$, $0 \leqq \lambda < 1$, *for* $|z| < 1$, *then* $f(z)$ *is univalent and starlike for* $|z| < R^{1/n}$, *where* $R = \{[(2n + \lambda - n\lambda)^2 + (1 - \lambda^2)]^{1/2} - (2n + \lambda - n\lambda)\}/(1 + \lambda)$.

*Proof.* Let

$$h(z) = f(z)/[\lambda f(z) + (1 - \lambda) g(z)] = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots ,$$

then $h(z)$ is analytic and $\operatorname{Re}(h(z)) > 0$ for $|z| < 1$. Now

$$f(z)\,[1 - \lambda h(z)] = (1 - \lambda) h(z) z p(z) , \tag{10}$$

where $p(z) = g(z)/z = 1 + b_{n+1} z^n + b_{n+2} z^{n+1} + \cdots$. Multiplying the logarithmic derivative of both sides of equation (10) by $z$ we have

$$z f'(z)/f(z) = 1 + z p'(z)/p(z) + z h'(z)/\{h(z)[1 - \lambda h(z)]\} . \tag{11}$$

Equation (11) is valid for those $z$ for which $1 - \lambda h(z) \neq 0$ and $|z| < 1$. Since $|h(z)| \leqq (1 + |z|^n)/(1 - |z|^n)$, $1 - \lambda h(z) \neq 0$ in particular if $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$. Now from equation (11), we have

$$|z f'(z)/f(z) - 1| \leqq |z p'(z)/p(z)| + |z h'(z)/h(z)|\,|1 - \lambda h(z)|^{-1}$$

and by using Lemma 2 with $\alpha = 0$ and Lemma 4, this gives

$$\begin{aligned} |z f'(z)/f(z) - 1| &\leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{2n|z|^n}{(1 - |z|^{2n}) - \lambda(1 + |z|^n)^2} , \\ &= \frac{2n|z|^n[(1 - |z|^n) - \lambda(1 + |z|^n) + (1 - |z|^n)]}{(1 - |z|^{2n})[(1 - |z|^n) - \lambda(1 + |z|^n)]} \end{aligned} \tag{12}$$

provided that $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$.

The fact that $|z f'(z)/f(z) - 1| < 1$ implies that $\operatorname{Re}(z f'(z)/f(z)) > 0$, it follows from the inequality (12) that $\operatorname{Re}(z f'(z)/f(z) > 0$ if

$$|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$$

and if

$$G(|z|^n) \equiv (1+\lambda)|z|^{3n} + (4n + 2n\lambda + \lambda - 1)|z|^{2n} + (2n\lambda - 4n - \lambda - 1)|z|^n + (1-\lambda) > 0 . \tag{13}$$

Let $|z|^n = t$ and consider the cubic polynomial $G(t)$ for $0 \leqq t \leqq 1$. $G(t)$ has at most two positive zeros. Since $G(0) = (1-\lambda) > 0$, $G[(1-\lambda)/(1+\lambda)] = -4\lambda n(1-\lambda)/(1+\lambda)^2 < 0$ and $G(1) = 4\lambda n > 0$, it follows that $G(t_1) = 0$ for some $t_1$ such that $0 < t_1 < (1-\lambda)/(1+\lambda)$ and $G(t) > 0$ for $0 \leqq t < t_1$ and $G(t) < 0$ for $t_1 < t < (1-\lambda)/(1+\lambda)$. Hence $\operatorname{Re}(zf'(z)/f(z)) > 0$ for those $z$ for which only the inequality (13) is true. Now the inequality (13) holds if, in particular

$$(1+\lambda)|z|^{3n} + (4n - 2n\lambda + \lambda - 1)|z|^{2n} + (2n\lambda - 4n - \lambda - 1)|z|^n + (1-\lambda) > 0$$

or,

$$(|z|^n - 1)[(1+\lambda)|z|^{2n} + (4n - 2n\lambda + 2\lambda)|z|^n + (\lambda - 1)] > 0$$

or,

$$(1+\lambda)|z|^{2n} + (4n - 2n\lambda + 2\lambda)|z|^n + (\lambda - 1) < 0 .$$

The last inequality holds if

$$|z|^n < \{[(2n + \lambda - n\lambda)^2 + (1-\lambda^2)]^{1/2} - (2n + \lambda - n\lambda)\}/(1+\lambda) . \tag{14}$$

Since $f(z)$ is univalent and starlike for those $z$ for which

$$\operatorname{Re}(zf'(z)/f(z)) > 0 ,$$

we have that $f(z)$ is univalent and starlike for $|z| < R^{1/n}$, where $R$ is the right side of (14).

If we put $\lambda = 0$ in Theorem 1 we obtain the following result which, when $n = 1$, reduces to a result of Ratti [5, Theorem 1].

COROLLARY 1. *Suppose that* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2}\cdots$, *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *are analytic and* $\operatorname{Re}(g(z)/z) > 0$ *for* $|z| < 1$. *If* $\operatorname{Re}(f(z)/g(z)) > 0$ *for* $|z| < 1$ *then* $f(z)$ *is univalent and starlike for* $|z| < [(4n^2+1)^{1/2} - 2n]^{1/n}$.

The functions $f(z) = z(1-z^n)^2/(1+z^n)^2$ and $g(z) = z(1-z^n)/(1+z^n)$ satisfy the hypothesis of Corollary 1 and it is easy to see that the derivative of $f(z)$ vanishes at $z = [(4n^2+1)^{1/2} - 2n]^{1/n}$ and hence $[(4n^2+1)^{1/2} - 2n]^{1/n}$ is in fact the radius of univalence for such functions $f(z)$. This shows that Corollary 1 is sharp and hence Theorem 1 is sharp at least for $\lambda = 0$.

THEOREM 2. *Suppose* $f(z) = z + a_2z^2 + \cdots$, *and*

$$g(z) = z + b_2 z^2 + \cdots$$

*are analytic for* $|z| < 1$ *and* $\operatorname{Re}(g(z)/z) > 1/2$ *for* $|z| < 1$. *If*

$$\operatorname{Re}(f(z)/[\lambda f(z) + (1-\lambda)g(z)]) > 0 \qquad \text{for } |z| < 1$$

*then* $f(z)$ *is univalent and starlike for* $|z| < (1-\lambda)/(3+\lambda)$.

*Proof.* Let $h(z) = f(z)/[\lambda f(z) + (1-\lambda)g(z)] = 1 + c_1 z + c_2 z^2 + \cdots$. Now $h(z)$ is analytic and $\operatorname{Re}(h(z)) > 0$ for $|z| < 1$ and

$$f(z)[1 - \lambda h(z)] = (1-\lambda)h(z)g(z)\ . \tag{15}$$

If we let $g(z) = zp(z)$, then by applying Lemma 1 with $\alpha = 1/2$ and $n = 1$ we have that $p(z) = [1 + zu(z)]^{-1}$, where $u(z)$ is analytic and $|u(z)| \leqq 1$ for $|z| < 1$. Equation (15) now reduces to

$$f(z)[1 - \lambda h(z)] = (1-\lambda)zh(z)/[1 + zu(z)]\ .$$

Hence

$$\frac{zf'(z)}{f(z)} = \frac{1 - z^2u'(z)}{1 + zu(z)} + \frac{zh'(z)}{h(z)[1 - \lambda h(z)]}$$

and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \operatorname{Re}\left(\frac{1 - z^2u'(z)}{1 + zu(z)}\right) - \frac{|zh'(z)/h(z)|}{|1 - \lambda h(z)|}\ .$$

Using Lemmas 2 and 4 with $n = 1$, we get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \operatorname{Re}\left(\frac{1 - z^2u'(z)}{1 + zu(z)}\right) - \frac{2|z|}{(1 - |z|^2) - \lambda(1 + |z|)^2}$$

for $|z| < (1-\lambda)/(1+\lambda)$.

Hence $\operatorname{Re}(zf'(z)/f(z)) > 0$ if $|z| < (1-\lambda)/(1+\lambda)$ and

$$T(|z|)\operatorname{Re}[(1 - z^2u'(z))(1 + \overline{zu(z)})] - 2|z|\operatorname{Re}[(1 + zu(z))(1 + \overline{zu(z)})] > 0\ ,$$

where $T(|z|) = (1 - |z|^2) - \lambda(1 + |z|)^2$. The last inequality holds if

$$\begin{aligned} &T(|z|)\operatorname{Re}(1 + \overline{zu(z)}) - T(|z|)\operatorname{Re}[z^2u'(z)(1 + \overline{zu(z)})] \\ &\quad + 2|z|\operatorname{Re}[(1 - zu(z))(1 + \overline{zu(z)})] - 4|z|\operatorname{Re}(1 + \overline{zu(z)}) > 0\ , \end{aligned}$$

or if

$$\begin{aligned} &[4|z| - T(|z|)]\operatorname{Re}(1 + \overline{zu(z)}) + T(|z|)\operatorname{Re}[z^2u'(z)(1 + \overline{zu(z)})] \\ &< 2|z|(1 - |z|^2|u(z)|^2) \end{aligned}$$

or

$$| 4 | z | - T(| z |) | (1 + | z | | u(z) |) + T(| z |) | z |^2 | u'(z) | (1 + | z | | u(z) |) < 2 | z | (1 - | z |^2 | u(z) |^2) .$$

This inequality holds, in view of (5) with $n = 1$ if

$$| 4 | z | - T(| z |) | + T(| z |) | z |^2 (1 - | u(z) |^2)(1 - | z |^2)^{-1} < 2 | z | (1 - | z | | u(z) |) . \tag{16}$$

Two cases arise according as $4 | z | - T(| z |)$ is nonnegative or not.

*Case* 1. $4 | z | - T(| z |) \geqq 0$, i.e. $| z | \geqq [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. Since $[(4\lambda + 5)^{1/2} - (\lambda + 2)] < (1 - \lambda)$ for $0 \leqq \lambda < 1$, it follows, in view of inequality (16), that $\operatorname{Re}(zf'(z)/f(z)) > 0$ for those $z$ for which $[(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda) \leqq | z | < (1 - \lambda)/(1 + \lambda)$ and

$$4 | z | - T(| z |) + T(| z |) | z |^2 (1 - | u(z) |^2)(1 - | u(z) |^2)^{-1} < 2 | z | (1 - | z | | u(z) |) .$$

The last inequality holds, because of the original value of $T(| z |)$, if

$$2 | z | + 2 | z |^2 - 1 + \lambda(1 + | z |)^2 - \lambda | z |^2(1 + | z |)/(1 - | z |) < | z |^2 | u(z) |^2 - \lambda| z |^2 | u(z) |^2 (1 + | z |)/(1 - | z |) - 2 | z |^2 | u(z) | . \tag{17}$$

Since $| u(z) | \leqq 1$, the right side of inequality (17)

$$\geqq | z |^2 | u(z) |^2 - 2 | z |^2 | u(z) | - \lambda | z |^2 (1 + | z |)/(1 - | z |) .$$

Hence inequality (17) holds, if in particular

$$2 | z | + 2 | z |^2 - 1 + \lambda(1 + | z |)^2 < | z |^2 | u(z) |^2 - 2 | z |^2 | u(z) | . \tag{18}$$

If we let $F(x) = x^2 | z |^2 - 2x | z |^2$, where $x = | u(z) |$, $0 \leqq x \leqq 1$, then $F(x)$ is a decreasing function of $x$ for $0 \leqq x \leqq 1$, and hence

$$F(x) \geqq F(1) = -| z |^2 \qquad \text{for } 0 \leqq x \leqq 1 .$$

Hence inequality (18) holds if $2 | z | + 2 | z |^2 - 1 + \lambda(1 + | z |)^2 < -| z |^2$ or $(3 | z | - 1)(| z | + 1) + \lambda(1 + | z |)^2 < 0$ or $3 | z | - 1 + \lambda(1 + | z |) < 0$ or if $| z | < (1 - \lambda)/(3 + \lambda)$. Since $(1 - \lambda)/(3 + \lambda) < (1 - \lambda)/(1 + \lambda)$, we have shown that

$$\operatorname{Re}(zf'(z)/f(z)) > 0 \quad \text{for } [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda) \leqq | z | < (1 - \lambda)/(3 + \lambda) . \tag{19}$$

*Case* 2. $4 | z | - T(| z |) < 0$, i.e. $| z | < [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. We intend to show that $\operatorname{Re}(zf'(z)/f(z)) > 0$ in this case also. Since $f(z)$ and $g(z)$ satisfy, in particular, the hypothesis of Theorem 1 with $n = 1$, it follows from Theorem 1 that

$$\operatorname{Re}(zf'(z)/f(z)) > 0 \quad \text{for } |z| < [(5-\lambda^2)^{1/2}-2]/(1+\lambda) .$$

It is easy to see that

$$[(4\lambda+5)^{1/2}-(\lambda+2)] \leqq (5-\lambda^2)^{1/2}-2 \quad \text{for } 0 \leqq \lambda \leqq 1$$

and hence in particular

$$\operatorname{Re}(zf'(z)/f(z)) > 0 \quad \text{for } |z| < [(4\lambda+5)^{1/2}-(\lambda+2)]/(1+\lambda) .$$

In view of the above and (19), it now follows that $f(z)$ is univalent and starlike for $|z| < (1-\lambda)/(3+\lambda)$ and this completes the proof.

For $\lambda = 0$ the above result reduces to a result of Ratti [5, Theorem 2] and improves a result of MacGregor [2, Theorem 4] since $\operatorname{Re}(g(z)/z) > 1/2$ does not necessarily imply that $g(z)$ is convex [7]. The functions $f(z) = z(1-z)/(1+z)^2$ and $g(z) = z/(1+z)$ satisfy the hypothesis of Theorem 2 with $\lambda = 0$ and $f(z)$ is univalent in no circle $|z| < r$ with $r > 1/3$ since $f'(z)$ vanishes at $z = 1/3$. This shows that Theorem 2 is sharp at least for $\lambda = 0$.

A function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is said to be starlike of order $\alpha$, $0 \leqq \alpha < 1$, for $|z| < 1$ if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ for $|z| < 1$, we now prove the following result.

THEOREM 3. *Let* $f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ *and* $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ *be analytic for* $|z| < 1$ *and* $g(z)$ *be starlike of order* $\alpha$, $0 \leqq \alpha < 1$, *for* $|z| < 1$. *If* $\operatorname{Re}(f(z)/[\lambda f(z) + (1-\lambda)g(z)]) > 0$ *for* $|z| < 1$, *then* $f(z)$ *is univalent and starlike for*

(i) $$|z| < [(1-\lambda)/(1+\lambda+2n)]^{1/n} \qquad \text{if } \alpha = 1/2 ;$$

*and*

(ii) $$|z| < R^{1/n} , \qquad \text{if } \alpha \neq 1/2 ,$$

*where*

$$R = \{[A^2 + 4(1-\lambda^2)(2\alpha-1)]^{1/2} - A\}/[2(1+\lambda)(2\alpha-1)]$$

*with* $A = 2n + \lambda + 1 - (2\alpha-1)(1-\lambda)$.

*Proof.* Proceeding as in the proof of Theorem 1 we get

$$\operatorname{Re}(zf'(z)/f(z)) \geqq \operatorname{Re}(zg'(z)/g(z)) - |zh'(z)/h(z)|\,|1-\lambda h(z)|^{-1} .$$

Applying Lemma 3 (to $zg'(z)/g(z)$) and Lemmas 2 and 4 we get,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \frac{1+(2\alpha-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{(1-|z|^{2n}) - \lambda(1+|z|^n)^2} \tag{20}$$

provided that $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$.

Hence $\operatorname{Re}(zf'(z)/f(z)) > 0$ for those $z$ for which $|z| < [1-\lambda)/(1+\lambda)]^{1/n}$ and the right side of inequality (20) is greater than zero. The latter holds if

$$G(|z|^n) \equiv (1+\lambda)(2\alpha - 1)|z|^{2n} + [2n + \lambda + 1 - (2\alpha - 1)(1 - \lambda)]|z|^n - (1-\lambda) < 0 . \tag{21}$$

Let $|z|^n = t$ and consider the quadratic $G(t)$ for $0 \leqq t \leqq 1$. Since $G(0) = \lambda - 1 < 0$, $G[(1-\lambda)/(1+\lambda)] = 2n(1-\lambda)/(1+\lambda) > 0$, it follows that $G(t_1) = 0$ for some $t_1$ such that $0 < t_1 < (1-\lambda)/(1+\lambda)$ and $G(t) < 0$ for $0 \leqq t < t_1$ and $G(t) > 0$ for $t_1 < t < (1-\lambda)/(1+\lambda)$. Hence $f(z)$ is univalent and starlike for those $z$ for which only the inequality (21) holds. Now the inequality (21) holds if

$$|z| < [(1-\lambda)/(1+\lambda+2n)]^{1/n}$$

when $\alpha = 1/2$ and

$$|z| < \{[A^2 + 4(1-\lambda^2)(2\alpha - 1)]^{1/2} - A\}^{1/n}/[2(1+\lambda)(2\alpha-1)]^{1/n}$$

when $\alpha \neq 1/2$, where $A = 2n + \lambda + 1 - (2\alpha - 1)(1-\lambda)$ and this completes the proof.

If we put $\lambda = 0, n = 1$ and $\alpha = 0$ in the above result then we see that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ under the modified hypothesis is univalent and starlike for $|z| < 2 - \sqrt{3}$, a result obtained by MacGregor [2, Theorem 3]. On the other hand if $\lambda = 0$ and $n = 1$, Theorem 3 reduces to a result of Ratti [5, Theorem 3]. The functions

$$f(z) = z(1-z^n)/(1+z^n)^{\frac{2-2\alpha}{n}+1} \quad \text{and} \quad g(z) = z/(1+z^n)^{\frac{2-2\alpha}{n}}$$

show that Theorem 3 is sharp at least for $\lambda = 0$ and arbitrary $n$, since the derivative of $f(z)$ vanishes at

$$z = \{[(n+1-\alpha) - ((n+1-\alpha)^2 - (1-2\alpha))^{1/2}]/(1-2\alpha)\}^{1/n}$$

for $\alpha \neq 1/2$ and at $z = -1/(2n+1)$ when $\alpha = 1/2$.

4. Let $S(R)$ denote the functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic and satisfy $|zf'(z)/f(z) - 1| < 1$ for $|z| < R$. Obviously every member of $S(R)$ is univalent and starlike for $|z| < R$. We now prove the following result.

THEOREM 4. *Let* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$, *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *be analytic and satisfy* $\operatorname{Re}(g(z)/z) > 0$ *for* $|z| < 1$. *If* $|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1| < 1$, $0 \leqq \lambda < 1$, *for* $|z| < 1$, *then* $f(z) \in S(R^{1/n})$, *where* $R$ *is the smallest positive root of the equation* $(2n\lambda + \lambda - n - 1)R^2 - (3n + \lambda - 2n\lambda)R + (1-\lambda) = 0$.

*Proof.* Let

$$h(z) = f(z)/[\lambda f(z) + (1 - \lambda)g(z)] - 1 = c_n z^n + c_{n+1}z^{n+1} + \cdots . \tag{22}$$

By hypothesis, $h(z)$ is analytic and $|h(z)| < 1$ for $|z| < 1$ and hence by a result of Goluzin [1] we have that for $|z| < 1$

$$|h'(z)| \leqq n|z|^{n-1}(1 - |h(z)|^2)/(1 - |z|^{2n}) \tag{23}$$

and by Schwarz's lemma for $|z| < 1$

$$|h(z)| \leqq |z|^n . \tag{24}$$

If we let $g(z) = zp(z)$, then we have from (22)

$$f(z)[1 - \lambda - \lambda h(z)] = (1 - \lambda)zp(z)[1 + h(z)] .$$

Hence,

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{[1 + h(z)][1 - \lambda - \lambda h(z)]}$$

and this gives

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \left|\frac{zp'(z)}{p(z)}\right| + \frac{|zh'(z)|}{|1 + h(z)||1 - \lambda - \lambda h(z)|} .$$

Applying Lemma 2, with $\alpha = 0$, we get, in view of (23), for $|z| < 1$

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{n|z|^n(1 - |h(z)|^2)}{(1 - |z|^{2n})\,|1 + h(z)|\,|1 - \lambda - \lambda h(z)|} \\ &\leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{n|z|^n(1 + |h(z)|)}{(1 - |z|^{2n})\,|1 - \lambda - \lambda h(z)|}\end{aligned}$$

by using (24), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{n|z|^n}{(1 - |z|^n)(1 - \lambda - \lambda|z|^n)}$$

valid for $|z| < [(1 - \lambda)/\lambda]^{1/n}$. Hence $|zf'(z)/f(z) - 1| < 1$ if

$$|z| < [(1 - \lambda)/\lambda]^{1/n}$$

and

$$2n|z|^n(1 - \lambda - \lambda|z|^n) + n|z|^n(1 + |z|^n) < (1 - |z|^{2n})(1 - \lambda - \lambda|z|^n) .$$

The last inequality holds if

$$\begin{aligned}G(|z|^n) \equiv \lambda|z|^{3n} &+ (2n\lambda + \lambda - n - 1)|z|^{2n} \\ &- (3n + \lambda - 2n\lambda)|z|^n + (1 - \lambda) > 0 .\end{aligned} \tag{25}$$

Let $|z|^n = t$ and consider the cubic polynomial $G(t)$ for $0 \leqq t \leqq 1$.

$G(t)$ has at most two positive zeros. Since $G(0) = (1 - \lambda) > 0$ and $G((1-\lambda)/\lambda) = -(n(1-\lambda)/\lambda^2 < 0$, it follows that $G(t_1) = 0$ for some $t_1$ such that $0 < t_1 < (1-\lambda)/\lambda$ and $G(t) > 0$ for $0 \leqq t < t_1$ and $G(t) < 0$ for some values of $t$ between $t_1$ and $(1-\lambda)/\lambda$. Hence

$$|zf'(z)/f(z) - 1| < 1$$

for those values of $z$ for which only the inequality (25) holds. Now inequality (25) holds if, in particular

$$(2n\lambda + \lambda - n - 1)\,|z|^{2n} - (3n + \lambda - 2n\lambda)\,|z|^n + (1-\lambda) > 0$$

and this completes the proof.

If we set $\lambda = 0$ and $n = 1$ in the above result we have the following.

COROLLARY 2. *Suppose* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *and* $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ *are analytic and satisfy* $\mathrm{Re}\,(g(z)/z) > 0$ *for* $|z| < 1$. *If* $|f(z)/g(z) - 1| < 1$ *for* $|z| < 1$, *then* $|zf'(z)/f(z) - 1| < 1$ *for* $|z| < 1/4(\sqrt{17} - 3)$.

It may be noted that Corollary 2 implies, in particular, that $f(z)$ is univalent and starlike for $|z| < 1/4\,(\sqrt{17} - 3)$ and hence includes a result of Ratti [5, Theorem 4]. If we take $f(z) = z(1-z^n)^2/(1+z^n)$ and $g(z) = z(1-z^n)/(1+z^n)$, it is easy to see that these functions satisfy the hypothesis of Theorem 4 with $\lambda = 0$. We see that $f'(z)$ vanishes at $z_0 = [-3n + (9n^2 + 4n + 4)^{1/2}]/(2n+2)$ and hence

$$|z_0f'(z_0)/f(z_0) - 1| = 1\,.$$

This shows that Theorem 4 is sharp for at least $\lambda = 0$ and also that Corollary 2 is sharp.

THEOREM 5. *Let* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *be analytic for* $|z| < 1$ *and* $g(z)$ *be starlike of order* $\alpha$ *for* $|z| < 1$, $0 \leqq \alpha < 1$. *If*

$$|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1 < 1\,,\quad 0 \leqq \lambda < 1\,,\quad \textit{for } |z| < 1\,,$$

*then* $f(z)$ *is univalent and starlike for* $|z| < R^{1/n}$, *where* $R$ *is the smallest positive root of the equation*

$$(26)\qquad \begin{aligned}(2\alpha - 1)\lambda R^3 &- (n + 2\alpha - 1 - \lambda)R^2 \\ &+ (2\alpha - 2 - 2\alpha\lambda + \lambda - n)R + (1-\lambda) = 0\,.\end{aligned}$$

*Proof.* Proceeding as in the proof of Theorem 4 we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{[1 + h(z)]\,[1 - \lambda - \lambda h(z)]}\,.$$

Hence,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) - \frac{|\,zh'(z)\,|}{|\,1 + h(z)\,|\,|\,1 - \lambda - \lambda h(z)\,|}\,.$$

Since $\mathrm{Re}\,(zg'(z)/g(z)) > \alpha$ and $zg'(z)/g(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \cdots$, we have by Lemma 3 and inequalities (23) and (24) that

$$\begin{aligned}\mathrm{Re}\,(zf'(z)/f(z)) \geqq {} & [1 + (2\alpha - 1)\,|\,z\,|^n]/(1 + |\,z\,|^n) \\ & - n\,|\,z\,|^n/[(1 - |\,z\,|^n)\,(1 - \lambda - \lambda\,|\,z\,|^n)]\end{aligned} \tag{27}$$

valid for $|\,z\,| < [(1 - \lambda)/\lambda]^{1/n}$.

Hence $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ if $|\,z\,| < [(1 - \lambda)/\lambda]^{1/n}$ and if (in view of inequality (27))

$$\begin{aligned}G(|\,z\,|^n) \equiv {} & (2\alpha - \ )\,\lambda\,|\,z\,|^{3n} \\ & - (n + 2\alpha - 1 - \lambda)\,|\,z\,|^{2n} \\ & + (2\alpha - 2 - 2\alpha\lambda + \lambda - n)\,|\,z\,|^n \\ & + (1 - \lambda) > 0\,.\end{aligned} \tag{28}$$

Let $|\,z\,| = t$ and consider the cubic polynomial $G(t)$ for $0 \leqq t \leqq 1$. Since $G(0) = 1 - \lambda > 0$ and $G((1 - \lambda)/\lambda) = (-n(1 - \lambda))/\lambda^2 < 0$, it follows that $G(t_1) = 0$ for some $t_1$ such that $0 < t_1 < (1 - \lambda)/\lambda$ and $G(t) > 0$ for $0 \leqq t < t_1$ and $G(t) < 0$ for some $t$ between $t_1$ and $(1 - \lambda)/\lambda$. Hence $f(z)$ is starlike and univalent for $|\,z\,| < R^{1/n}$, in view of inequality (28), where $R$ is the smallest positive root of the equation (26).

The case when $\lambda = 0$ in Theorem 5 is of special interest. In this case equation (26) becomes

$$(n + 2\alpha - 1)R^2 - (2\alpha - 2 - n)R - 1 = 0$$

which gives $R = 1/3$ in case $\alpha = 0$ and $n = 1$ and

$$R = \{(2\alpha - 2 - n) + [(2\alpha - 2 - n)^2 + 4(n + 2\alpha - 1)]^{1/2}\}/[2(n + 2\alpha - 1)] \tag{29}$$

if $\alpha \neq 0$. This proves the following result, which includes a result of Ratti [5, Theorem 6].

COROLLARY 3. *Suppose* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *are analytic for* $|\,z\,| < 1$ *and* $g(z)$ *is starlike of order* $\alpha$ *for* $|\,z\,| < 1$, $0 \leqq \alpha < 1$. *If* $|\,f(z)/g(z) - 1\,| < 1$ *for* $|\,z\,| < 1$ *then* $f(z)$ *is univalent and starlike for*

( i ) $|\,z\,| < 1/3$ *if* $\alpha = 0$ *and* $n = 1$

(ii) $|z| < R^{1/n}$, *where* $R$ *is given by* (29) *if* $\alpha \neq 0$.

It is easy to see that the functions $f(z) = z(1 - z^n)/(1 + z^n)^{(2-2\alpha)/n}$ and $g(z) = z/(1 + z^n)^{(2-2\alpha)/n}$ satisfy the hypothesis of Corollary 3 and also that the derivative of $f(z)$ vanishes at $z = 1/3$ if $\alpha = 0$ and $n = 1$, and at $z = \{[(n + 2 - 2\alpha)^2 + 4(n + 2\alpha - 1)]^{1/2} - (n + 2 - 2\alpha)\}^{1/n}/[2(n + 2\alpha - 1)]^{1/n}$ if $\alpha \neq 0$. This shows that Corollary 3 is sharp.

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THE UNIVERSITY OF WISCONSIN — WAUKESHA

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# CRITERIA FOR BANACH SPACES

J. E. VALENTINE AND S. G. WAYMENT

**It is well known in euclidean geometry that the quadrilateral obtained from an arbitrary quadrilateral by joining its midpoints is a parallelogram. The purpose of this paper is to show that a complete metric space with a unique metric line joining any pair of its distinct points is a Banach space if and only if it has the above mentioned property.**

Let $p, q, r$, and $s$ be distinct points in a Banach space such that no three are linear let $m_1, m_2, m_3$, and $m_4$ be the midpoints of the algebraic segments joining $p$ and $q, q$ and $r, r$ and $s$, and $s$ and $p$, respectively. It is well known that $m_3 - m_2 = m_4 - m_1$ and $m_2 - m_1 = m_3 - m_4$. In Euclidean space one usually refers to this result by saying that the midpoints $m_1, m_2, m_3, m_4$ form a parallelogram. If the Banach space does not have unique segments joining pairs of distinct points, then the restriction that the $\{m_i\}$ be midpoints of algebraic segments is easily seen to be necessary. We shall say that the metric space $M$ satisfies the quadrilateral midpoint postulate provided that if $p, q, r, s$ are points of $M$ such that no three are linear and if $m_1, m_2, m_3, m_4$ are the respective midpoints, then $m_1m_2 = m_3m_4$ and $m_2m_3 = m_1m_4$. Hereafter we shall assume that $M$ is a complete metric space with a unique metric line joining any pair of its distinct points and show that the Quadrilateral Midpoint Postulate characterizes the class of Banach spaces among such metric spaces.

The technique will be to show that a complete metric space with a unique metric line joining any pair of its distinct points satisfies the Quadrilateral Midpoint Postulate if and only if it satisfies the Young Postulate which may be stated as follows.

*The Young Postulate.* If $p, q, r$ are points of a metric space $M$ and $q'$ and $r'$ are the midpoints of $p$ and $q$, and $p$ and $r$, respectively, then $q'r' = qr/2$.

The result will then follow, for Andalafte and Blumenthal [1] have shown that a complete metric space with a unique metric line joining any pair of its distinct points is a Banach space if and only if it satisfies the Young Postulate.

That the Young Postulate implies the Quadrilateral Midpoint Postulate is almost immediate. For if a complete metric space with a metric line joining any pair of its distinct points satisfies the Young Postulate, then it is a Banach space and consequently satisfies the Quadrilateral Midpoint Postulate.

Suppose $M$ satisfies the Quadrilateral Midpoint Postulate and $p$, $q$, $r$, are non-linear points of $M$ with $m_1$, $m_2$ the midpoints of $p$ and $q$, $q$ and $r$, respectively.

LEMMA 1. *There exists a number $k$, depending only on $p$ and $r$, such that if $q$, $m_1$, $m_2$ are as above, then $m_1m_2 = kpr$.*

*Proof.* Let $s$ be a point such that no three of $p, q, r, s$ are collinear, and let $m_3$, $m_4$ be the midpoints of the segments joining $r$ and $s$, $s$ and $p$, respectively. Let $k = m_3m_4/pr$. Then since $M$ satisfies the quadrilateral midpoint property, $m_1m_2 = m_3m_4 = kpr$. We see immediately that $k$ does not depend on $q$.

LEMMA 2. *The $k$ in Lemma 1 is 1/2.*

*Proof.* Let $\{x_i\}$ be a sequence of points tending to $x$ on the segment between $p$ and $r$ with $p \neq x \neq r$ and such that for each $i$ we have $p$, $x_i$, $r$ non-collinear. Let $\{p_i\}$ and $\{r_i\}$ be the sequences such that $p_i$ and $r_i$ are the midpoints of the segments determined by $p$ and $x_i$, $r$ and $x_i$, respectively. Then $\lim p_ix_i = 1/2 \lim px_i = 1/2\, px$ and similarly $\lim r_ix_i = 1/2\, rx$. This, along with the triangle inequality $p_ix_i + x_ir_i \geqq p_ir_i = kpr$, implies $k \leqq 1/2$. However, the inequality $pr \leqq pp_i + p_ir_i + r_ir = pp_i + kpr + r_ir$ and the aforementioned limits imply $k \geqq 1/2$. Hence $k$ is 1/2.

THEOREM. *A complete metric space with a unique line joining any two of its distinct points is a normed linear space (Banach Space) if and only if it satisfies the Quadrilateral Midpoint Postulate.*

*Proof.* We have shown that the Quadrilateral Midpoint Postulate implies the Young Postulate; that is, if $p'$ and $r'$ are midpoints of $p$ and $q$, and $q$ and $r$, respectively, then $p'r' = (1/2)pr$. Thus an application of the Andalafte-Blumenthal result [1] completes the proof.

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UTAH STATE UNIVERSITY

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# LINEARLY STRATIFIABLE SPACES

*Dedicated to Professor John H. Roberts on the occasion of his sixty-fifth birthday*

J. E. VAUGHAN

**The purpose of this paper is to introduce a new class of spaces, called linearly stratifiable spaces, which contains the class of stratifiable spaces and is contained in the class of hereditarily paracompact spaces. The notion of linearly stratifiable spaces is related to several of the concepts most recently studied by the late Professor Hisahiro Tamano, and also to questions raised by A. H. Stone and E. A. Michael concerning the normality and paracompactness of certain product spaces.**

The class of linearly stratifiable spaces is composed of special subclasses called $\alpha$-stratifiable spaces (where $\alpha$ is an infinite cardinal number) of which the class of stratifiable spaces is the subclass corresponding to the first infinite cardinal. Many results which hold for stratifiable spaces can be extended to linearly stratifiable spaces (see § 4) because the importance of the "countability" inherent in stratifiable spaces is often due only to the well-ordering of the natural numbers and not to their cardinality. One notable exception is that while, as is known, the subclass of stratifiable spaces is preserved by countable products, the other subclasses are preserved only by finite products. In addition, the subclass of $\alpha$-stratifiable spaces is preserved by box products provided there are fewer than $\alpha$ factors in the product. An analogous extension of the concept of a Nagata space is given in § 6, and some examples are given in § 7.

Stratifiable spaces (originally called $M_3$-spaces) and Nagata spaces were introduced in 1961 by J. G. Ceder [6] along with several other generalizations of metrizability. In 1966 C. J. R. Borges used an equivalent definition of $M_3$-space to show that Ceder's $M_3$-spaces had many important features, and, thinking they deserved a better name, he called them stratifiable spaces. Since then many authors have considered this class of spaces, and recently, A. Arhangel'skii [1, pp. 139-142] and Borges [4], [5] have given surveys of results on stratifiable spaces. A further generalization of metrizable spaces, called perfectly paracompact spaces, was announced in two abstracts [14], [15] in 1968 by H. Tamano, and he stated two interesting product theorems for this class of spaces. His definition, however, allows non-paracompact spaces to be perfectly paracompact (see Example 3.1), which was not his intention. (In light of this fact and current terminology, it seems better to reserve the term "perfectly paracompact" for the class of paracompact spaces in which every closed set is a countable intersection of open sets. Nevertheless, in this paper we shall use the term "perfectly paracompact" in the sense in which it was used by Professor Tamano.) It seems reasonable (see §3) to suppose that Tamano was interested in a concept similar to linearly stratifiable spaces. If we substitute the words "linearly stratifiable" for "perfectly paracompact" in the product theorems given in Tamano's abstracts, we get the statements below, which seem to be plausible conjectures. In fact, the author had considered the first conjecture before becoming aware of Tamano's abstracts. The definition of the box topology can be found in [11, p. 107].

*Conjecture* 1. The product of two linearly stratifiable spaces is paracompact.

*Conjecture* 2. Any product of linearly stratifiable spaces with the box topology is paracompact.

One reason that Tamano was interested in Conjecture 2 is that it would (if true) provide an affirmative answer to A. H. Stone's question [12, p. 54]: Is a product of real lines with the box topology normal? In this direction, M. E. Rudin [23] has recently proved that, under the assumption of the continuum hypothesis, the box product of countably many locally compact, $\sigma$-compact, metric spaces is paracompact.

In this paper, we shall show that Conjecture 1 and a form of Conjecture 2 are true for $\alpha$-stratifiable spaces. These results are given in §5, and the definitions of these spaces are given in §2. Most of these results were announced in [18], [19], and [20]. The fact that Conjecture 1 holds for the subclass of stratifiable spaces follows from results of Ceder [3, Thm. 2.2, Thm. 2.4].

## 2. Definitions and characterizations.

DEFINITION 2.1. An ordinal number $\alpha$ is called an *initial ordinal* provided for every ordinal $\beta < \alpha$, there exists an injection from $\beta$ to $\alpha$, but there does not exist an injection from $\alpha$ to $\beta$. We assume that cardinal numbers and initial ordinal numbers are the same. Let $\omega$ stand for the first infinite ordinal.

DEFINITION 2.2. Let $(X, \mathscr{T})$ be a $T_1$-topological space and let $\alpha$ be an initial ordinal, $\alpha \geqq \omega$. The space $(X, \mathscr{T})$ is said to be *stratifiable over* $\alpha$ or *linearly stratifiable* provided there exists a map $S: \alpha \times \mathscr{T} \to \mathscr{T}$ (called an *$\alpha$-stratification*) which satisfies the following (where we denote $S(\beta, U)$ by $U_\beta$).

$LS_I$ : $\bar{U}_\beta \subset U$ for all $\beta < \alpha$ and all $U \in \mathscr{T}$.

$LS_{II}$ : $\cup \{U_\beta: \beta < \alpha\} = U$ for all $U \in \mathscr{T}$.

$LS_{III}$: If $U \subset W$, then $U_\beta \subset W_\beta$ for all $\beta < \alpha$.

$LS_{IV}$ : If $\gamma < \beta < \alpha$, then $U_\gamma \subset U_\beta$ for all $U \in \mathscr{T}$.

DEFINITION 2.3. A $T_1$-space $X$ is called *$\alpha$-stratifiable* provided $\alpha$ is the smallest initial ordinal for which $X$ is stratifiable over $\alpha$. A space which is stratifiable over $\omega$ is called *stratifiable*, and the map $S$ is called a *stratification*.

REMARK 2.4. In the case of a stratifiable space, our definition above agrees with that of Borges [3, p. 1] because (as he noted) if $S$ is a stratification which satisfies $LS_I$, $LS_{II}$, and $LS_{III}$, then there is a stratification which satisfies all four conditions $LS_I$—$LS_{IV}$. Example 7.5 shows this is not true in general for $\alpha > \omega$.

DEFINITION 2.5. A collection $\boldsymbol{P}$ of pairs $P = (P_1, P_2)$ of subsets of a topological space $(X, \mathscr{T})$ is said to be a *linearly cushioned collection of pairs with respect to a linear order* $\leqq$ provided $\leqq$ is a linear order on $\boldsymbol{P}$ such that $(\cup \{P_1: P = (P_1, P_2) \in \boldsymbol{P}'\})^- \subset \cup \{P_2: P = (P_1, P_2) \in \boldsymbol{P}'\}$ for every subset $\boldsymbol{P}'$ of $\boldsymbol{P}$ which is majorized (i.e., has an upper bound) with respect to $\leqq$.

DEFINITION 2.6. (Ceder) A collection $\boldsymbol{P}$ of pairs is called a *pair-base* for $(X, \mathscr{T})$ provided (1) for each $P = (P_1, P_2) \in \boldsymbol{P}$, $P_1$ is open and (2) for every $x$ in $X$ and every open set $W$ containing $x$, there exists $P = (P_1, P_2) \in \boldsymbol{P}$ such that $x \in P_1 \subset P_2 \subset W$.

THEOREM 2.7. *If $(X, \mathscr{T})$ is a $T_1$-topological space and $\alpha$ an infinite initial ordinal, then the following are equivalent.*

( i ) *$(X, \mathscr{T})$ is stratifiable over $\alpha$.*

( ii ) *$(X, \mathscr{T})$ has a linearly cushioned pair-base $\boldsymbol{P}$ and $\alpha$ is cofinal with $\boldsymbol{P}$.*

(iii) *There exists a family $\{g_\beta: \beta < \alpha\}$ of functions with domain $X$ and range $\mathscr{T}$ such that the following hold.*

(a) *$x \in g_\beta(x)$ for all $\beta < \alpha$.*

(b) *For every $F \subset X$, if $y \in [\cup \{g_\beta(x): x \in F\}]^-$ for all $\beta < \alpha$, then $y \in \bar{F}$.*

(c) *If $\beta < \gamma < \alpha$, then $g_\beta(x) \supset g_\gamma(x)$ for all $x$.*

*Proof.* (i) $\rightarrow$ (ii). Let $S: \alpha \times \mathscr{T} \rightarrow \mathscr{T}$ be an $\alpha$-stratification for $(X, \mathscr{T})$. Give $\mathscr{T}$ any well-order and define

$$\boldsymbol{P} = \{P_{(\beta,U)} = (U_\beta, U) \colon (\beta, U) \in \alpha \times_{\text{lex}} \mathscr{T}\}$$

where $\alpha \times_{\text{lex}} \mathscr{T}$ denotes the product set $\alpha \times \mathscr{T}$ with the lexicographic order. It is easy to verify that $\boldsymbol{P}$ is a linearly cushioned pair-base for $X$.

(ii) $\rightarrow$ (iii). Let $\boldsymbol{P}$ be a linearly cushioned pair-base for $X$ and $\{P_\beta \colon \beta < \alpha\}$ a subset of $\boldsymbol{P}$ such that for every $P \in \boldsymbol{P}$ there exists $\beta < \alpha$ such that $P < P_\beta$. For each $x$ in $X$ and each $\beta < \alpha$ define

$$g_\beta(x) = X - [\cup \{P_1 \colon x \notin P_2 \text{ and } P = (P_1, P_2) \leqq P_\beta\}]^- .$$

Clearly (a) and (c) hold. To see that (b) holds note if $y \notin \bar{F}$ then there exists $P \in \boldsymbol{P}$ such that $y \in P_1 \subset P_2 \subset X - \bar{F}$. Let $\beta < \alpha$ be such that $P = (P_1, P_2) \leqq P_\beta$; then $P_1$ is a neighborhood of $y$ which misses $g_\beta(x)$ for all $x \in F$. Thus $y \notin [\cup \{g_\beta(x) \colon x \in F\}]^-$.

(iii) $\rightarrow$ (i). For each $\beta < \alpha$ and each open set $U$ define an open set

$$U_\beta = X - [\cup \{g_\beta(x) \colon x \in X - U\}]^- .$$

The correspondence $S(\beta, U) = U_\beta$ is easily seen to satisfy $LS_I$—$LS_{III}$, and $LS_{IV}$ follows from (c). This completes the proof.

For the stratifiable case, Ceder is credited with showing (i) $\leftrightarrow$ (ii) in [3, p. 2, footnote 1], and (i) $\leftrightarrow$ (iii) is due to Heath [10].

REMARK 2.8. A dual characterization for linearly stratifiable spaces can be given by stating Definition 2.2 in terms of closed sets rather than open sets.

The next characterization justifies the terminology "linearly" stratifiable.

PROPOSITION 2.9. *Let $(X, \mathscr{T})$ be a $T_1$-space. $X$ is linearly stratifiable if and only if there exists a linearly ordered set $A$ and a map $S \colon A \times \mathscr{T} \rightarrow \mathscr{T}$ which satisfies $LS_I$—$LS_{IV}$.*

*Proof.* Let $\alpha$ be the smallest ordinal which is cofinal with $A$; then $\alpha$ is regular (i.e., there exists no strictly smaller ordinal which is cofinal with $\alpha$) and $S'$, the restriction of $S$ to any cofinal subset of $A$, will satisfy $LS_I$—$LS_{IV}$.

The proof of this proposition also shows that if $X$ is an $\alpha$-stratifiable space, then $\alpha$ is a regular initial ordinal number.

The next result, though not a characterization, is useful in examples.

PROPOSITION 2.10. *If* $(X, \mathscr{T})$ *is stratifiable over a regular infinite initial ordinal* $\alpha$, *then every subset* $F$ *of* $X$ *whose cardinality is strictly less than* $\alpha$ *is a closed discrete subspace.*

*Proof.* Let $\boldsymbol{P}$ be a linearly cushioned pair-base for $X$ such that the regular initial ordinal $\alpha$ is cofinal with $\boldsymbol{P}$. It suffices to show that $F$ has no accumulation points. If $x_0 \in X$ then for every $x \in F - \{x_0\}$ there exists $P_x \in \boldsymbol{P}$ such that $x \in (P_x)_1$ and $x_0 \notin (P_x)_2$. Then $\{P_x: x \in F\}$ must have an upper bound in $\boldsymbol{P}$, because it is not cofinal. Hence

$$X - [\cup \{(P_x)_1: x \in F - \{x_0\}\}]^-$$

is a neighborhood of $x_0$ which misses $F - \{x_0\}$.

From this proposition it is clear that a space stratifiable over a regular initial ordinal can not possess any property which requires any countable set to have an accumulation point unless the space is stratifiable. For example, if such a space is a $k$-space or a separable space it must be stratifiable. We also note that Proposition 2.10 holds in particular for $\alpha$-stratifiable spaces.

We now recall some definitions.

DEFINITIONS. 2.11. The *character* of a point $x$ in a space $X$ is the smallest cardinal number $\chi(x, X)$ such that $x$ has a fundamental system of neighborhoods of cardinality $\chi(x, X)$. The *character* of the space $X$ is the cardinal number $\chi X = \sup\{\chi(x, X): x \in X\}$. The *pseudocharacter* of $x$ is the smallest cardinal number $\psi(x, X)$ such that $x$ is the intersection of a collection of open sets which has cardinality $\psi(x, X)$. The *pseudocharacter* of $X$ is the cardinal number $\psi X = \sup\{\psi(x, X): x \in X\}$.

COROLLARY 2.12. *If* $X$ *is a non-discrete,* $\alpha$*-stratifiable space, then* $\psi X \leqq \alpha \leqq \chi X$.

**3. Pair-base versus pair of bases.** As was mentioned in the introduction, H. Tamano has defined [14] a class of spaces which seems to be closely related to linearly stratifiable spaces. His definition is essentially as follows. Tamano called a space $X$ *perfectly paracompact* provided there exist two bases $\mathscr{U}, \mathscr{V}$ for the topology of $X$, a map $\phi: \mathscr{V} \to \mathscr{U}$ such that $\phi(\mathscr{V})$ is also a base, and a well-order on $\mathscr{V}$ such that for every bounded subcollection $\mathscr{V}^* \subset \mathscr{V}$ we have

$$(\cup\{V: V \in \mathscr{V}^*\})^- \subset \cup\{\phi(V): V \in \mathscr{V}^*\} .$$

In short, the space has a "pair of bases", one of which is linearly cushioned in the other. We shall show below that this concept is weaker than the concept of a linearly cushioned "pair-base" as defined in §2 in that, for regular spaces, the latter notion implies paracompactness (Theorem 4.11 C) while the former does not. From the abstract [14] it is clear that Tamano was interested in a class of paracompact spaces, and from [16] we know that he was aware of the "pair-base" type of definition (he used it to define elastic spaces, which are paracompact). It seems probable, therefore, that the type of base Tamano wanted was a linearly cushioned pair-base. By Theorem 2.7 a $T_1$-space having such a base is linearly stratifiable.

EXAMPLE 3.1. A perfectly paracompact space which is not normal. The desired space is the well-known example of V. Niemytzki. Let $X = \{(x, y): x$ and $y$ are real numbers and $y \geqq 0\}$, $X_1 = \{(x, y) \in X: y = 0\}$, and $X_2 = X - X_1$. For each $p = (p_1, p_2) \in X$, let $B(p, r)$ denote the set of points of $X$ which lie inside the circle with center $p$ and radius $r > 0$. Then $\{B(p, r): r > 0\}$ is taken as a fundamental system of neighborhoods of points $p \in X_2$. For $p = (p_1, 0) \in X_1$, let $U(p, r) = B((p_1, r), r) \cup \{p\}$ and let $\{U(p, r): r > 0\}$ be a fundamental system of neighborhoods of points $p \in X_1$. We now define a base $\mathscr{V}$ for the Niemytzki topology on $X$. Let $\mathscr{V}_1 = \{U(p, r): p \in X_1, r > 0\}$ and $\mathscr{V}_n = \{B(p, p_2/n): p = (p_1, p_2) \in X_2$ and $1/n \leqq p_2\}$ for $n = 2, 3, \cdots$. Clearly $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}_n$ is a base for $X$. Next, we define a second base $\mathscr{U}$ for $X$. Let $\mathscr{U}_1 = \mathscr{V}_1 \cup \{X\}$, and $\mathscr{U}_{2k+1} = \{B(p, 2p_2/(2k+1)): p = (p_1, p_2) \in X_2\}$ for $k = 1, 2, \cdots$. Set $\mathscr{U} = \bigcup_{k=0}^{\infty} \mathscr{U}_{2k+1}$. Now let $\leqq_n$ be any well-order on $\mathscr{V}_n$ for $n \geqq 1$, and define a well-order $\leqq$ on $\mathscr{V}$ as follows. For $V, V' \in \mathscr{V}$, we say $V \leqq V'$ iff (1) there exists a natural number $n$ such that $V, V' \in \mathscr{V}_n$ and $V \leqq_n V'$, or (2) $V \in \mathscr{V}_n$, $V' \in \mathscr{V}_m$ and $n < m$. We define a map $\phi: \mathscr{V} \to \mathscr{U}$ by

$$\phi(V) = \begin{cases} X \text{ if } V \in \mathscr{V}_1 \\ U((p_1, 0), p_2) \text{ if } V = B\left(p, \dfrac{p_2}{n}\right) \text{ and } n \text{ is even} \\ B\left(p, \dfrac{2p_2}{n}\right) \text{ if } V = B\left(p, \dfrac{p_2}{n}\right) \text{ and } n \geqq 3 \text{ is odd} . \end{cases}$$

It is clear that $\phi(\mathscr{V})$ is a base since $\phi(\mathscr{V}) = \mathscr{U}$. Finally, we shall show that $\mathscr{V}$ is linearly cushioned in $\mathscr{U}$. Let $\mathscr{V}^*$ be a bounded subcollection of $\mathscr{V}$. We must show that

$$(\cup \{V: V \in \mathscr{V}^*\})^- \subset \cup \{\phi(V): V \in \mathscr{V}^*\} .$$

If $\mathscr{V}^*$ contains any member of $\mathscr{V}_1$, the inclusion is trivial. Thus we assume that $\mathscr{V}^* \cap \mathscr{V}_1 = \varnothing$. Since $\mathscr{V}^*$ is bounded, $\{n: \mathscr{V}^* \cap \mathscr{V}_n \neq \varnothing\}$ has a largest element $N$. For each $V \in \mathscr{V}^*$, we have that $V$ and $\phi(V)$ are (essentially) the insides of circles with the same center and the circle for $\phi(V)$ has at least twice the radius of the circle for $V$. The desired inclusion now follows from the fact that if $V$ is in $\mathscr{V}^*$, then $V$ does not reach below the line of height $1/(2N)$, and does not have a radius of less than $(1/N)^2$.

**4. Additional results.** We shall now give some important results for linearly stratifiable spaces which easily extend from the analogous results for stratifiable spaces.

THEOREM 4.1. *Let $X$ be stratifiable over $\alpha$.*

A. *Every open set in $X$ is a union of a collection $\mathscr{C}$ of closed sets with the cardinality of $\mathscr{C}$ less than or equal to $\alpha$.*

B. *Every subspace of $X$ is stratifiable over $\alpha$.*

C. *$X$ is paracompact (hence hereditarily paracompact).*

D. *Every closed continuous image of $X$ is stratifiable over $\alpha$.*

E. *$X$ is completely monotonically normal (see [21] or [22]).*

F. *$X$ has a network $N = \cup\{N_\beta\colon \beta < \alpha\}$ such that each $N_\beta$ is a discrete collection in $X$.*

*Proof.* Clearly (A) and (B) follow from the definition. The proof of (C) follows from Theorem 1 in [17]. Proofs of (D), (E), and (F) can be given in a manner similar to the proofs of [3, Thm. 3.1, p. 5], [22, Prop. A] and [9] respectively.

We conclude this section with two more interesting results.

THEOREM 4.2. *A space is stratifiable over $\alpha$ iff it is dominated by a collection of closed subsets, each of which is stratifiable over $\alpha$* [3, Thm. 7.2, p. 13].

THEOREM 4.3. *If $X$ and $Y$ are stratifiable over $\alpha$ and $A$ is a closed subset of $X$ and $f\colon A \to Y$ a continuous function, then $X \cup_f Y$ (the adjunction space) is stratifiable over $\alpha$* [3, Thm. 6.2, p. 11].

**5. Products.** In [6, Theorem 4.5, p. 107] J. Ceder proved that a countable product of stratifiable spaces is a stratifiable space. In this section, we shall prove that a finite product of spaces stratifiable over the same $\alpha$ is again stratifiable over $\alpha$. Example 7.4 shows that if $\alpha > \omega$ then a countable product of spaces stratifiable over $\alpha$ need not be linearly stratifiable.

It follows from our product theorem (Theorem 5.2A) and Theorem 4.1C that Conjecture 1 is true in the special case that both spaces are stratifiable over the same initial ordinal. We also prove (Theorem 5.2D) that certain products (with the box topology [11, p. 107]) of spaces stratifiable over the same $\alpha$ is again stratifiable over $\alpha$. This result yields a special case in which Conjecture 2 is true.

LEMMA 5.1. *Let $\alpha$ be an infinite initial ordinal number, and let $\{A_\lambda: \lambda \in \Lambda\}$ be a family of linearly ordered sets such that $\alpha$ has cardinality strictly greater than that of $\Lambda$, and $\alpha$ is cofinal with $A_\lambda$ for all $\lambda \in \Lambda$. If $\Lambda$ is finite or if $\alpha$ is a regular ordinal, then $A = \Pi\{A_\lambda: \lambda \in \Lambda\}$ can be well-ordered so that for every majorized $H \subset A$ we have $Pr_\lambda(H)$ (i.e., the $\lambda$th projection) is majorized in $A_\lambda$ for all $\lambda \in \Lambda$, and $\alpha$ is cofinal in $A$. Further, if $\alpha$ is the smallest initial ordinal cofinal with each $A_\lambda$, then $\alpha$ is the smallest initial ordinal cofinal with $A$.*

*Proof*. For convenience we assume that $\alpha$ is a subset of each $A_\lambda$. Let $\Lambda$ be ordered as its cardinal number $\alpha(\Lambda)$. Define $T_{\mu,\beta} = \{a = (a_\lambda) \in A: a_\mu \leqq \beta\}$ for all $\beta < \alpha$ and $\mu < \alpha(\Lambda)$. Let $R_\beta = \cap\{T_{\mu,\beta}: \mu < \alpha(\Lambda)\}$ for all $\beta < \alpha$, and let $D_\beta = R_\beta - \cup\{R_\gamma: \gamma < \alpha$ and $\gamma < \beta\}$ for all $\beta < \alpha$. Then $\{D_\beta: \beta < \alpha\}$ is a partition of $A$ because if $a = (a_\lambda) \in A$, then for each $a_\lambda$ there exists $\beta_\lambda < \alpha$ such that $a_\lambda \leqq \beta_\lambda$. Now $\{\beta_\lambda: \lambda < \alpha(\Lambda)\}$ has an upper bound in $\alpha$ because either $\alpha(\Lambda)$ is finite, or $\alpha$ is regular and $\alpha(\Lambda) < \alpha$. Call the smallest upper bound $\beta'$, then $a = (a_\lambda) \in D_{\beta'}$. Let $\leqq_\beta$ be any well-order on $D_\beta$ and define a well-order on $A$ as follows. For $x$ and $y$ in $A$, we say $x \leqq y$ iff either

(1) there exists $\beta < \alpha$ such that $x$ and $y$ are in $D_\beta$ and $x \leqq_\beta y$, or

(2) there exists $\beta < \gamma < \alpha$ such that $x \in D_\beta$ and $y \in D_\gamma$.

If $H$ is a majorized subset of $A$, then there exists $\beta < \alpha$ such that $b = (b_\lambda)$ and $b_\lambda = \beta$ for all $\lambda \in \Lambda$, and $b$ is an upper bound for $H$. Hence $\beta$ is an upper bound for $Pr_\lambda(H)$ in $A_\lambda$ for all $\lambda$. The remaining assertions follow easily from the definition of $\leqq$.

THEOREM 5.2. *Let $\alpha$ be an initial ordinal number $\alpha \geqq \omega$. Let $X_i$ be stratifiable over $\alpha$ for each $i < \omega$. Then the following hold:*

A. *$\Pi\{X_i: i \leqq n\}$ is stratifiable over $\alpha$ for all $n < \omega$.*

B. *If each $X_i$ is $\alpha$-stratifiable, then $\Pi\{X_i: i \leqq n\}$ is $\alpha$-stratifiable for each $n < \omega$.*

C. *(Ceder) If each $X_i$ is stratifiable, then $\Pi\{X_i: i < \omega\}$ is stratifiable.*

D. *If each $X_\lambda$ is stratifiable over the regular initial ordinal $\alpha$ for all $\lambda \in \Lambda$ and $\alpha$ is strictly larger than the cardinality of $\Lambda$, then $\Pi\{X_\lambda: \lambda \in \Lambda\}$ with the box topology is stratifiable over $\alpha$.*

*Proof*. By Theorem 2.7, each $X_i$ has a linearly cushioned pair-base $\boldsymbol{P}_i$ such that $\alpha$ is cofinal with $\boldsymbol{P}_i$. For each $n < \omega$ and each $Q = (P^1, \cdots, P^n) \in \prod\{\boldsymbol{P}_i: i \leqq n\}$ define $\prod_{i=1}^n P_1^i = \{x = (x_i): x_i \in P_1^i$ for $i \leqq n\}$, and similarly define $\prod_{i=1}^n P_2^i$. Set $B_{Q1} = \prod_{i=1}^n P_1^i$, $B_{Q2} = \prod_{i=1}^n P_2^i$, and $\boldsymbol{B}_n = \{B_Q = (B_{Q1}, B_{Q2}): Q \in \prod \{\boldsymbol{P}_i: i \leqq n\}\}$ and order the index set of $\boldsymbol{B}_n$ as in Lemma 5.1 so that $\alpha$ is cofinal with $\boldsymbol{B}_n$. Clearly $\boldsymbol{B}_n$ is a pair-base for $\prod \{X_i: i \leqq n\}$, and if we consider $(x_i) \in \prod \{X_i: i < \omega\}$, then $\boldsymbol{B} = \cup \{\boldsymbol{B}_n: n < \omega\}$ is a pair-base for $\prod\{X_i: i < \omega\}$. We now show that each $\boldsymbol{B}_n$ is a linearly cushioned collection of pairs in $X = \prod\{X_i: i \leqq n\}$. Suppose $H$ is a majorized subset of $\prod_{i=1}^n \boldsymbol{P}_i$ and $x \notin \cup \{B_{Q2}: Q \in H\}$. Let $N_i = X_i - (\cup \{P_1: P = (P_1, P_2) \in Pr_i(H)$ and $x_i \notin P_2\})^-$. Then $N_i$ is an open neighborhood of $x_i$ in $X_i$ because $Pr_i(H)$ is a majorized subset of $\boldsymbol{P}_i$. Finally, $\prod_{i=1}^n N_i$ is a neighborhood of $x$ in $X$ which misses $\cup \{B_{Q1}: Q \in H\}$. Thus $(\cup \{B_{Q1}: Q \in H\})^- \subset \cup \{B_{Q2}: Q \in H\}$, and this completes the proof of (A). The proof of (B) follows from (A) and Proposition 4.1B. To see that (C) holds, assume that each linearly cushioned pair-base $\boldsymbol{P}_i$ of $X_i$ has a countable cofinal subset (this is equivalent to $\boldsymbol{P}_i$ being a $\sigma$-cushioned pair-base). The preceding argument shows that each $\boldsymbol{B}_n$ is linearly cushioned with a countable cofinal subset, and is, therefore, a $\sigma$-cushioned collection. Thus $\boldsymbol{B} = \cup\{\boldsymbol{B}_n: n < \omega\}$ is a $\sigma$-cushioned pair-base for $\prod \{X_i: i < \omega\}$. The proof of (D) is similar to the proof of (B) by use of Lemma 5.1.

Example 7.2 shows that if $X_1$ and $X_2$ are stratifiable over different $\alpha_1$ and $\alpha_2$ respectively, then $X_1 \times X_2$ need not be linearly stratifiable.

In [13] E. Michael asked several questions concerning product spaces. In particular, he asked whether or not there is a space $X$ such that $X^n$ (the product of $X$ with itself $n$ times) is hereditarily paracompact for all finite cardinals $n$, but $X^\omega$ is not normal. We raise a related question: If $X$ is stratifiable over $\alpha > \omega$, is $X^\omega$ normal? For such a space $X$, it would follow from Theorem 5.2A and Theorem 4.1C, that $X^n$ is hereditarily paracompact for all finite $n$. Thus a negative answer to the preceding question would provide a negative answer to Michael's question.

**6. $\alpha$-Nagata spaces.** The concept of a Nagata space was introduced by Ceder in [6, p. 109]. In this section we shall extend this concept and give some basic results. One important difference between Nagata spaces and the generalization presented here should be mentioned. Ceder proved that the Nagata spaces are exactly the first countable stratifiable spaces [6, Theorem 3.1, p. 109]. The $\alpha$-Nagata spaces, however, form a smaller class of spaces than the $\alpha$-stratifiable spaces of character $\alpha$. The difference is that the $\alpha$-Nagata spaces have, for each point, a fundamental system of neighborhoods which is well-ordered with respect to reverse inclusion (see $N_{III}$ below), while an $\alpha$-stratifiable space of character $\alpha$ need not have such neighborhood systems (see Example 7.3).

DEFINITION 6.1. A $T_1$-space $X$ is called a *Nagata space over* $\alpha$ (where $\alpha$ is an initial ordinal and $\alpha \geqq \omega$) provided for every $x \in X$ there exist collections of neighborhoods of $x$, $\{U_\beta(x): \beta < \alpha\}$ and $\{S_\beta(x): \beta < \alpha\}$, such that

$N_I$ : for each $x \in X$, $\{U_\beta(x): \beta < \alpha\}$ is a fundamental system of neighborhoods of $x$,

$N_{II}$ : for every $x, y \in X$, $S_\beta(x) \cap S_\beta(y) \neq \emptyset$ implies $x \in U_\beta(y)$

$N_{III}$: if $\beta < \gamma < \alpha$ then $S_\beta(x) \supset S_\gamma(x)$ for all $x$.

The set of ordered pairs

$$\{(\{U_\beta(x): \beta < \alpha\}, \{S_\beta(x): \beta < \alpha\}): x \in X\}$$

is called an *$\alpha$-Nagata structure* for $X$ provided for each $x$ in $X$, $\{U_\beta(x): \beta < \alpha\}$ and $\{S_\beta(x): \beta < \alpha\}$ are systems of neighborhoods of $x$ which satisfy $N_I$, $N_{II}$, and $N_{III}$ of 6.1.

DEFINITION 6.2. A $T_1$-space is called an *$\alpha$-Nagata space* provided $\alpha$ is the smallest initial ordinal for which $X$ has an $\alpha$-Nagata structure. A space which is an $\omega$-Nagata space is simply called a *Nagata space*, and its $\omega$-Nagata structure is called a *Nagata structure*. This last definition agrees with the one given by Ceder [6, p. 109] because in Ceder's definition we may assume without loss of generality that $S_n(x) \supset S_{n+1}(x)$ for all $n < \omega$ and $x$ in $X$.

We now give some characterizations of Nagata spaces over $\alpha$ which extend the analogous results due to Ceder [6, Theorem 3.1, p. 109] and Heath [8, Theorem 5, p. 94].

THEOREM 6.3. *Let $(X, \mathscr{T})$ be a $T_1$-space, and let $\alpha$ be an infinite initial ordinal number. The following are equivalent.*

(i) *$X$ is a Nagata space over $\alpha$.*

(ii) *$X$ is stratifiable over $\alpha$ and for each $x$ in $X$ there exists a fundamental system of neighborhoods of $x$ $\{W_\beta(x): \beta < \alpha\}$ such that $\beta < \gamma < \alpha$ implies $W_\beta(x) \supset W_\gamma(x)$.*

(iii) *There exists a family $\{g_\beta: \beta < \alpha\}$ of functions with domain $X$ and range $\mathscr{T}$ such that the following hold:*

(a) *$\{g_\beta(x): \beta < \alpha\}$ is a fundamental system of open neighborhoods of $x$ for every $x$ in $X$,*

(b) *for every neighborhood $U$ of $x$ there exists $\beta < \alpha$ such that $g_\beta(x) \cap g_\beta(y) \neq \emptyset$ implies that $y \in U$,*

(c) *if $\beta < \gamma < \alpha$, then $g_\beta(x) \supset g_\gamma(x)$ for all $x$ in $X$.*

*Proof.* Let $X$ have an $\alpha$-Nagata structure

$$\{(\{U_\beta(x)\colon \beta < \alpha\}, \{S_\beta(x)\colon \beta < \alpha\})\colon x \in X\}\ ,$$

and define $g_\beta(x)$ to be the interior of $S_\beta(x)$ for all $x$ in $X$ and all $\beta < \alpha$. It is easy to check that (a), (b) and (c) of (iii) hold. This proves (i) $\rightarrow$ (iii). To see that (iii) $\rightarrow$ (ii), we note that each $x$ in $X$ clearly has the desired fundamental system of neighborhoods. We need only show that $X$ is stratifiable over $\alpha$, and to do this we will show that Theorem 2.7 (iii) holds. Let $\{g_\beta\colon \beta < \alpha\}$ be the family of functions given by hypothesis. Clearly 2.7 (iii) (a) and (c) hold. To see that (b) is also true, assume $y \notin \bar{F}$. Then there exists $\beta < \alpha$ such that $g_\beta(y) \cap g_\beta(x) \neq \emptyset$ implies $x \notin \bar{F}$. Hence $y \notin [\cup \{g_\beta(x)\colon x \notin \bar{F}\}]^-$.

The proof that (ii) implies (i) is a slight elaboration of Ceder's proof of Theorem 3.1 in [6, p. 109].

COROLLARY 6.4. *The closed continuous image $X$ of a Nagata space over $\alpha$ is a Nagata space over $\alpha$ iff for each point $x \in X$ there exists a fundamental system of neighborhoods $\{W_\beta(x)\colon \beta < \alpha\}$ such that $\beta < \gamma < \alpha$ implies $W_\beta(x) \supset W_\gamma(x)$.*

LEMMA 6.5. *Let $\alpha$ be a regular initial ordinal. If $X$ is a Nagata space over $\alpha$, then for every $x$ in $X$ either $x$ is isolated or $\psi(x, X) = \chi(x, X) = \alpha$.*

*Proof.* If $\alpha = \omega$ the result is clear. If $\alpha > \omega$, then the result follows from Theorem 6.3 (ii) and the observation that the intersection of fewer than $\alpha$ neighborhoods of a point $x$ will still be a neighborhood of $x$.

We can now give an analogue to Ceder's result that the class of Nagata spaces is the same as the class of first countable stratifiable spaces.

THEOREM 6.6. *A $T_1$-space $X$ is an $\alpha$-Nagata space iff it is $\alpha$-stratifiable and there exists for each $x$ in $X$ a fundamental system of neighborhoods $\{W_\beta(x)\colon \beta < \alpha\}$ such that $\beta < \gamma < \alpha$ implies $W_\beta(x) \supset W_\gamma(x)$.*

*Proof.* If $X$ is an $\alpha$-Nagata space, then by Theorem 6.3, we know $X$ is stratifiable over $\alpha$ and has the desired fundamental system of neighborhoods. We need only show that $X$ is not stratifiable over $\gamma$ for $\omega \leqq \gamma < \alpha$. This is clear if $\alpha = \omega$, and follows from Lemma 6.5 for $\alpha > \omega$ since a space stratifiable over $\gamma$ has pseudocharacter $\leqq \gamma$. The proof of the other half of the theorem is clear.

One can easily check that every subspace of a space which is Nagata over $\alpha$ is itself Nagata over $\alpha$, and that a finite product of spaces Nagata over $\alpha$ is Nagata over $\alpha$.

The reader will probably recall that the well-known extension theorem of Dugundji [7] was generalized from metric spaces to Nagata spaces by Ceder [6, Theorem 3.2, p. 110] and from Nagata spaces to stratifiable spaces by Borges [3, Theorem 4.3, p. 7]. We do not know, however, if Dugundji's theorem can be generalized to all $\alpha$-Nagata spaces.

**7. Examples.** In this section we denote the first uncountable ordinal by $\Omega$.

EXAMPLE 7.1. An $\Omega$-Nagata space (hence an $\Omega$-stratifiable space) which is not stratifiable. Let $X = [0, \Omega]$ and give $X$ the smallest topology larger than the order topology for which every point is isolated except $\Omega$. Let $\mathscr{B} = \{V_\alpha = (\alpha, \Omega): \alpha < \Omega\} \cup \{W_\alpha = \{\alpha\}: \alpha < \Omega\}$ and order $\mathscr{B}$ so that every $V_\alpha$ precedes every $W_\alpha$ and $\alpha < \beta < \Omega$ implies $V_\alpha < V_\beta$ and $W_\alpha < W_\beta$. Then $\mathscr{B}$ is a "linearly closure preserving base" for $X$, and $\{(B, B): B \in \mathscr{B}\}$ forms a linearly cushioned pair-base. $X$ is not stratifiable because the point $\Omega$ is not a $G_\delta$.

EXAMPLE 7.2. A stratifiable space $Y$ and an $\Omega$-stratifiable space $X$ such that $X \times Y$ is not linearly stratifiable. Let $X$ be the space of Example 7.1. Let $Y = [0, \omega]$ with the order topology. Then $Y$ is a stratifiable space (in fact, $Y$ is a compact metric space). It is known that if the point $(\Omega, \omega)$ is removed from this space, the resulting subspace is not normal. This can be seen by using the techniques of Exercise F on page 132 of [11]. Thus $X \times Y$ is not hereditarily normal and by Theorem 4.1.C it is not linearly stratifiable.

EXAMPLE 7.3. An $\Omega$-stratifiable space of character $\Omega$ which is not an $\Omega$-Nagata space. Let $X$ be the space described in 7.1. Let $Y = X$, but give $Y$ a topology stronger than the topology on $X$ as follows: Let $L_0$ be the set of limit ordinals in $[0, \Omega)$ and define inductively, for each $n < \omega$, $L_n$ as the set of ordinals which have a member of $L_{n-1}$ as immediate predecessor. (This idea was used by C. Aull [2, p. 50] for a different example.) Define $W(\alpha, n) = \cup \{(\alpha, \Omega) \cap L_k: k \geqq n\} \cup \{\Omega\}$ and $\mathscr{W} = \{W(\alpha, n): \alpha < \Omega \text{ and } n < \omega\}$. Then $\mathscr{W}$ is taken as a fundamental system of neighborhoods of $\Omega$ and all the other points in $Y$ are isolated. Note that $\Omega$ is a $G_\delta$ in $Y$. As in 7.1 we see that $Y$ is stratifiable over $\Omega$. (Also, one can easily show that $Y$ is stratifiable.) By Theorem 5.2 $X \times Y$ is stratifiable over $\Omega$, and since $X \times Y$ has subspaces which are not stratifiable, we know $X \times Y$ is $\Omega$-stratifiable. Clearly, $X \times Y$ has character $\Omega$, and has some points which are not isolated, but have pseudocharacter $\omega$ (i.e., $G_\delta$-points). It follows from Lemma 6.5 that $X \times Y$ is not $\Omega$-Nagata, and $X \times Y$ is not a Nagata space over $\alpha$ for any $\alpha \geqq \omega$.

EXAMPLE 7.4. A countable product of $\Omega$-stratifiable spaces need not be linearly stratifiable. Let $X_i$ be the space in 7.1 for each $i < \omega$. Since each $X_i$ has isolated points, $X = \prod\{X_i: i < \omega\}$ has convergent sequences, and also non-stratifiable subspaces. Hence, $X$ is not linearly stratifiable by Proposition 2.10.

EXAMPLE 7.5. Every regular space $(X, \mathscr{T})$ has a "stratification map" $S: \alpha \times \mathscr{T} \rightarrow \mathscr{T}$ which satisfies $LS_I$, $LS_{II}$ and $LS_{III}$ of 2.2. Take $\alpha$ to be the cardinal number of $\mathscr{T}$, let $\mathscr{T} = \{T_\beta: \beta < \alpha\}$, and define

$$S(\beta, U) = \begin{cases} T_\beta \text{ if } \bar{T}_\beta \subset U \\ \varnothing \text{ otherwise} . \end{cases}$$

It is easy to see that $S$ satisfies $LS_I$, $LS_{II}$, $LS_{III}$. Now if this map $S$ also satisfied $LS_{IV}$, then $X$ would be paracompact by Theorem 4.1 C.

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# ON SPACES OF DISTRIBUTIONS STRONGLY REGULAR WITH RESPECT TO PARTIAL DIFFERENTIAL OPERATORS

Z. ZIELEZNY

**A distribution $T$ in $\Omega$ is said to be strongly regular with respect to the differential operator $P(D)$, if $P^k(D)T$, $k = 0, 1, \cdots$, are of bounded order in any open set $\Omega' \subset\subset \Omega$. Necessary and sufficient conditions on the polynomials $P$ and $Q$ are established in order that a distribution $T$ strongly regular with respect to $P(D)$ be strongly regular with respect to $Q(D)$.**

Let $P(D)$ be a partial differential operator in $R^n$ with constant coefficients and $P^k(D)$, $k = 1, 2, \cdots$, its successive iterations. The following result is due to L. Hörmander ([3], Theorem 3.6 and Remark on p. 233):

If $P(D)$ is hypoelliptic and $T$ is a distribution such that $P^k(D)T$, $k = 1, 2, \cdots$, have a bounded order in any relatively compact open subset of $R^n$, then $T$ is a $C^\infty$-function.

In other words, the space $\mathscr{E}_P$ of distributions in $R^n$ "strongly regular with respect to $P(D)$" is contained in the space $\mathscr{E}$ of $C^\infty$-functions; in this case $\mathscr{E}_P = \mathscr{E}$. The concept of strong regularity with respect to $P(D)$ coincides with that of strong regularity in some variables (see [6], p. 453), when $P(D)$ is the Laplace operator in those variables.

Suppose now that given are two arbitrary partial differential operators $P(D)$ and $Q(D)$. Then the question arises: Under what conditions on $P$ and $Q$ is $\mathscr{E}_P \subset \mathscr{E}_Q$? In particular, if $P(D)$ is "$Q$-hypoelliptic," i.e. all solutions $U \in \mathscr{D}'$ of the equation

$$P(D)U = 0$$

are in $\mathscr{E}_Q$, must then be $\mathscr{E}_P \subset \mathscr{E}_Q$? The $Q$-hypoelliptic operators were studied (in a slightly different but equivalent version) and characterized by E. A. Gorin and V. V. Grušin [2].

In this paper we give necessary and sufficient conditions for the inclusion $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, where $\mathscr{E}_P(\Omega)$ and $\mathscr{E}_Q(\Omega)$ are the spaces of "strongly regular" distributions on an arbitrary open set $\Omega \subset R^n$. These conditions are, in general, stronger than the $Q$-hypeollipticity of $P(D)$. If the inclusion in question holds for every $Q$-hypoelliptic operator $P(D)$, then $Q(D)$ must be hypoelliptic and the problem reduces to that in Hörmander's theorem stated above.

## 1. The spaces $\mathscr{E}_P(\Omega)$ and $C_P^{\mu,\infty}(\Omega)$.

Let $\Omega$ be a nonempty open subset of $R^n$. A distribution $T \in \mathscr{D}'(\Omega)$ will be called strongly regular with respect to the differential operator $P(D)$, if to every open set $\Omega'$ having compact closure contained in $\Omega$ (we express this by writing $\Omega' \subset\subset \Omega$) there exists an integer $m \geqq 0$ such that $P^k(D)T$, $k = 0, 1, \cdots$, are all of order $\leqq m$ in $\Omega'$, i.e. the restrictions of $P^k(D)T$ to $\Omega'$ are all in $\mathscr{D}'^m(\Omega')$[1]. We denote by $\mathscr{E}_P(\Omega)$ the space of all distributions in $\Omega$, which are strongly regular with respect to $P(D)$. We also denote by $C_P^{\mu,\infty}(\Omega)$, where $\mu$ is an integer $\geqq 0$, the space of all $C^\mu$-functions in $\Omega$ such that $P^k(D)D^\alpha f$, $|\alpha| \leqq \mu$, $k = 0, 1, \cdots$, are continuous functions; here $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Consider now the spaces $\mathscr{E}_P(\Omega)$ and $\mathscr{E}_Q(\Omega)$ corresponding to the differential operators $P(D)$ and $Q(D)$ respectively.

THEOREM 1. *If $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, then to any open set $\Omega' \subset\subset \Omega$ there exists an integer $\mu \geqq 0$ such that the restriction mapping $f \to f \mid \Omega'$ maps $C_P^{\mu,\infty}(\Omega)$ into $C_Q^{0,\infty}(\Omega')$.*

*Proof.* Let $\Omega'$ be an open set satisfying the assumption $\Omega' \subset\subset \Omega$. We first prove the existence of nonnegative integers $\nu$ and $m$ such that

$$(1) \qquad \{Q^k(D)f \mid \Omega' : f \in C_P^{\nu,\infty}(\Omega),\ k = 0, 1, \cdots\} \subset \mathscr{D}'^m(\Omega') .$$

Suppose that inclusion (1) does not hold for any $\nu$ and $m$. Then to every $\nu$ and $m$ there exist a function $f \in C_P^{\nu,\infty}(\Omega)$ and a $k$ such that $Q^k(D)f \mid \Omega' \notin \mathscr{D}'^m(\Omega')$. Thus we can find strictly increasing sequences of positive integers $\nu_i$, $m_i$ and $k_i$, and a sequence of functions $f_i$ with the following properties:

$$(2) \qquad f_i \in C_P^{\nu_i,\infty}(\Omega) ,$$

$$(3) \qquad Q^k(D)f_i \mid \Omega' \in \mathscr{D}'^{m_i}(\Omega'),\ k = 0, 1, \cdots,$$

$$(4) \qquad Q^{k_i}(D)f_i \mid \Omega' \text{ is of order } m_i,$$

$$(5) \qquad qk_i < \nu_{i+1} ,$$

where $i = 1, 2, \cdots$, and $q$ is the order of the operator $Q(D)$.

We denote by $\Omega_i$, $i = 1, 2, \cdots$, open subsets of $\Omega$ such that

$$(6) \qquad \Omega_i \subset\subset \Omega_{i+1} \text{ and } \bigcup_{i=1}^{\infty} \Omega_i = \Omega .$$

Next we set

$$a_1 = 1 \text{ and } a_i = 2^{-i}M_i^{-1} , \qquad i = 2, 3, \cdots,$$

[1] $P^0(D)$ is the identity operator, i.e. $P^0(D)T = P$.

where

$$M_i = \sup \{| P^k(D)f_i(x) | + | Q^l(D)f_i(x) | + 1\}$$

and the supremum is taken over all $x \in \Omega_i$ and $k, l = 0, 1, \cdots, k_{i-1}$. Note that $Q^l(D)f_i$, $l = 0, 1, \cdots, k_{i-1}$, are continuous functions in $\Omega$, because of (5).

The function

$$f = \sum_{i=1}^{\infty} a_i f_i$$

is defined and continuous in $\Omega$, since the $f_i$'s are continuous in $\Omega$ and the series converges there almost uniformly. Moreover, for any $k$ we have (distributionally)

$$P^k(D)f = \sum_{i=1}^{\infty} a_i P^k(D)f_i \, . \tag{7}$$

But each term of the last series is a continuous function in $\Omega$, by (1). Also

$$a_i \sup_{x \in \Omega_j} | P^k(D)f_i(x) | \leqq 2^{-i}$$

whenever $k < i$ and $j \leqq i$, by the definition of $a_i$. Hence it follows that the series (7) converges almost uniformly in $\Omega$, for any $k$. Consequently $f \in C_P^{0,\infty}(\Omega) \subset \mathscr{E}_P(\Omega)$.

We now show that $f$ is not in $\mathscr{E}_Q(\Omega)$, which is a contradiction to our hypothesis. We write

$$g_j = \sum_{i=1}^{j} a_i f_i \text{ and } h_j = \sum_{i=j+1}^{\infty} a_i f_i \, .$$

In view of (3) and (4), the restriction of $Q^{k_j}(D)g_j$ to $\Omega'$ is a distribution of order $m_j$. On the other hand, $Q^{k_j}(D)f_i$, $i = j + 1, j + 2, \cdots$, are continuous functions in $\Omega$, because of (2) and (5). Furthermore, by the definition of the $a_i$'s, the series

$$\sum_{i=j+1}^{\infty} a_i Q^{k_j}(D)f_i$$

converges almost uniformly in $\Omega$, and so $Q^{k_j}(D)h_j$ is in $\Omega$ a continuous function. Thus

$$Q^{k_j}(D)f = Q^{k_j}(D)g_j + Q^{k_j}(D)h_j$$

is in $\Omega'$ a distribution of order $m_j$. Since $m_j \to \infty$, $f$ is not in $\mathscr{E}_Q(\Omega)$. This contradiction proves (1).

Consider now the fundamental solution $E$ of the iterated Laplace equation, i.e.

$$\Delta^{\gamma}E = \delta .$$

For sufficiently large $\gamma$, $E$ is $m$ times continuously differentiable. Therefore every distribution $T$ on $\Omega'$ such that $\Delta^{\gamma}T \in \mathscr{D}'^m(\Omega')$ is, in fact, a continuous function (see [5], vol. 2, p. 47). We choose $\mu = 2\gamma + \nu$, where $\nu$ is the integer occurring in (1). Then, if $f \in C_P^{\mu,\infty}(\Omega)$, it follows that $\Delta^{\gamma}f \in C_P^{\nu,\infty}(\Omega)$ whence, in view of (1), $Q^k(D)\Delta^{\gamma}f \mid \Omega' = \Delta^{\gamma}Q^k(D)f \mid \Omega' \in \mathscr{D}'^m(\Omega')$. Thus, by what we said before, $Q^k(D)f \mid \Omega'$ is a continuous function, for every $k = 0, 1, \cdots$, i.e. $f \mid Q' \in C_Q^{0,\infty}(\Omega')$. The proof is complete.

**2. Necessary conditions.** We proceed to derive necessary conditions for the inclusion $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$. In view of Theorem 1 it suffices to find necessary conditions for the inclusion

$$\{f \mid \Omega' : f \in C_P^{\mu,\infty}(\Omega)\} \subset C_Q^{0,\infty}(\Omega') . \tag{8}$$

We accomplish this by means of the standard argument based on the closed graph theorem and the Seidenberg-Tarski theorem (see [1]).

Let $\Omega_j$, $j = 1, 2, \cdots$, be open sets satisfying conditions (6). We define the topology in $C_P^{\mu,\infty}(\Omega)$ by means of the semi-norms

$$v_j(f) = \sup | P^k(D)D^{\alpha}f(x) | ,$$

where the supremum is taken over all $x \in \Omega_j$, $|\alpha| \leqq \mu$ and $k \leqq j$. Similarly, if $\Omega'_j$, $j = 1, 2, \cdots$, are open sets satisfying conditions analogous to (6) with $\Omega$ replaced by $\Omega'$, we define the topology in $C_Q^{0,\infty}(\Omega')$ by means of the semi-norms

$$w_j(f) = \sup_{x \in \Omega'_j, k \leqq j} | Q^k(D)f(x) | .$$

Then $C_P^{\mu,\infty}(\Omega)$ and $C_Q^{0,\infty}(\Omega')$ become Fréchet spaces. Moreover, the restriction mapping $C_P^{\mu,\infty}(\Omega) \rightarrow C_Q^{0,\infty}(\Omega')$ is closed and therefore continuous, by the closed graph theorem. Hence, to every integer $l > 0$, there exists an integer $k > 0$ and a constant $C > 0$ such that

$$w_l(f) \leqq C \max_{1 \leqq j \leqq k} v_j(f) , \tag{9}$$

for every $f \in C_P^{\mu,\infty}(\Omega)$. Applying condition (9) to the function

$$f(x) = e^{i\langle x, \zeta\rangle} ,$$

where $\zeta = \xi + i\eta$ and $\xi, \eta \in R^n$, we obtain the following lemma[2].

LEMMA 1. *If the inclusion* (8) *holds then, for every integer* $l > 0$, *we can find an integer* $k > 0$ *and constants* $C, c > 0$ *such that*

[2] We assume that $D^{\alpha} = D_1^{\alpha_1}D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, where $D_j = -i(\partial/\partial x_j)$.

$$(10) \qquad |Q^l(\zeta)| \leqq C(1 + |\xi|^{\mu})(1 + |P^k(\zeta)|)e^{c|\eta|} .$$

We denote by $N(P, a)$, $V_a$ and $W_a$ the sets of all $\zeta = \xi + i\eta \in C^n$ such that $|P(\zeta) \leqq a$, $|\eta| \leqq a$ and $|\xi| \leqq a$, respectively.

LEMMA 2. *If condition* (10) *is satisfied, then* $Q(\zeta)$ *is bounded on every set* $N(P, a) \cap V_b$, $a, b \geqq 0$.

*Proof.* Suppose there are $a, b \geqq 0$ such that $Q(\zeta)$ is not bounded on $N(P, a) \cap V_b$. Then the function

$$s(t) = \sup_{\zeta \in N(P,a) \cap V_b \cap W_t} |Q(\zeta)|$$

is defined and continuous for sufficiently large $t$, and

$$(11) \qquad s(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty .$$

But, for a given $t$, $s(t)$ is the largest of all $s$ such that the equations and inequalities

$$(12) \qquad \begin{gathered} |P(\xi + i\eta)|^2 \leqq a^2, |\eta|^2 \leqq b^2 , \\ |Q(\xi + i\eta)|^2 = s^2, |\xi|^2 \leqq t^2, s \geqq 0, t \geqq 0 , \end{gathered}$$

have a solution $\xi, \eta \in R^n$. Applying to (12) the Seidenberg-Tarski theorem and next a well-known argument (see [4], p. 276, or [6], p. 317) one shows easily that, for sufficiently large $t$, $s(t)$ is an algebraic function. We now expand $s(t)$ in a Puiseux series in a neighborhood of infinity and make use of (11). It follows that

$$s(t) > t^h$$

for some $h > 0$ and all $t$ sufficiently large. On the other hand, $s(t)$ is assumed for some $\xi = \xi(t)$, $\eta = \eta(t)$, and

$$|\xi(t)| \leqq t .$$

Choosing in (10) $l > \mu h^{-1}$ we obtain a contradiction, which proves the lemma.

THEOREM 2. *If* $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, *then the following equivalent conditions are satisfied:*

($I_1$) $Q(\zeta)$ *is bounded on every set* $N(P, a) \cap V_b$.

($I_2$) *For any* $a \geqq 0$ *there are constants* $C, h > 0$ *such that*

$$|Q(\zeta)|^h \leqq C(1 + |\eta|), \text{ for all } \zeta \in N(P, a) .$$

($I_3$) *For any* $b \geqq 0$ *there are constants* $C', h' > 0$ *such that*

$$|Q(\zeta)|^{h'} \leqq C'(1 + |P(\zeta)|), \text{ for all } \zeta \in V_b .$$

*Proof.* In view of Theorem 1, Lemma 1 and Lemma 2, we need only to show that conditions $(I_1)$-$(I_3)$ are equivalent. Also the implications $(I_2) \Rightarrow (I_1)$ and $(I_3) \Rightarrow (I_1)$ are obvious. We prove that $(I_1) \Rightarrow (I_2)$.

Consider the real polynomial

$$W(\xi, \eta, r, s, t) \\ = (a^2 - |P(\xi + i\eta)|^2 - r^2)^2 + (s^2 - |\eta|^2)^2 + (t^2 - |Q(\xi + i\eta)|^2)^2$$

of $2n + 3$ real variables. If $\xi, \eta \in R^n$ lie on the surface

$$W(\xi, \eta, r, s, t) = 0 , \tag{13}$$

then $\zeta = \xi + i\eta \in N(P, a)$. Moreover, by condition $(I_1)$, the surface (13) is contained in a domain defined by an inequality

$$|s| > \varphi(|t|),$$

where $\varphi(\tau) \to \infty$ as $\tau \to \infty$. Applying now a theorem of Gorin ([1], Theorem 4.1) we conclude that there exist constants $C, h > 0$ satisfying condition $(I_2)$. Thus $(I_1) \Rightarrow (I_2)$. The proof of the implication $(I_1) \Rightarrow (I_3)$ is similar.

**3. Sufficient conditions.** We now prove that conditions $(I_1)$-$(I_3)$ are sufficient for the inclusion under consideration. Our first goal is to construct a sequence of suitable fundamental solutions for the operators $P^k(D)$, $k = 1, 2, \cdots$. We achieve this by modifying the construction of a fundamental solution for $P(D)$ given in [2].

In what follows $p$ and $q$ denote the orders of the differential operators $P(D)$ and $Q(D)$, respectively.

LEMMA 3. *Suppose that conditions* $(I_1)$-$(I_3)$ *are satisfied. Then there exist continuous functions* $F_k$, $k = 1, 2, \cdots$, *in* $R^n$ *with the following properties*:

(a) *For* $\nu = p + q + n$ *and any* $k$,

$$E_k = (\lambda - \Delta)^\nu F_k$$

*is a fundamental solution for* $P^k(D)$, *i.e.*

$$P^k(D)E_k = \delta$$

(b) $P^j(D)F_k = F_{k-j}$, *for* $j = 1, 2, \cdots, k - 1$.

(c) $Q^l(D)F_k$, $k, l = 1, 2, \cdots$, *are continuous functions in* $R^n \backslash \{0\}$.

(d) *For any* $l$ *there is a* $k$ *such that* $Q^l(D)F_k$ *is a continuous function in* $R^n$.

*Proof.* For any $\xi' = (\xi_1, \cdots, \xi_{n-1}) \in R^{n-1}$, consider the subset of the complex $\zeta_n$-plane

$$U(\xi') = \{\zeta_n \in C: |P(\xi', \zeta_n)| \leqq 1 \text{ or } |\lambda + |\xi'|^2 + \zeta_n^2| \leqq 1\},$$

where $\lambda > 2p$. There exist constants $C, h > 0$ such that

$$|Q(\xi', \zeta_n)| \leqq C(1 + |\eta_n|^h), \tag{14}$$

for all $\xi' \in R^{n-1}$ and $\zeta_n = \xi_n + i\eta \in U(\xi')$. This follows from $(\mathrm{I}_2)$, when $|P(\xi', \zeta_n)| \leqq 1$ and can be easily verified in the other case.

Let $U^-(\xi')$ be the union of all connected components of $U(\xi')$ having nonempty intersections with $C^- = \{\zeta_n \in C: \eta_n < 0\}$. We denote by $L(\xi')$ the boundary of $C^- \cup U^-(\xi')$.

If $\zeta_n \in L(\xi')$, we have

$$|P(\xi', \zeta_n)| \geqq 1; \tag{15}$$

also there are constants $C', h' > 0$ (independent of $\xi'$) such that

$$|Q(\xi', \zeta_n)| \leqq C' |P(\xi', \zeta_n)|^{h'}. \tag{16}$$

Inequality (16) is implied by $(\mathrm{I}_3)$ and (15), since $(\xi', \zeta_n) \in V_{2p}$, when $\zeta_n \in L(\xi')$.

For $k = 1, 2, \cdots$, we set

$$F_k(x) = \frac{1}{(2\pi)^n} \int_{R^{n-1}} \left\{ \int_{L(\xi')} \frac{e^{i\langle x, \zeta\rangle}}{(\lambda + |\xi'|^2 + \zeta_n^2)^\nu P^k(\zeta)} d\zeta_n \right\} d\xi'.$$

The functions $F_k$ are obviously continuous, because of (15). We claim that they satisfy the conditions (a)-(d).

Conditions (a) and (b) follow from general properties of the Fourier transforms of distributions.

The verification of condition (c) can be carried out in the same way as in [2] (see the proof of Lemma 4). We give a brief sketch of the argument.

Suppose first that, for a given $k$, $F_k^{(j)}$ is a function obtained by a construction as above, where the contour of integration (corresponding to $L(\xi')$) lies in the complex $\zeta_j$-plane; in particular $F_k^{(n)} = F_k$. Then

$$Q^l(D)[F_k - F_k^{(j)}],\ j = 1, \cdots, n-1;\ l = 1, 2, \cdots,$$

are continuous functions in $R^n$; we omit the easy proof of this fact. Thus condition (c) will be verified, if we show that $Q^l(D)F_k^{(j)}$, $l = 1, 2, \cdots$, are continuous for $x_j \neq 0$ $(j = 1, \cdots, n)$.

Consider, for example, the function $F_k$ and let $x_n < 0$. In this case the contour $L(\xi')$ can be replaced by the boundary $V^-(\xi')$ of $U^-(\xi')$. By (14), there are positive constants $C_1$ and $C_2$ such that

$$\eta_n \leqq -C_1 |Q(\xi', \zeta_n)|^{1/h} + C_2$$

for all $\xi' \in R^{n-1}$ and $\zeta_n \in V^-(\xi')$. Hence, if $\zeta = (\xi', \zeta_n)$, we have

$$| Q^l(\zeta)e^{i\langle x,\zeta\rangle} | \leqq | Q(\zeta) |^l \exp\{x_n(C_1 | Q(\zeta) |^{1/h} - C_2)\} .$$

It follows that the integral

$$\int_{R^{n-1}} \left\{\int_{V^-(\xi')} \frac{Q^l(\zeta)e^{i\langle x,\zeta\rangle}}{(\lambda + | \xi' |^2 + \zeta_n^2)^\nu P^k(\zeta)} d\zeta_n\right\} d\xi'$$

converges absolutely and coincides with $Q^l(D)F_k(x)$, for every $l$.

In case $x_n > 0$ we can reason similarly, replacing $L(\xi')$ by a contour $V^+(\xi')$ lying entirely in the half-plane $\eta_n \geqq 0$.

Condition (d) is a consequence of inequality (16). In fact,

$$\frac{Q^l(\xi', \zeta_n)}{P^k(\xi', \zeta_n)}$$

is bounded for $\xi' \in R^{n-1}$, $\zeta_n \in L(\xi')$, whenever $k \geqq h'l$.

Lemma 3 is now established.

THEOREM 3. *If conditions* $(\mathrm{I}_1) - (\mathrm{I}_3)$ *are satisfied, the* $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, *for any open set* $\Omega \subset R^n$.

*Proof.* Assume that $T \in \mathscr{E}_P(\Omega)$ and fix an arbitrary open set $\Omega' \subset\subset \Omega$. We have to show that the restrictions of $Q^l(D)T$, $l = 0, 1, \cdots$, to $\Omega'$ are all in a space $\mathscr{D}'^m(\Omega')$.

By Lemma 3, there are fundamental solutions $E_k$ for the operators $P^k(D)$, $k = 1, 2, \cdots$, representable according to (a) with the functions $F_k$ satisfying conditions (b) – (d). Let $l$ be given and let $k$ be the integer corresponding to $l$ in condition (d).

There are open sets $\Omega_j$, $j = 0, 1, \cdots, k + 1$, such that

$$\Omega' \subset\subset \Omega_{k+1} \subset\subset \Omega_k \subset\subset \cdots \subset\subset \Omega_0 \subset\subset \Omega . \tag{17}$$

Since $T \in \mathscr{E}_P(\Omega)$, the restrictions of $P^j(D)T$, $j = 0, 1, \cdots$, to $\Omega_0$ are all of order $\leqq m_0$, say. For every $j = 1, 2, \cdots, k + 1$, we now choose a function $\varphi_j \in \mathscr{D}(\Omega_{j-1})$ such that $\varphi = 1$ on $\Omega_j$. Then the distributions

$$S_1 = \varphi_1 T,\ S_j = \varphi_j P(D) S_{j-1},\ j = 2, 3, \cdots, k + 1 ,$$

are all of order $\leqq m_0$. Moreover

$$S_1 = T \text{ on } \Omega_1 \tag{18}$$

and

$$P(D)S_j - S_{j+1} = 0 \text{ on } \Omega_{j+1} , \qquad j = 1, \cdots, k. \tag{19}$$

Making use of (a) we may write

$$S_1 = \sum_{j=1}^{k} [P(D)S_j - S_{j+1}] * E_j + S_{k+1} * E_k ,$$

whence

$$(20) \qquad Q^l(D)S_1 = \sum_{j=1}^{k} [P(D)S_j - S_{j+1}] * Q^l(D)E_j + S_{k+1} * Q^l(D)E_k \,;$$

here $*$ denotes the convolution. By (19), the "values" on $\Omega'$ of each convolution

$$[P(D)S_j - S_{j+1}] * Q^l(D)E_j$$

depend on the values of $Q^l(D)E_j$ outside a neighborhood of the origin (see [5], Chapter VI, Theorem III). Therefore the restriction to $\Omega'$ of the sum in (20) is a distribution of order $\leqq m_0 + p + 2\nu$. On the other hand, the last term in (20) is of order $\leqq m_0 + p + 2\nu$, because of (a) and (d). Hence the restriction of $Q^l(D)S_1$ to $\Omega'$ is of order $\leqq m = m_0 + p + 2\nu$ and $m_0$ can be chosen the same for all $l$. Since, by (18), the restrictions of $Q^l(D)S_1$ and $Q^l(D)T$ to $\Omega'$ coincide, the theorem is proved.

Combining Theorem 2 with Theorem 3 we obtain the following corollary.

COROLLARY. *Each of the conditions* $(\mathrm{I}_1) - (\mathrm{I}_3)$ *is necessary and sufficient for the inclusion* $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, *where* $\Omega$ *is any nonempty open set.*

REMARK. Suppose that

$$Q(\zeta) = P(\zeta) \sum_{j=1}^{n} \zeta_j^2$$

where $P(\zeta)$ is an arbitrary polynomial. Then the operator $P(D)$ is $Q$-hypoelliptic (see [2], Theorem 1), but condition $(\mathrm{I}_3)$ is not satisfied, unless $P(D)$ (and consequently $Q(D)$) is hypoelliptic.

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