DERIVED ALGEBRAS IN $L_1$ OF A COMPACT GROUP

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Let $G$ be a compact topological group. In this paper, it
is shown that the derived algebra $D_p$ of $L_p(G)$ (for $1 \leq p < \infty$) is contained in the ideal $S_p$ of functions in $L_p(G)$ with unconditionally convergent Fourier series. It is also noted that this inclusion can be strict if $G$ is nonabelian. Finally, it is shown that the derived algebra of the center of $L_p(G)$ is always equal to the center of $S_p$, generalizing a known result that $D_p = S_p$ when $G$ is compact and abelian.

In general, let $(A, \| \|_A)$ be a Banach algebra which is an essential left Banach $L_1(G)$-module in $L_1(G)$ under convolution. For convenience and with no loss of generality it is assumed that $\| f \|_A \geq \| f \|_1$ for every $f \in A$.

This paper investigates the relationship between the derived algebra of $A$ and the ideal in $A$ of functions with unconditionally convergent Fourier series. Bachelis has shown in [1] that in case $G$ is abelian and $A$ is equal to $L_p(G)$, for $1 \leq p < \infty$, the two algebras coincide.

Bachelis' result is generalized to the derived algebra of the center of $L_p(G)$ and it is shown that for the compact group $\mathcal{S}_3^\infty$ and $A = L_p(\mathcal{S}_3^\infty)$ with $p \neq 2$, the derived algebra is strictly contained in the ideal of functions in $L_p(\mathcal{S}_3^\infty)$ whose Fourier series converge unconditionally.

Notation throughout will be as in [4]. $\Sigma$ will denote the dual object of $G$, the set of equivalence classes of continuous irreducible unitary representations of $G$. For each $\sigma \in \Sigma$, $H_\sigma$ will denote the representation space of $\sigma$ (of finite dimension $d_\sigma$) and $\mathcal{E}(\Sigma)$ will denote the product space $\prod_{\sigma \in \Sigma} B(H_\sigma)$. Important subspaces of $\mathcal{E}(\Sigma)$ referred to in the text include:

(i) $\mathcal{E}_0(\Sigma) = \{ E = \{ E_\sigma \}; \| E_\sigma \|_{\sigma p} \text{ is small off finite sets} \}$
(ii) $\mathcal{E}_1(\Sigma) = \{ E = \{ E_\sigma \}; \| E \|_1 = \sum_{\sigma \in \Sigma} d_\sigma \| E_\sigma \|_{\sigma 1} < \infty \}$
(iii) $\mathcal{E}_2(\Sigma) = \{ E = \{ E_\sigma \}; \| E \|_2^2 = \sum_{\sigma \in \Sigma} d_\sigma \| E_\sigma \|_{\sigma 2}^2 < \infty \}$.

For $f \in L_1(G)$, $f$ has Fourier series $f \sim \sum_{\sigma \in \Sigma} d_\sigma tr(A_\sigma U^{(\sigma)})$ where $A_\sigma \in B(H_\sigma)$, $U^{(\sigma)} \in \sigma$. The Fourier transform $\hat{f}$ of $f$ has the property that $\hat{f}(\sigma) = A_\sigma$ and hence:

$$\| \hat{f} \|_\infty = \sup_{\sigma \in \Sigma} \| A_\sigma \|_{\sigma p}.$$ 

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1. The derived algebra. We begin by defining the derived algebra $D_A$ for an essential left Banach $L_1(G)$-module $A$, and noting a few of its properties.

**Definition 1.1.** If $f \in A$, we define

$$\| f \|_{D_A} = \sup_{g \in A} \| f * g \|_A$$

and let

$$D_A = \{ f \in A : \| f \|_{D_A} < \infty \}.$$

The following facts are easy to check.

**Proposition 1.2.** (i) $(D_A, \| \cdot \|_{D_A})$ is a Banach algebra and a left Banach $L_1(G)$-module in $L_1(G)$ under convolution.

(ii) $\| f \|_A \leq \| f \|_{D_A}$ for every $f \in A$.

(iii) If we denote the set of trigonometric polynomials by $T(G)$ then we have

$$\| f \|_{D_A} = \sup_{g \in T(G)} \| f * g \|_A$$

for every $f \in A$.

We next give a characterization of $D_A$ which is due essentially to Helgason ([3], Theorem 2).

**Theorem 1.3.** (Helgason)

$$D_A = \{ f \in A : \hat{f}E \in \hat{A}, \text{ for every } E \in \mathcal{S}_0(\Sigma) \}.$$

**Proof.** Suppose $f \in A$ and that for $E \in \mathcal{S}_0(\Sigma)$, $\hat{f}E = \hat{g}_E$ for some $g_E \in A$. Then the linear map $E \to g_E$ of $\mathcal{S}_0(\Sigma)$ into $A$ has closed graph and is therefore continuous. In particular, there exists a constant $k > 0$ such that

$$\| f * h \|_A \leq k \| \hat{h} \|_\infty$$

for every $h \in A$.

Consequently, $f$ belongs to $D_A$.

Conversely, if $f \in D_A$ then the continuous map $\hat{g} \to f * g$ of $\hat{A}$ into $A$ extends to a continuous map $E \to \hat{h}_E$ of $\mathcal{S}_0(\Sigma)$ into $A$. Then the element $\hat{f}E = \hat{h}_E$ belongs to $\hat{A}$ for every $E \in \mathcal{S}_0(\Sigma)$. 
This characterization of $D_A$ gives two more properties of $D_A$.

**Corollary 1.4.** (i) $D_A$ is an ideal in $L_1(G)$ and
(ii) $\hat{D}_A$ is a right ideal in $\mathscr{C}_c(\Sigma)$.

We denote by $C(G)$ the algebra of continuous complex valued functions on $G$, and by $K(G)$ the algebra of functions on $G$ with absolutely convergent Fourier series (see [4], Sect. 34).

For $1 \leq p < \infty$, the derived algebra of $L_p(G)$ is denoted by $D_p$.

**Examples 1.5.** (i) $D_{K(G)} = K(G)$,
(ii) $D_{C(G)} = K(G)$, and
(iii) $D_p = L_2(G)$ for $1 \leq p \leq 2$.

**Proof.** First we show (i). Let $f$ belong to $K(G)$ and $g$ to $T(G)$. Then $\| f * g \|_K = \| \hat{f} \hat{g} \|_1 \leq \| \hat{f} \|_1 \| \hat{g} \|_\infty = \| f \|_K \| \hat{g} \|_\infty$. Hence, by (1.2), $f$ belongs to $D_{K(G)}$.

To see (ii), observe that since $\| \|_w \leq \| \|_{K(G)}$ on $K(G)$, it follows that $K(G) = D_{K(G)} \subset D_{C(G)}$. Conversely, let $f \in D_{C(G)}$ with Fourier series given by

$$f \sim \sum_{\sigma \in \Sigma} d_\sigma tr(A_\sigma U^{(\sigma)}).$$

For each $\sigma \in \Sigma$, let $V_\sigma$ be the unitary matrix such that $V_\sigma A_\sigma = |A_\sigma|$. For $F \subset \Sigma$, a finite set, define:

$$g = \sum_{\sigma \in F} d_\sigma tr(V_\sigma U^{(\sigma)}).$$

Then $g \in T(G)$, $\| \hat{g} \|_\infty = 1$ and we have:

$$\sum_{\sigma \in F} d_\sigma \| A_\sigma \|_1 = \sum_{\sigma \in F} d_\sigma tr |A_\sigma| = |f * g(e)| \leq \| f * g \|_w \leq \| f \|_{D_C}. \quad \text{(1)}$$

Hence $\| f \|_{K(G)} \leq \| f \|_{D_{C(G)}}$ and $f \in K(G)$.

To prove (iii), we use the facts (see [4], 36.10, 36.12) that $D_1 = L_2(G)$ and

$$2^{-1/2} \| f \|_2 \leq \| f \|_{D_1} \leq \| f \|_2 \quad \text{for every } f \in L_2(G). \quad \text{(2)}$$

It $1 < p \leq 2$ and $f \in L_2(G)$, then for $g \in T(G)$ we see that

$$\| f * g \|_p \leq \| f * g \|_2 = \| \hat{f} \hat{g} \|_2 \leq \| \hat{f} \|_2 \| \hat{g} \|_\infty = \| f \|_2 \| \hat{g} \|_\infty. \quad \text{(3)}$$

Hence, we conclude that $\| f \|_{D_p} \leq \| f \|_2$ and

$$\| f \|_{D_p} \geq \| f \|_{D_1} \geq 2^{-1/2} \| f \|_2. \quad \text{(4)}$$

2. The ideal in $A$ of functions with unconditionally con-
vergent Fourier series. Let \( \mathcal{F} \) denote the family of all nonvoid finite subsets of \( \Sigma \). For \( F \in \mathcal{F} \), let \( D(F) = \sum_{\sigma \in F} d_{\sigma} \chi_{\sigma} \). For \( f \) in \( L_1(G) \), \( f * D(F) \) is the finite partial sum of the Fourier series of \( f \) consisting only of terms involving elements of \( F \). We say that \( f \) in \( A \) has unconditionally convergent Fourier series in \( A \) whenever

\[
\lim_{F \in \mathcal{F}} ||f - f * D(F)||_A = 0.
\]

We denote by \( S_A \) the family of all functions in \( A \) with this property. If we also define

\[
||f||_{S_A} = \sup_{F \in \mathcal{F}} ||f * D(F)||_A,
\]

then the following facts are easily verified.

**Proposition 2.1.** (i) \( f \in S_A \), then \( ||f||_{S_A} < \infty \).

(ii) \( (S_A, ||||_{S_A}) \) is a Banach algebra.

(iii) \( ||f||_A \leq ||f||_{S_A} \) for every \( f \in A \).

(iv) If \( f \in S_A \), then \( \lim_{F \in \mathcal{F}} ||f - f * D(F)||_{S_A} = 0 \).

(v) \( S_A \) is an essential left Banach \( L_1(G) \)-module in \( L_1(G) \) under convolution.

Since \( S_A \) satisfies the conditions we have postulated for \( A \), we may compute its derived algebra.

**Theorem 2.2.** (i) \( D_{S_A} = D_A \cap S_A \) and \( ||f||_{D_{S_A}} = ||f||_{D_A} \) for \( f \in D_{S_A} \).

(ii) \( S_{S_A} = S_A \) (isometry).

**Proof.** Suppose \( f \) belongs to \( D_{S_A} \). Then for \( f \in S_A \) and \( g \in T(G) \) we have

\[
||f * g||_A \leq ||f * g||_{S_A} \leq ||f||_{D_{S_A}}.
\]

Hence we have \( ||f||_{D_A} \leq ||f||_{D_{S_A}} < \infty \), and thus \( f \) belongs to \( D_A \cap S_A \).

Conversely, if \( f \in D_A \cap S_A \) then for \( g \in T(G) \) and \( F \in \mathcal{F} \), we have

\[
||f * g * D(F)||_A \leq ||f * g * D(F)||_{S_A} \leq ||f||_{D_A}.
\]

Thus it follows that \( ||f||_{D_{S_A}} \leq ||f||_{D_A} < \infty \), and \( f \) belongs to \( D_{S_A} \).

Part (ii) follows immediately from (2.1, iv).

3. Central derived algebras. Let \( A^z \) denote the center of \( A \). Then \( A^z = L_1^\ast(G) \cap A \) and \( (A^z, ||||_A) \) is an essential Banach \( L_1^\ast \)-module.
in $L^1(G)$ under convolution. Before we investigate the derived algebra of $A^\ast$, we prove a useful proposition.

**Proposition 3.1.** For $E \in \mathcal{E}_\omega(\Sigma)$, define a function $\varphi_E$ on $\Sigma$ by:

$$\varphi_E(\sigma) = 1/d_\sigma \text{tr}(E_\sigma)$$

for every $\sigma \in \Sigma$. The map $E \mapsto \varphi_E$ is an isometric isomorphism of

(i) $\mathcal{E}_\omega(\Sigma)$ onto $l_\omega(\Sigma)$,

(ii) $\mathcal{E}_\sigma(\Sigma)$ onto $c_\sigma(\Sigma)$, and

(iii) $\mathcal{E}_0(\Sigma)$ onto $c_0(\Sigma)$.

For $f \in L^1(G)$, let $\hat{f}(\sigma) = 1/d_\sigma \text{tr}(\hat{f}(\sigma)) = \varphi_f(\sigma)$, so that $f$ has Fourier series $\sum_{\sigma \in \Sigma} d_\sigma \hat{f}(\sigma) \chi_\sigma$. Then the map $f \mapsto \hat{f}$ is the Gel'fand transform $A^\ast$; $\Sigma$ is the maximal ideal space of $A^\ast$, and

(iv) $\|\hat{f}\|_\infty = \|f\|_\infty$ for every $f \in L^1(G)$.

**Proof.** Let $E$ belong to $\mathcal{E}_\omega(\Sigma)$. By Schur's lemma we have

$$E_\sigma = \varphi_E(\sigma) I_{d_\sigma}$$

for $\sigma \in \Sigma$.

It follows that

$$\|E\|_\infty = \|\varphi_E\|_\infty.$$

Clearly the map $E \mapsto \varphi_E$ is linear and carries $\mathcal{E}_\omega(\Sigma)$ isometrically onto $l_\omega(\Sigma)$. By (1), $E \mapsto \varphi_E$ is multiplicative. By (2), the image of $\mathcal{E}_\sigma(\Sigma)$ is $c_\sigma(\Sigma)$, and the image of $\mathcal{E}_0(\Sigma)$ is $c_0(\Sigma)$. The rest of the proof is analogous to ([4], 28.71).

**Definition 3.2.** For $f$ in $A^\ast$, let

$$\|f\|_{\mathcal{D}} = \sup_{g \in A^\ast} \|f \ast g\|_A.$$

The derived algebra $\mathcal{D}$ of $A^\ast$ is defined as

$$\mathcal{D} = \{f \in A^\ast : \|f\|_{\mathcal{D}} < \infty\}.$$

The following properties of $\mathcal{D}$ are easily proved.

**Proposition 3.3.** (i) $(\mathcal{D}, \|\|_{\mathcal{D}})$ is a Banach algebra and an $L^1(G)$-module under convolution.

(ii) $\|f\|_A \leq \|f\|_{\mathcal{D}}$ for every $f \in A^\ast$.

(iii) $\|f\|_{\mathcal{D}} = \sup_{g \in L^1(G)} \|f \ast g\|_A$ for every $f \in A^\ast$.

(iv) $D_{\mathcal{D}} = \mathcal{D}$. Helgason's characterization (1.3) has an analogue in the central case. We omit the proof since it is exactly like that of (1.3).
THEOREM 3.4. (Helgason)

\[ D_A = \{ f \in A^\circ : \hat{f} \hat{\varphi} \in (A^\circ)^\circ \text{ for every } \varphi \in c_0(\Sigma) \}. \]

We next prove that the center \( S_i^* \) of \( S_A \) is always contained in \( D_A \). To do so, we use the following well known fact which follows from a theorem of Seever ([6]).

FACT 3.5. Let \( X \) be a discrete topological space and \( M \) a Banach space. If \( T : M \to l_\infty(X) \) is a bounded linear map whose image contains the characteristic function of every subset of \( X \), then \( T \) is onto.

We also use the following lemma which states that every element of \( l_\infty(\Sigma) \) is a multiplier for \( S_i^* \).

LEMMA 3.6. If \( f \in S_i^* \) and \( \varphi \in l_\infty(\Sigma) \), then there exists \( g \in S_i^* \) such that \( \hat{g} = \hat{\varphi} \hat{f} \).

Proof. Let \( f \) belong to \( S_i^* \), and denote by \( M \) the collection of all \( \varphi \in l_\infty(\Sigma) \) such that \( \varphi \hat{f} \in (S_i^*)^\circ \). Then \( M \) is a Banach space under the norm

\[ \| \varphi \| = \| \varphi \|_\infty + \| g \|_{S_A} \text{ where } g = \varphi \hat{f}. \]

To show \( M = l_\infty(\Sigma) \), it suffices by (3.5) to show that for \( \Delta \subset \Sigma \), the characteristic function \( \varphi \) of \( \Delta \) is an element of \( M \). To establish this, we note that the net \( \{ f * D(E) : E \text{ finite} \subset \Delta \} \) is Cauchy in \( S_i^* \), so there is a function \( g \) in \( S_i^* \) such that

\[ \lim_{E \text{ finite} \subset \Delta} \| g - f * D(E) \|_{S_A} = 0. \]

We conclude that \( \hat{g} = \hat{\varphi} \hat{f} \) and hence, \( \varphi \) belongs to \( M \).

THEOREM 3.7. \( S_i^* \subset D_A \).

Proof. Suppose \( f \) belongs to \( S_i^* \). Then for \( \varphi \in c_0(\Sigma) \subset l_\infty(\Sigma) \), \( \hat{\varphi} \hat{f} \) belongs to \( (S_i^*)^\circ \) and hence to \( (A^\circ)^\circ \) by (3.6). Therefore \( f \in D_A \) by (3.4).

We now restrict our attention to the case of \( A = L_p(G) \) for \( 1 \leq p < \infty \). As before we write \( D_A = D_p \); we also write \( S_A = S_p \) and \( D_A = D_p \). To compare \( D_p \) and \( S_p \) we use the following.
**Lemma 3.8.** Let $1 \leq p < \infty$. If $f \in L_p(G)$ and $\|f\|_{S_p} < \infty$, then $f \in S_p$.

**Proof.** Let $f$ belong to $L_p(G)$ with $\|f\|_{S_p} < \infty$. Suppose $f$ has Fourier series

$$f \sim \sum_{j=1}^{\infty} d_{\alpha_j} tr(A_{\alpha_j} U^{(\alpha_j)}).$$

For $\varphi \in L_p(G)^*$ and any nonvoid finite $F \subset \mathbb{Z}^+$, we have

$$\left| \sum_{j \in F} \varphi(d_{\alpha_j} tr(A_{\alpha_j} U^{(\alpha_j)})) \right| \leq \|f\|_{S_p} \| \varphi \|_{\alpha_p}.$$

Hence, we see

$$\sup_{F_{\text{finite}} \subset \mathbb{Z}^+} \left| \sum_{j \in F} \varphi(d_{\alpha_j} tr(A_{\alpha_j} U^{(\alpha_j)})) \right| < \infty,$$

which implies

$$\sum_{j=1}^{\infty} | \varphi(d_{\alpha_j} tr(A_{\alpha_j} U^{(\alpha_j)})) | < \infty.$$

Thus the Fourier series of $f$ is weakly subseries Cauchy and, since $L_p(G)$ is weakly complete, the series is weakly subseries convergent. Therefore, by the Orlicz-Pettis theorem ([2], p. 60, or [6], p. 19) it is norm convergent and unconditionally convergent to some $g \in L_p(G)$. Comparing transforms, we see that $f = g$ and consequently, $f$ belongs to $S_p$.

Finally, we state the main result of this section, generalizing the abelian result of Bachelis.

**Theorem 3.9.** Let $1 \leq p < \infty$. Then we have

(i) $D_p \subset S_p$, and

(ii) $\mathcal{D}_p = S^s_p$.

**Proof.** Observe that $\|f\|_{S_p} \leq \|f\|_{\alpha_p}$ for every $f \in D_p$, and that $\|f\|_{S_p} \leq \|f\|_{\alpha_p}$ for every $f \in \mathcal{D}_p$. The theorem now follows from (3.8).

**4.** $\mathcal{S}^\infty_3$ as a source of examples. Throughout this section $G$ will denote $\mathcal{S}^\infty_3 = \prod_{\mathbb{N}} \mathcal{S}_3$, where $\mathcal{S}_3$ is the symmetric group on three symbols. Using this group we demonstrate that Bachelis' result does not extend to the non-abelian case.

**Theorem 4.1.** Let $G = \mathcal{S}^\infty_3$ and $1 \leq p < \infty$. Then

(i) $D_p = S_p$ if and only if $p = 2$, and
(ii) \( D_p = L_p \) if and only if \( p = 2 \).

**Proof.** By (1.5, iii) and (3.9), we have

\[
L_2(G) = D_2 \subset S_2 \subset L_2(G).
\]

Suppose \( p \neq 2 \). Observe that (ii) follows from (i) because

\[
D_p \subset S_p \subset L_p.
\]

Note also that \( \|f\|_{S_p} \leq \|f\|_{D_p} \) for every \( f \in D_p \). Hence to prove that \( D_p \neq S_p \), it is enough to find sequences \( \{f^{(n)}\} \) in \( D_p \) and \( \{g^{(n)}\} \) in \( T(G) \) such that

\[
\frac{\|f^{(n)} * g^{(n)}\|_p}{\|g^{(n)}\|_\infty \|f^{(n)}\|_{S_p}} \to \infty \quad \text{as} \quad n \to \infty.
\]

We select these sequences as follows. Let \( \sigma \) be the representation class on \( S^\sigma \) of dimension 2 (see [4], 27.61). For \( f \) and \( g \) in \( T_{\sigma}(S) \) which will be specified later, form

\[
f^{(n)}(\underline{x}) = \prod_{k=1}^n f(x_k)
\]

and

\[
g^{(n)}(\underline{x}) = \prod_{k=1}^n g(x_k),
\]

where \( \underline{x} \in G \) is given by \( \underline{x} = (x_1, x_2, \cdots) \). Then \( f^{(n)} \) and \( g^{(n)} \) are elements of \( T_{\sigma^{(n)}}(G) \) where \( \sigma^{(n)} \) is the element of \( \Sigma_G \) given by

\[
U_{\sigma^{(n)}} = U_{x_1} \otimes \cdots \otimes U_{x_n} \quad \text{for every} \quad \underline{x} \in G.
\]

It is easily verified that

\[
\|f^{(n)}\|_{S_p} = \|f^{(n)}\|_p = \|f\|^n_p,
\]

\[
\|f^{(n)} * g^{(n)}\|_p = \|f * g\|^n_p,
\]

and

\[
\|g^{(n)}\|_\infty = \|g\|_\infty^n.
\]

Hence, to show (1) it suffices to find \( f \) and \( g \) in \( T_{\sigma}(S) \) such that

\[
\frac{\|f * g\|_p}{\|g\|_\infty \|f\|_p} > 1.
\]

Let \( g = 2u_{x_1}^{(\sigma)} + 2iu_{x_2}^{(\sigma)} \) and note that \( \|g\|_\infty = 1 \). The rest of the argument divides into two cases.
Case 1. $1 \leq p < 2$. In this case we let $f = 2\chi$, so that $f \ast g = g$, and we compute

$$\|f\|_p = 2 \left[\frac{2^p + 2}{6}\right]^{1/p} \quad \text{(see [4], 27.61)}.$$

Also, we have

$$\|g\|_p = 2 \left[\frac{(1 + 2^{1-p}) 2 \sqrt{2^{1-p}}}{6}\right]^{1/p},$$

and therefore we conclude

$$\frac{\|f \ast g\|_p}{\|\hat{\mathcal{G}}\|_\infty \|f\|_p} = 2^{1/p - 1/2} > 1.$$

Case 2. $2 \leq p < \infty$. In this case we let $f = 2iu_{12}^p + 2u_{21}^p$. Then $f \ast g = -2w_{12}^p + 2u_{21}^p$ and so we have

$$\|f\|_p = \sqrt{6} \left(\frac{2}{3}\right)^{1/p} \quad \text{and} \quad \|f \ast g\|_p = 2 \sqrt{3} \left(\frac{1}{3}\right)^{1/p}.$$

Therefore, we conclude

$$\frac{\|f \ast g\|_p}{\|\hat{\mathcal{G}}\|_\infty \|f\|_p} = 2^{1/2 - 1/p} > 1.$$

The question naturally arises as to whether $D_{\lambda}$ is equal to $\mathcal{D}_\lambda$. The next example shows that in some cases the answer is no.

**Theorem 4.2.** If $G = \mathcal{S}^\omega$ and $1 \leq p < 4$, then $D_{\lambda} = \mathcal{D}_\lambda$ if and only if $p = 2$.

**Proof.** By (1.5, iii) and (3.3, iv) we have

$$D_{\lambda} = \mathcal{D}_\lambda = L^1(G).$$

Suppose $p \neq 2$. Since $D_{\lambda} \subset \mathcal{D}_\lambda$ and $\|x\|_p \leq \|x\|_{D_{\lambda}}$ on $D_{\lambda}$, to show that $D_{\lambda} \neq \mathcal{D}_\lambda$, it suffices to find sequences $\{f^{(n)}\}$ in $D_{\lambda}$ and $\{g^{(n)}\}$ in $T(G)$ such that

$$\frac{\|f^{(n)} \ast g^{(n)}\|_p}{\|g^{(n)}\|_\infty \|f^{(n)}\|_p} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

As in the proof of (4.1) we construct the sequences by choosing $f$ and $g$ on $\mathcal{S}_\lambda$ as follows. First, let $f = 2\chi$. Then $f \ast g = g$ for any $g \in T_s(\mathcal{S}_\lambda)$, and $\|f\|_p = 2 [(2^p + 2)/6]^{1/p}$. Also we have $f^{(n)} = 2^n\chi_{\mathcal{S}_\lambda^{(n)}}$ and
\[ \| f^{(n)} \|_{p} = \| f^{(n)} \|_{p} = \| f \|_{p}^{n}. \]

As before, it suffices to find \( g \in T_{\sigma}(\mathcal{S}) \) with the property that

\[ \frac{\| g \|_{p}}{\| \hat{g} \|_{\infty} \| f \|_{p}} > 1. \]

Again we consider two cases.

**Case 1.** \( 1 \leq p < 2 \). Let \( g = 2u_{12}^{(\sigma)} + 2i\bar{u}_{21}^{(\sigma)} \). Then as in (4.1), Case 1, we have

\[ \frac{\| g \|_{p}}{\| \hat{g} \|_{\infty} \| f \|_{p}} = 2^{1/p - 1/2} > 1. \]

**Case 2.** \( 2 < p < 4 \). Let \( g = 2u_{12}^{(\sigma)} + 2u_{21}^{(\sigma)} \). Then \( \| \hat{g} \|_{\infty} = 1 \) and

\[ \| g \|_{p} = 2 \left[ \frac{2\sqrt{3}^{p}}{6} \right]^{1/p}. \]

Therefore we see

\[ \frac{\| g \|_{p}}{\| \hat{g} \|_{\infty} \| f \|_{p}} = \left[ \frac{2 \cdot 3^{p/2}}{2^{p} + 2} \right]^{1/p} > 1. \]

Finally, we observe that for \( G = \mathcal{S}^{\infty} \) we have the following.

**THEOREM 4.3.** \( K(G) \subseteq S_{C(G)}. \)

**Proof.** Since \( \| f \|_{u} \leq \| f \|_{K(G)} \) for \( f \) is \( K(G) \), it follows that

\[ K(G) = S_{K(G)} \subseteq S_{C(G)}. \]

Also, since \( \| f \|_{S_{C(G)}} \leq \| f \|_{K(G)} \) for \( f \) in \( K(G) \), to show that \( K(G) \neq S_{C(G)} \), we need only find \( f \in T_{\sigma}(\mathcal{S}) \) such that

\[ \frac{\| f \|_{K(\mathcal{S})}}{\| f \|_{\infty}} > 1. \]

If we let \( f = u_{12}^{(\sigma)} + u_{21}^{(\sigma)} \), then we have \( \| f \|_{\infty} = \sqrt{3} \) and \( \| f \|_{K(\mathcal{S})} = 2 \). Hence, the proof is complete.

The techniques used to prove (4.1) — (4.3) can also be applied to show the following.

**THEOREM 4.4.** If \( G = \mathcal{S}^{\infty} \) and \( 1 \leq p < \infty \), then

\[ \mathcal{D}_{p}(G) = L_{p}(G) \text{ if and only if } p = 2 . \]
5. Open questions.

(5.1) Is \(T(G)\) dense in \(D_A\)? If so, then it can easily be shown that \(D_{DA}\) is isometrically isomorphic to \(D_A\). One easily shows that the density of \(T(G)\) is equivalent to the condition that \(S_{DA} = D_A\).

(5.2) Another question of interest is whether or not \(D_A\) is self-adjoint (that is, closed under \(f \rightarrow \overline{f}\), where \(\overline{f}(x) = \overline{f(x^{-1})}\)) whenever \(A\) is. Equivalently, is \(\hat{D_A}\) a left ideal in \(C_0(\Sigma)\) when \(A\) is self-adjoint?

(5.3) Are there any conditions on a compact non-abelian group \(G\) sufficient to imply that \(D_p = S_p\) for \(p \neq 2\)?

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