DERIVED ALGEBRAS IN $L_1$ OF A COMPACT GROUP

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Let $G$ be a compact topological group. In this paper, it is shown that the derived algebra $D_p$ of $L_p(G)$ (for $1 \leq p < \infty$) is contained in the ideal $S_p$ of functions in $L_p(G)$ with unconditionally convergent Fourier series. It is also noted that this inclusion can be strict if $G$ is nonabelian. Finally, it is shown that the derived algebra of the center of $L_p(G)$ is always equal to the center of $S_p$, generalizing a known result that $D_p = S_p$ when $G$ is compact and abelian.

In general, let $(A, || \cdot ||_A)$ be a Banach algebra which is an essential left Banach $L_1(G)$-module in $L_1(G)$ under convolution. For convenience and with no loss of generality it is assumed that

$$|| f ||_A \geq || f ||_1$$

for every $f \in A$.

This paper investigates the relationship between the derived algebra of $A$ and the ideal in $A$ of functions with unconditionally convergent Fourier series. Bachelis has shown in [1] that in case $G$ is abelian and $A$ is equal to $L_p(G)$, for $1 \leq p < \infty$, the two algebras coincide.

Bachelis’ result is generalized to the derived algebra of the center of $L_p(G)$ and it is shown that for the compact group $\mathcal{G}^\sigma$ and $A = L_p(\mathcal{G}^\sigma)$ with $p \neq 2$, the derived algebra is strictly contained in the ideal of functions in $L_p(\mathcal{G}^\sigma)$ whose Fourier series converge unconditionally.

Notation throughout will be as in [4]. $\Sigma$ will denote the dual object of $G$, the set of equivalence classes of continuous irreducible unitary representations of $G$. For each $\sigma \in \Sigma$, $H_\sigma$ will denote the representation space of $\sigma$ (of finite dimension $d_\sigma$) and $\mathcal{E}(\Sigma)$ will denote the product space $\prod_{\sigma \in \Sigma} B(H_\sigma)$. Important subspaces of $\mathcal{E}(\Sigma)$ referred to in the text include:

(1) $\mathcal{E}_0(\Sigma) = \{E = \{E_\sigma\}: || E_\sigma ||_{sp} \text{ is small off finite sets}\}$

(2) $\mathcal{E}_1(\Sigma) = \{E = \{E_\sigma\}: || E ||_1 = \sum_{\sigma \in \Sigma} d_\sigma \text{ || } E_\sigma ||_{sp} < \infty\}$

(3) $\mathcal{E}_2(\Sigma) = \{E = \{E_\sigma\}: || E ||_2 = \sum_{\sigma \in \Sigma} d_\sigma \text{ || } E_\sigma ||_{p_2} < \infty\}$.

For $f \in L_1(G)$, $f$ has Fourier series $f \sim \sum_{\sigma \in \Sigma} d_\sigma \text{ tr}(A_{\sigma} \mathcal{U}(\sigma))$ where $A_{\sigma} \in B(H_{\sigma})$, $U(\sigma) \in \sigma$. The Fourier transform $\hat{f}$ of $f$ has the property that $\hat{f}(\sigma) = A_{\sigma}$ and hence:

$$|| \hat{f} ||_\infty = \sup_{\sigma \in \Sigma} || A_{\sigma} ||_{sp}.$$

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1. The derived algebra. We begin by defining the derived algebra $D_A$ for an essential left Banach $L_1(G)$-module $A$, and noting a few of its properties.

**Definition 1.1.** If $f \in A$, we define

$$||f||_{D_A} = \sup_{g \in A} \frac{||f * g||_A}{||g||_\infty}$$

and let

$$D_A = \{f \in A : ||f||_{D_A} < \infty\}.$$

The following facts are easy to check.

**Proposition 1.2.** (i) $(D_A, ||||_{D_A})$ is a Banach algebra and a left Banach $L_1(G)$-module in $L_1(G)$ under convolution.

(ii) $||f||_A \leq ||f||_{D_A}$ for every $f \in A$.

(iii) If we denote the set of trigonometric polynomials by $T(G)$ then we have

$$||f||_{D_A} = \sup_{g \in T(G)} \frac{||f * g||_A}{||g||_\infty}$$

for every $f \in A$.

We next give a characterization of $D_A$ which is due essentially to Helgason ([3], Theorem 2).

**Theorem 1.3.** (Helgason)

$$D_A = \{f \in A : \hat{f}E \in \hat{A}, \text{ for every } E \in \mathcal{E}_0(\Sigma)\}.$$

**Proof.** Suppose $f \in A$ and that for $E \in \mathcal{E}_0(\Sigma)$, $\hat{f}E = \hat{g}_E$ for some $g_E \in A$. Then the linear map $E \to g_E$ of $\mathcal{E}_0(\Sigma)$ into $A$ has closed graph and is therefore continuous. In particular, there exists a constant $k > 0$ such that

$$||f * h||_A \leq k ||\hat{h}||_\infty$$

for every $h \in A$.

Consequently, $f$ belongs to $D_A$.

Conversely, if $f \in D_A$ then the continuous map $\hat{g} \to f * g$ of $\hat{A}$ into $A$ extends to a continuous map $E \to h_E$ of $\mathcal{E}_0(\Sigma)$ into $A$. Then the element $\hat{f}E = \hat{h}_E$ belongs to $\hat{A}$ for every $E \in \mathcal{E}_0(\Sigma)$.
This characterization of $D_A$ gives two more properties of $D_A$.

**Corollary 1.4.**

(i) $D_A$ is an ideal in $L_1(G)$ and

(ii) $D_A$ is a right ideal in $\mathfrak{g}(G)$.

We denote by $C(G)$ the algebra of continuous complex valued functions on $G$, and by $K(G)$ the algebra of functions on $G$ with absolutely convergent Fourier series (see [4], Sect. 34).

For $1 \leq p < \infty$, the derived algebra of $L_p(G)$ is denoted by $D_p$.

**Examples 1.5.**

(i) $D_{K(G)} = K(G)$,

(ii) $D_{C(G)} = K(G)$, and

(iii) $D_p = L_p(G)$ for $1 \leq p \leq 2$.

**Proof.** First we show (i). Let $f$ belong to $K(G)$ and $g$ to $T(G)$. Then $\| f \ast g \|_K = \| \hat{f} \hat{g} \|_K \leq \| \hat{f} \|_1 \| \hat{g} \|_\infty = \| f \|_K \| \hat{g} \|_\infty$. Hence, by (1.2), $f$ belongs to $D_{K(G)}$.

To see (ii), observe that since $\| \| \cdot \|_u \leq \| \cdot \|_{K(G)}$ on $K(G)$, it follows that $K(G) = D_{K(G)} \subset D_{C(G)}$. Conversely, let $f \in D_{C(G)}$ with Fourier series given by

$$ f \sim \sum_{\sigma \in \Sigma} d_\sigma tr(A_\sigma U^{(\sigma)}). $$

For each $\sigma \in \Sigma$, let $V_\sigma$ be the unitary matrix such that $V_\sigma A_\sigma = |A_\sigma|$. For $F \subset \Sigma$, a finite set, define:

$$ g = \sum_{\sigma \in F} d_\sigma tr(V_\sigma U^{(\sigma)}). $$

Then $g \in T(G)$, $\| \hat{g} \|_\infty = 1$ and we have:

$$ \sum_{\sigma \in F} \| A_\sigma \|_{d_1} = \sum_{\sigma \in F} d_\sigma tr(A_\sigma) = \| f \ast g(e) \|_u \leq \| f \ast g \|_u \leq \| f \|_{D_C}. $$

Hence $\| f \|_{K(G)} \leq \| f \|_{D_C}$ and $f \in K(G)$.

To prove (iii), we use the facts (see [4], 36.10, 36.12) that $D_1 = L_1(G)$ and

$$ 2^{-1/2} \| f \|_2 \leq \| f \|_{D_1} \leq \| f \|_2 \quad \text{for every } f \in L_2(G). $$

It $1 < p \leq 2$ and $f \in L_2(G)$, then for $g \in T(G)$ we see that

$$ \| f \ast g \|_p \leq \| f \ast g \|_2 = \| \hat{f} \hat{g} \|_2 \leq \| \hat{f} \|_2 \| \hat{g} \|_\infty = \| f \|_2 \| \hat{g} \|_\infty. $$

Hence, we conclude that $\| f \|_{D_p} \leq \| f \|_2$ and

$$ \| f \|_{D_p} \geq \| f \|_{D_1} \geq 2^{-1/2} \| f \|_2. $$

2. The ideal in $A$ of functions with unconditionally con-
vergent Fourier series. Let \( \mathscr{F} \) denote the family of all nonvoid finite subsets of \( \Sigma \). For \( F \in \mathscr{F} \), let \( D(F) = \sum_{\sigma \in F} d_{\sigma} \chi_{\sigma} \). For \( f \) in \( L_1(G) \), \( f \ast D(F) \) is the finite partial sum of the Fourier series of \( f \) consisting only of terms involving elements of \( F \). We say that \( f \) in \( A \) has unconditionally convergent Fourier series in \( A \) whenever 

\[
\lim_{F \in \mathscr{F}} \| f - f \ast D(F) \|_A = 0.
\]

We denote by \( S_A \) the family of all functions in \( A \) with this property. If we also define 

\[
\| f \|_{S_A} = \sup_{F \in \mathscr{F}} \| f \ast D(F) \|_A,
\]

then the following facts are easily verified.

**Proposition 2.1.** (i) If \( f \in S_A \), then \( \| f \|_{S_A} < \infty \).

(ii) \((S_A, \| \|_{S_A})\) is a Banach algebra.

(iii) \( \| f \|_A \leq \| f \|_{S_A} \) for every \( f \in A \).

(iv) If \( f \in S_A \), then \( \lim_{F \in \mathscr{F}} \| f - f \ast D(F) \|_{S_A} = 0 \).

(v) \( S_A \) is an essential left Banach \( L_1(G) \)-module in \( L_1(G) \) under convolution.

Since \( S_A \) satisfies the conditions we have postulated for \( A \), we may compute its derived algebra.

**Theorem 2.2.** (i) \( D_{S_A} = D_A \cap S_A \) and \( \| f \|_{D_{S_A}} = \| f \|_{D_A} \) for \( f \in D_{S_A} \).

(ii) \( S_{S_A} = S_A \) (isometry).

**Proof.** Suppose \( f \) belongs to \( D_{S_A} \). Then for \( f \in S_A \) and \( g \in T(G) \) we have 

\[
\| f \ast g \|_A \leq \| f \|_{S_A} \| g \|_\infty \leq \| f \|_{D_{S_A}}.
\]

Hence we have \( \| f \|_{D_A} \leq \| f \|_{D_{S_A}} < \infty \), and thus \( f \) belongs to \( D_A \cap S_A \).

Conversely, if \( f \in D_A \cap S_A \) then for \( g \in T(G) \) and \( F \in \mathscr{F} \), we have 

\[
\| f \ast g \ast D(F) \|_A \leq \| f \ast g \ast D(F) \|_{D_A} \leq \| f \|_{D_A} \ast D(F).
\]

Thus it follows that \( \| f \|_{D_{S_A}} \leq \| f \|_{D_A} < \infty \), and \( f \) belongs to \( D_{S_A} \).

Part (ii) follows immediately from (2.1, iv).

3. Central derived algebras. Let \( A^* \) denote the center of \( A \). Then \( A^* = L_1(G) \cap A \) and \((A^*, \| \|_A)\) is an essential Banach \( L_1 \)-module
in $L^1_G$ under convolution. Before we investigate the derived algebra of $A^\circ$, we prove a useful proposition.

**Proposition 3.1.** For $E \in \mathcal{C}_0^\infty(\Sigma)$, define a function $\mathcal{P}_E$ on $\Sigma$ by:

$\mathcal{P}_E(\sigma) = 1/d_\sigma \text{tr}(E_\sigma)$ for every $\sigma \in \Sigma$. The map $E \rightarrow \mathcal{P}_E$ is an isometric isomorphism of

(i) $\mathcal{C}_0^\infty(\Sigma)$ onto $l_\infty(\Sigma)$,
(ii) $\mathcal{C}_0^\circ(\Sigma)$ onto $c_0(\Sigma)$, and
(iii) $\mathcal{C}_0^\circ(\Sigma)$ onto $c_0(\Sigma)$.

For $f \in L^1_G$, let $\hat{f}(\sigma) = 1/d_\sigma \text{tr}(\hat{f}(\sigma)) = \mathcal{P}_\hat{f}(\sigma)$, so that $f$ has Fourier series $\sum_{\sigma \in \Sigma} d_\sigma \hat{f}(\sigma) \chi_\sigma$. Then the map $f \rightarrow \hat{f}$ is the Gel'fand transform $A^\circ$, $\Sigma$ is the maximal ideal space of $A^\circ$, and

(iv) $\|f\|_\infty = \|\hat{f}\|_\infty$ for every $f \in L^1_G$.

**Proof.** Let $E$ belong to $\mathcal{C}_0^\infty(\Sigma)$. By Schur’s lemma we have

(1) $E_\sigma = \mathcal{P}_E(\sigma) I_{d_\sigma}$ for $\sigma \in \Sigma$.

It follows that

(2) $\|E\|_\infty = \|\mathcal{P}_E\|_\infty$.

Clearly the map $E \rightarrow \mathcal{P}_E$ is linear and carries $\mathcal{C}_0^\infty(\Sigma)$ isometrically onto $l_\infty(\Sigma)$. By (1), $E \rightarrow \mathcal{P}_E$ is multiplicative. By (2), the image of $\mathcal{C}_0^\circ(\Sigma)$ is $c_0(\Sigma)$, and the image of $\mathcal{C}_0^\circ(\Sigma)$ is $c_0(\Sigma)$. The rest of the proof is analogous to ([4], 28.71).

**Definition 3.2.** For $f$ in $A^\circ$, let

$\|f\|_{D_A} = \sup_{g \in A^\circ} \frac{\|f \ast g\|_A}{\|g\|_\infty}$.

The derived algebra $D_A$ of $A^\circ$ is defined as

$D_A = \{ f \in A^\circ : \|f\|_{D_A} < \infty \}$.

The following properties of $D_A$ are easily proved.

**Proposition 3.3.** (i) $(D_A, \|\cdot\|_{D_A})$ is a Banach algebra and an $L^1_G$-module under convolution.

(ii) $\|f\|_A \leq \|f\|_{D_A}$ for every $f \in A^\circ$.

(iii) $\|f\|_{D_A} = \sup_{g \in \tau(G)} \|f \ast g\|_A/\|g\|_\infty$ for every $f \in A^\circ$.

(iv) $D_A' \subset D_A$.

Helgason’s characterization (1.3) has an analogue in the central case. We omit the proof since it is exactly like that of (1.3).
THEOREM 3.4. (Helgason)

$$\mathcal{D}_A = \{ f \in A^\circ : f^\circ \varphi \in (A^\circ)^\circ \text{ for every } \varphi \in c_0(\Sigma) \}.$$ 

We next prove that the center $S_A^1$ of $S_A$ is always contained in $\mathcal{D}_A$. To do so, we use the following well known fact which follows from a theorem of Seever ([6]).

**FACT 3.5.** Let $X$ be a discrete topological space and $M$ a Banach space. If $T: M \to l_\omega(X)$ is a bounded linear map whose image contains the characteristic function of every subset of $X$, then $T$ is onto.

We also use the following lemma which states that every element of $l_\omega(\Sigma)$ is a multiplier for $S_A^1$.

**LEMMA 3.6.** If $f \in S_A^1$ and $\varphi \in l_\omega(\Sigma)$, then there exists $g \in S_A^1$ such that $\check{g} = \hat{\varphi} f$.

**Proof.** Let $f$ belong to $S_A^1$, and denote by $M$ the collection of all $\varphi \in l_\omega(\Sigma)$ such that $\varphi \check{f} \in (S_A^1)^0$. Then $M$ is a Banach space under the norm

$$|| \varphi || = || \varphi ||_\omega + || g ||_{S_A} \text{ where } \check{g} = \hat{\varphi} f.$$ 

To show $M = l_\omega(\Sigma)$, it suffices by (3.5) to show that for $\Delta \subset \Sigma$, the characteristic function $\varphi$ of $\Delta$ is an element of $M$. To establish this, we note that the net $\{ f^\circ D(E) : E \text{ finite } \subset \Delta \}$ is Cauchy in $S_A^1$, so there is a function $g$ in $S_A^1$ such that

$$\lim_{E \text{ finite } \subset \Delta} || g - f^\circ D(E) ||_{S_A} = 0.$$ 

We conclude that $\check{g} = \hat{\varphi} f$ and hence, $\varphi$ belongs to $M$.

**THEOREM 3.7.** $S_A^1 \subset \mathcal{D}_A$. 

**Proof.** Suppose $f$ belongs to $S_A^1$. Then for $\varphi \in c_0(\Sigma) \subset l_\omega(\Sigma)$, $\check{f} \hat{\varphi}$ belongs to $(S_A^1)^0$ and hence to $(A^\circ)^0$ by (3.6). Therefore $f \in \mathcal{D}_A$ by (3.4).

We now restrict our attention to the case of $A = L_p(G)$ for $1 \leq p < \infty$. As before we write $D_A = D_p$; we also write $S_A = S_p$ and $\mathcal{D}_A = \mathcal{D}_p$. To compare $D_p$ and $S_p$ we use the following.
**Lemma 3.8.** Let $1 \leq p < \infty$. If $f \in L_p(G)$ and $\| f \|_{S_p} < \infty$, then $f \in S_p$.

**Proof.** Let $f$ belong to $L_p(G)$ with $\| f \|_{S_p} < \infty$. Suppose $f$ has Fourier series

$$f \sim \sum_{j=1}^{\infty} d_{aj} tr(A_{aj} U^{\sigma j}).$$

For $\varphi \in L_p(G)^*$ and any nonvoid finite $F \subset Z^+$, we have

$$\left| \sum_{j \in F} \varphi(d_{aj} tr(A_{aj} U^{\sigma j})) \right| \leq \| f \|_{S_p} \| \varphi \|_{\mathcal{S}p}.$$

Hence, we see

$$\sup_{\text{finite } Z^+} \left| \sum_{j \in F} \varphi(d_{aj} tr(A_{aj} U^{\sigma j})) \right| < \infty,$$

which implies

$$\sum_{j=1}^{\infty} | \varphi(d_{aj} tr(A_{aj} U^{\sigma j})) | < \infty.$$

Thus the Fourier series of $f$ is weakly subseries Cauchy and, since $L_p(G)$ is weakly complete, the series is weakly subseries convergent. Therefore, by the Orlicz-Pettis theorem ([2], p. 60, or [6], p. 19) it is norm convergent and unconditionally convergent to some $g \in L_p(G)$. Comparing transforms, we see that $f = g$ and consequently, $f$ belongs to $S_p$.

Finally, we state the main result of this section, generalizing the abelian result of Bachelis.

**Theorem 3.9.** Let $1 \leq p < \infty$. Then we have

(i) $D_p \subseteq S_p$, and

(ii) $\mathcal{D}_p = S_p^*$.

**Proof.** Observe that $\| f \|_{S_p} \leq \| f \|_{D_p}$ for every $f \in D_p$, and that $\| f \|_{S_p} \leq \| f \|_{\mathcal{D}_p}$ for every $f \in \mathcal{D}_p$. The theorem now follows from (3.8).

4. $\mathcal{S}_3^\infty$ as a source of examples. Throughout this section $G$ will denote $\mathcal{S}_3^\infty = \prod_{n \in \mathbb{N}} \mathcal{S}_3$, where $\mathcal{S}_3$ is the symmetric group on three symbols. Using this group we demonstrate that Bachelis' result does not extend to the non-abelian case.

**Theorem 4.1.** Let $G = \mathcal{S}_3^\infty$ and $1 \leq p < \infty$. Then

(i) $D_p = S_p$ if and only if $p = 2$, and
(ii) $D_p = L_p$ if and only if $p = 2$.

Proof. By (1.5, iii) and (3.9), we have
$$L_2(G) = D_2 \subset S_2 \subset L_2(G).$$

Suppose $p \neq 2$. Observe that (ii) follows from (i) because
$$D_p \subset S_p \subset L_p.$$

Note also that $\|f\|_{S_p} \leq \|f\|_{D_p}$ for every $f \in D_p$. Hence to prove that $D_p \neq S_p$ it is enough to find sequences $\{f^{(n)}\}$ in $D_p$ and $\{g^{(n)}\}$ in $T(G)$ such that

$$\lim_{n \to \infty} \frac{\|f^{(n)} * g^{(n)}\|_p}{\|g^{(n)}\|_\infty \|f^{(n)}\|_{S_p}} = \infty.$$

We select these sequences as follows. Let $\sigma$ be the representation class on $\mathcal{S}_q$ of dimension 2 (see [4], 27.61). For $f$ and $g$ in $T_{\sigma}(\mathcal{S}_q)$ which will be specified later, form

$$f^{(n)}(\underline{x}) = \prod_{k=1}^{n} f(x_k)$$

and

$$g^{(n)}(\underline{x}) = \prod_{k=1}^{n} g(x_k),$$

where $\underline{x} \in G$ is given by $\underline{x} = (x_1, x_2, \cdots)$. Then $f^{(n)}$ and $g^{(n)}$ are elements of $T_{\sigma^{(n)}}(G)$ where $\sigma^{(n)}$ is the element of $\Sigma_\sigma$ given by

$$U_\underline{x}^{(n)} = U_{x_1}^{(n)} \otimes \cdots \otimes U_{x_n}^{(n)}$$

for every $\underline{x} \in G$.

It is easily verified that

$$\|f^{(n)}\|_{S_p} = \|f^{(n)}\|_p = \|f\|_p^n,$$

and

$$\|f^{(n)} * g^{(n)}\|_p = \|f * g\|_p^n,$$

and

$$\|g^{(n)}\|_\infty = \|\hat{g}\|_\infty^n.$$

Hence, to show (1) it suffices to find $f$ and $g$ in $T_{\sigma}(\mathcal{S}_q)$ such that

$$\frac{\|f * g\|_p}{\|\hat{g}\|_\infty \|f\|_p} > 1.$$

Let $g = 2w_1^{(n)} + 2i w_2^{(n)}$ and note that $\|\hat{g}\|_\infty = 1$. The rest of the argument divides into two cases.
Case 1. $1 \leq p < 2$. In this case we let $f = 2\chi_{\sigma}$ so that $f \ast g = g$, and we compute
\[
\|f\|_p = 2 \left[ \frac{2^p + 2}{6} \right]^{1/p} \quad \text{(see [4], 27.61)}.
\]
Also, we have
\[
\|g\|_p = 2 \left[ \frac{(1 + 2^{1-p}) 2 \sqrt{2^p}}{6} \right]^{1/p},
\]
and therefore we conclude
\[
\frac{\|f \ast g\|_p}{\|g\|_\infty \|f\|_p} = 2^{1/p-1/2} > 1.
\]

Case 2. $2 < p < \infty$. In this case we let $f = 2i\mu_{12}^{(p)} + 2\mu_{21}^{(p)}$. Then $f \ast g = -2\mu_{12}^{(p)} + 2\mu_{21}^{(p)}$ and so we have
\[
\|f\|_p = \sqrt{6} \left( \frac{2}{3} \right)^{1/p} \quad \text{and} \quad \|f \ast g\|_p = 2 \sqrt{3} \left( \frac{1}{3} \right)^{1/p}.
\]
Therefore, we conclude
\[
\frac{\|f \ast g\|_p}{\|g\|_\infty \|f\|_p} = 2^{1/2-1/p} > 1.
\]

The question naturally arises as to whether $D_p$ is equal to $D_z$. The next example shows that in some cases the answer is no.

**Theorem 4.2.** If $G = \mathcal{K}_{\infty}$ and $1 \leq p < 4$, then $D_p = D_z$ if and only if $p = 2$.

**Proof.** By (1.5, iii) and (3.3, iv) we have
\[
D_z = D_z = L_z(G).
\]
Suppose $p \neq 2$. Since $D_z \subset D_p$ and $\|\cdot\|_{D_p} \leq \|\cdot\|_{D_z}$ on $D_z$, to show that $D_z \neq D_p$, it suffices to find sequences $\{f^{(n)}\}$ in $D_z$ and $\{g^{(n)}\}$ in $T(G)$ such that
\[
\|f^{(n)} \ast g^{(n)}\|_p \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.
\]
As in the proof of (4.1) we construct the sequences by choosing $f$ and $g$ on $\mathcal{K}_z$ as follows. First, let $f = 2\chi_{\sigma}$. Then $f \ast g = g$ for any $g \in T_0(\mathcal{K}_z)$, and $\|f\|_p = 2 \left[ (2^p+2)/6 \right]^{1/p}$. Also we have $f^{(n)} = 2^n\chi_{\sigma^{(n)}}$ and
\[ || f^{(n)} ||_{\mathcal{F}_p} = || f^{(n)} ||_p = || f ||_{\mathcal{F}_p}. \] As before, it suffices to find \( g \in T_\sigma(\mathcal{F}_\sigma) \) with the property that
\[
\frac{|| g ||_p}{|| \widehat{g} ||_{\infty} || f ||_p} > 1.
\]

Again we consider two cases.

**Case 1.** \( 1 \leq p < 2 \). Let \( g = 2u_{12}^{(q)} + 2iu_{21}^{(q)} \). Then as in (4.1), Case 1, we have
\[
\frac{|| g ||_p}{|| \widehat{g} ||_{\infty} || f ||_p} = 2^{1/p-1/2} > 1.
\]

**Case 2.** \( 2 < p < 4 \). Let \( g = 2u_{12}^{(q)} + 2u_{21}^{(q)} \). Then \( || \widehat{g} ||_{\infty} = 1 \) and
\[
|| g ||_p = 2 \left[ \frac{2\sqrt{3}^p}{6} \right]^{1/p}.
\]

Therefore we see
\[
\frac{|| g ||_p}{|| \widehat{g} ||_{\infty} || f ||_p} = \left[ \frac{2 \cdot 3^{p/2}}{2p+2} \right]^{1/p} > 1.
\]

Finally, we observe that for \( G = \mathcal{F}_\infty \), we have the following.

**Theorem 4.3.** \( K(G) \subseteq S_{C(G)} \).

**Proof.** Since \( || f ||_a \leq || f ||_{K(G)} \) for \( f \) is \( K(G) \), it follows that
\[
K(G) = S_{K(G)} \subseteq S_{C(G)}.
\]

Also, since \( || f ||_{S_{C(G)}} \leq || f ||_{K(G)} \) for \( f \) in \( K(G) \), to show that \( K(G) \neq S_{C(G)} \), we need only find \( f \in T_\sigma(\mathcal{F}_\sigma) \) such that
\[
\frac{|| f ||_{K(\mathcal{F}_\sigma)}}{|| f ||_{\infty}} > 1.
\]

If we let \( f = u_{12}^{(q)} + u_{21}^{(q)} \), then we have \( || f ||_{\infty} = \sqrt{3} \) and \( || f ||_{K(\mathcal{F}_\sigma)} = 2 \). Hence, the proof is complete.

The techniques used to prove (4.1) – (4.3) can also be applied to show the following.

**Theorem 4.4.** If \( G = \mathcal{F}_\infty \) and \( 1 \leq p < \infty \), then
\[
\mathcal{D}_{\mathcal{F}}(G) = L_p(G) \text{ if and only if } p = 2.
\]
5. Open questions.

(5.1) Is $T(G)$ dense in $D_A$? If so, then it can easily be shown that $D_{pA}$ is isometrically isomorphic to $D_A$. One easily shows that the density of $T(G)$ is equivalent to the condition that $S_{pA} = D_A$.

(5.2) Another question of interest is whether or not $D_A$ is self-adjoint (that is, closed under $f \rightarrow \widehat{f}$, where $\widehat{f}(x) = \frac{1}{|\mathcal{N}|} f(x^{-1})$) whenever $A$ is. Equivalently, is $D_A$ a left ideal in $\mathcal{E}_2(A)$ when $A$ is self-adjoint?

(5.3) Are there any conditions on a compact non-abelian group $G$ sufficient to imply that $D_p = S_p$ for $p \neq 2$?

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