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David S. Browder

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DAVID S. BROWDER

Let G be a compact topological group. In this paper, it is shown that the derived algebra D_p of $L_p(G)$ (for $1 \leqq p < \infty$) is contained in the ideal S_p of functions in $L_p(G)$ with unconditionally convergent Fourier series. It is also noted that this inclusion can be strict if G is nonabelian. Finally, it is shown that the derived algebra of the center of $L_p(G)$ is always equal to the center of S_p, generalizing a known result that $D_p = S_p$ when G is compact and abelian.

In general, let $(A, \|\ \|_A)$ be a Banach algebra which is an essential left Banach $L_1(G)$-module in $L_1(G)$ under convolution. For convenience and with no loss of generality it is assumed that

$$\| f \|_A \geqq \| f \|_1 \qquad \text{for every } f \in A\,.$$

This paper investigates the relationship between the derived algebra of A and the ideal in A of functions with unconditionally convergent Fourier series. Bachelis has shown in [1] that in case G is abelian and A is equal to $L_p(G)$, for $1 \leqq p < \infty$, the two algebras coincide.

Bachelis' result is generalized to the derived algebra of the center of $L_p(G)$ and it is shown that for the compact group $\mathscr{S}_3^\infty$ and $A = L_p(\mathscr{S}_3^\infty)$ with $p \neq 2$, the derived algebra is strictly contained in the ideal of functions in $L_p(\mathscr{S}_3^\infty)$ whose Fourier series converge unconditionally.

Notation throughout will be as in [4]. Σ will denote the dual object of G, the set of equivalence classes of continuous irreducible unitary representations of G. For each $\sigma \in \Sigma$, H_σ will denote the representation space of σ (of finite dimension d_σ) and $\mathscr{E}(\Sigma)$ will denote the product space $\prod_{\sigma \in \Sigma} B(H_\sigma)$. Important subspaces of $\mathscr{E}(\Sigma)$ referred to in the text include:

(i) $\mathscr{E}_0(\Sigma) = \{E = \{E_\sigma\}\colon \| E_\sigma \|_{op}$ is small off finite sets$\}$
(ii) $\mathscr{E}_1(\Sigma) = \{E = \{E_\sigma\}\colon \| E \|_1 = \sum_{\sigma \in \Sigma} d_\sigma \| E_\sigma \|_{\phi_1} < \infty\}$
(iii) $\mathscr{E}_2(\Sigma) = \{E = \{E_\sigma\}\colon \| E \|_2^2 = \sum_{\sigma \in \Sigma} d_\sigma \| E_\sigma \|_{\phi_2}^2 < \infty\}$.

For $f \in L_1(G)$, f has Fourier series $f \sim \sum_{\sigma \in \Sigma} d_\sigma\, tr(A_\sigma U^{(\sigma)})$ where $A_\sigma \in B(H_\sigma)$, $U^{(\sigma)} \in \sigma$. The Fourier transform $\hat{f}$ of f has the property that $\hat{f}(\sigma) = A_\sigma^t$ and hence:

$$\| \hat{f} \|_\infty = \sup_{\sigma \in \Sigma} \| A_\sigma \|_{op}\,.$$

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1. The derived algebra. We begin by defining the derived algebra D_A for an essential left Banach $L_1(G)$-module A, and noting a few of its properties.

DEFINITION 1.1. If $f \in A$, we define

$$\|f\|_{D_A} = \sup_{g \in A} \frac{\|f * g\|_A}{\|\hat{g}\|_\infty}$$

and let

$$D_A = \{f \in A: \|f\|_{D_A} < \infty\} .$$

The following facts are easy to check.

PROPOSITION 1.2. (i) *$(D_A, \|\ \|_{D_A})$ is a Banach algebra and a left Banach $L_1(G)$-module in $L_1(G)$ under convolution.*

(ii) *$\|f\|_A \leqq \|f\|_{D_A}$ for every $f \in A$.*

(iii) *If we denote the set of trigonometric polynomials by $T(G)$ then we have*

$$\|f\|_{D_A} = \sup_{g \in T(G)} \frac{\|f * g\|_A}{\|\hat{g}\|_\infty} \qquad \textit{for every } f \in A .$$

We next give a characterization of D_A which is due essentially to Helgason ([3], Theorem 2).

THEOREM 1.3. (Helgason)

$$D_A = \{f \in A: \hat{f}E \in \hat{A},\ \textit{for every } E \in \mathscr{E}_0(\Sigma)\} .$$

Proof. Suppose $f \in A$ and that for $E \in \mathscr{E}_0(\Sigma), \hat{f}E = \hat{g}_E$ for some $g_E \in A$. Then the linear map $E \to g_E$ of $\mathscr{E}_0(\Sigma)$ into A has closed graph and is therefore continuous. In particular, there exists a constant $k > 0$ such that

$$\|f * h\|_A \leqq k \|\hat{h}\|_\infty \qquad \text{for every } h \in A .$$

Consequently, f belongs to D_A.

Conversely, if $f \in D_A$ then the continuous map $\hat{g} \to f * g$ of $\hat{A}$ into A extends to a continuous map $E \to h_E$ of $\mathscr{E}_0(\Sigma)$ into A. Then the element $\hat{f}E = \hat{h}_E$ belongs to $\hat{A}$ for every $E \in \mathscr{E}_0(\Sigma)$.

This characterization of D_A gives two more properties of D_A.

COROLLARY 1.4. (i) *D_A is an ideal in $L_1(G)$ and*
(ii) *$\hat{D}_A$ is a right ideal in $\mathscr{E}_0(\Sigma)$.*

We denote by $C(G)$ the algebra of continuous complex valued functions on G, and by $K(G)$ the algebra of functions on G with absolutely convergent Fourier series (see [4], Sect. 34).

For $1 \leqq p < \infty$, the derived algebra of $L_p(G)$ is denoted by D_p.

EXAMPLES 1.5. (i) $D_{K(G)} = K(G)$,
(ii) $D_{C(G)} = K(G)$, *and*
(iii) $D_p = L_2(G)$ *for* $1 \leqq p \leqq 2$.

Proof. First we show (i). Let f belong to $K(G)$ and g to $T(G)$. Then $\|f * g\|_K = \|\hat{f}\hat{g}\|_1 \leqq \|\hat{f}\|_1 \|\hat{g}\|_\infty = \|f\|_K \|\hat{g}\|_\infty$. Hence, by (1.2), f belongs to $D_{K(G)}$.

To see (ii), observe that since $\| \ \|_u \leqq \| \ \|_{K(G)}$ on $K(G)$, it follows that $K(G) = D_{K(G)} \subset D_{C(G)}$. Conversely, let $f \in D_{C(G)}$ with Fourier series given by

$$f \sim \sum_{\sigma \in \Sigma} d_\sigma tr(A_\sigma U^{(\sigma)}) .$$

For each $\sigma \in \Sigma$, let V_σ be the unitary matrix such that $V_\sigma A_\sigma = |A_\sigma|$. For $F \subset \Sigma$, a finite set, define:

$$g = \sum_{\sigma \in F} d_\sigma tr(V_\sigma U^{(\sigma)}) .$$

Then $g \in T(G)$, $\|\hat{g}\|_\infty = 1$ and we have:

$$\sum_{\sigma \in F} d_\sigma \|A_\sigma\|_{\phi_1} = \sum_{\sigma \in F} d_\sigma tr\,|A_\sigma| = |f * g(e)| \leqq \|f * g\|_u \leqq \|f\|_{D_C} .$$

Hence $\|f\|_{K(G)} \leqq \|f\|_{D_{C(G)}}$ and $f \in K(G)$.

To prove (iii), we use the facts (see [4], 36.10, 36.12) that $D_1 = L_2(G)$ and

$$2^{-1/2} \|f\|_2 \leqq \|f\|_{D_1} \leqq \|f\|_2 \qquad \text{for every } f \in L_2(G) .$$

It $1 < p \leqq 2$ and $f \in L_2(G)$, then for $g \in T(G)$ we see that

$$\|f * g\|_p \leqq \|f * g\|_2 = \|\hat{f}\hat{g}\|_2 \leqq \|\hat{f}\|_2 \|\hat{g}\|_\infty = \|f\|_2 \|\hat{g}\|_\infty .$$

Hence, we conclude that $\|f\|_{D_p} \leqq \|f\|_2$ and

$$\|f\|_{D_p} \geqq \|f\|_{D_1} \geqq 2^{-1/2} \|f\|_2 .$$

2. The ideal in A of functions with unconditionally con-

vergent Fourier series. Let $\mathscr{F}$ denote the family of all nonvoid finite subsets of Σ. For $F \in \mathscr{F}$, let $D(F) = \sum_{\sigma \in F} d_\sigma \chi_\sigma$. For f in $L_1(G)$, $f * D(F)$ is the finite partial sum of the Fourier series of f consisting only of terms involving elements of F. We say that f in A has unconditionally convergent Fourier series in A whenever

$$\lim_{F \in \mathscr{F}} \| f - f * D(F) \|_A = 0 .$$

We denote by S_A the family of all functions in A with this property. If we also define

$$\| f \|_{S_A} = \sup_{F \in \mathscr{F}} \| f * D(F) \|_A ,$$

then the following facts are easily verified.

PROPOSITION 2.1. (i) *If $f \in S_A$, then $\| f \|_{S_A} < \infty$.*
(ii) *$(S_A, \| \ \|_{S_A})$ is a Banach algebra.*
(iii) *$\| f \|_A \leqq \| f \|_{S_A}$ for every $f \in A$.*
(iv) *If $f \in S_A$, then $\lim_{F \in \mathscr{F}} \| f - f * D(F) \|_{S_A} = 0$.*
(v) *S_A is an essential left Banach $L_1(G)$-module in $L_1(G)$ under convolution.*

Since S_A satisfies the conditions we have postulated for A, we may compute its derived algebra.

THEOREM 2.2. (i) *$D_{S_A} = D_A \cap S_A$ and $\| f \|_{D_{S_A}} = \| f \|_{D_A}$ for $f \in D_{S_A}$.*
(ii) *$S_{S_A} = S_A$ (isometry).*

Proof. Suppose f belongs to D_{S_A}. Then for $f \in S_A$ and $g \in T(G)$ we have

$$\frac{\| f * g \|_A}{\| \hat{g} \|_\infty} \leqq \frac{\| f * g \|_{S_A}}{\| \hat{g} \|_\infty} \leqq \| f \|_{D_{S_A}} .$$

Hence we have $\| f \|_{D_A} \leqq \| f \|_{D_{S_A}} < \infty$, and thus f belongs to $D_A \cap S_A$.
Conversely, if $f \in D_A \cap S_A$ then for $g \in T(G)$ and $F \in \mathscr{F}$, we have

$$\frac{\| f * g * D(F) \|_A}{\| \hat{g} \|_\infty} \leqq \frac{\| f * g * D(F) \|_A}{g * D(F)} \leqq \| f \|_{D_A} .$$

Thus it follows that $\| f \|_{D_{S_A}} \leqq \| f \|_{D_A} < \infty$, and f belongs to D_{S_A}.
Part (ii) follows immediately from (2.1, iv).

3. Central derived algebras. Let A^z denote the center of A. Then $A^z = L_1^z(G) \cap A$ and $(A^z, \| \ \|_A)$ is an essential Banach L_1^z-module in $L_1^z(G)$ under convolution. Before we investigate the derived algebra of A^z, we prove a useful proposition.

PROPOSITION 3.1. *For $E \in \mathscr{E}_\infty(\Sigma)$, define a function φ_E on Σ by: $\varphi_E(\sigma) = 1/d_\sigma tr(E_\sigma)$ for every $\sigma \in \Sigma$. The map $E \rightarrow \varphi_E$ is an isometric isomorphism of*

(i) *$\mathscr{E}_\infty^z(\Sigma)$ onto $l_\infty(\Sigma)$,*

(ii) *$\mathscr{E}_0^z(\Sigma)$ onto $c_0(\Sigma)$, and*

(iii) *$\mathscr{E}_{00}^z(\Sigma)$ onto $c_{00}(\Sigma)$.*

For $f \in L_1^z(G)$, let $\mathring{f}(\sigma) = 1/d_\sigma tr(\hat{f}(\sigma)) = \varphi_{\hat{f}}(\sigma)$, so that f has Fourier series $\sum_{\sigma \in \Sigma} d_\sigma \mathring{f}(\sigma)\chi_\sigma$. Then the map $f \rightarrow \mathring{f}$ is the Gel'fand transform A^z, Σ is the maximal ideal space of A^z, and

(iv) *$\|\mathring{f}\|_\infty = \|\hat{f}\|_\infty$ for every $f \in L_1^z(G)$.*

Proof. Let E belong to $\mathscr{E}_\infty^z(\Sigma)$. By Schur's lemma we have

$$E_\sigma = \varphi_E(\sigma) I_{d_\sigma} \qquad \text{for } \sigma \in \Sigma . \tag{1}$$

It follows that

$$\| E \|_\infty = \| \varphi_E \|_\infty . \tag{2}$$

Clearly the map $E \rightarrow \varphi_E$ is linear and carries $\mathscr{E}_\infty^z(\Sigma)$ isometrically onto $l_\infty(\Sigma)$. By (1), $E \rightarrow \varphi_E$ is multiplicative. By (2), the image of $\mathscr{E}_0^z(\Sigma)$ is $c_0(\Sigma)$, and the image of $\mathscr{E}_{00}^z(\Sigma)$ is $c_{00}(\Sigma)$. The rest of the proof is analogous to ([4], 28.71).

DEFINITION 3.2. For f in A^z, let

$$\|f\|_{\mathscr{D}_A} = \sup_{g \in A^z} \frac{\|f * g\|_A}{\|\mathring{g}\|_\infty} .$$

The derived algebra $\mathscr{D}_A$ of A^z is defined as

$$\mathscr{D}_A = \{f \in A^z : \|f\|_{\mathscr{D}_A} < \infty\} .$$

The following properties of $\mathscr{D}_A$ are easily proved.

PROPOSITION 3.3. (i) *$(\mathscr{D}_A, \| \ \|_{\mathscr{D}_A})$ is a Banach algebra and an $L_1^z(G)$-module under convolution.*

(ii) *$\|f\|_A \leqq \|f\|_{\mathscr{D}_A}$ for every $f \in A^z$.*

(iii) *$\|f\|_{\mathscr{D}_A} = \sup_{g \in T^z(G)} \|f * g\|_A / \|\mathring{g}\|_\infty$ for every $f \in A^z$.*

(iv) *$D_A^z \subset \mathscr{D}_A$.*

Helgason's characterization (1.3) has an analogue in the central case. We omit the proof since it is exactly like that of (1.3).

THEOREM 3.4. (Helgason)

$$\mathscr{D}_A = \{f \in A^z\colon \overset{\circ}{f}\varphi \in (A^z)^\circ \ \textit{for every}\ \varphi \in c_0(\Sigma)\}\ .$$

We next prove that the center S_A^z of S_A is always contained in $\mathscr{D}_A$. To do so, we use the following well known fact which follows from a theorem of Seever ([6]).

FACT 3.5. *Let X be a discrete topological space and M a Banach space. If $T\colon M \rightarrow l_\infty(X)$ is a bounded linear map whose image contains the characteristic function of every subset of X, then T is onto.*

We also use the following lemma which states that every element of $l_\infty(\Sigma)$ is a multiplier for S_A^z.

LEMMA 3.6. *If $f \in S_A^z$ and $\varphi \in l_\infty(\Sigma)$, then there exists $g \in S_A^z$ such that $\overset{\circ}{g} = \varphi\overset{\circ}{f}$.*

Proof. Let f belong to S_A^z, and denote by M the collection of all $\varphi \in l_\infty(\Sigma)$ such that $\varphi\overset{\circ}{f} \in (S_A^z)^0$. Then M is a Banach space under the norm

$$\|\varphi\| = \|\varphi\|_\infty + \|g\|_{S_A} \ \text{where}\ \overset{\circ}{g} = \varphi\overset{\circ}{f}\ .$$

To show $M = l_\infty(\Sigma)$, it suffices by (3.5) to show that for $\Delta \subset \Sigma$, the characteristic function φ of Δ is an element of M. To establish this, we note that the net $\{f * D(E)\colon E^{\text{finite}} \subset \Delta\}$ is Cauchy in S_A^z, so there is a function g in S_A^z such that

$$\lim_{E^{\text{finite}} \subset \Delta} \|g - f * D(E)\|_{S_A} = 0\ .$$

We conclude that $\overset{\circ}{g} = \varphi\overset{\circ}{f}$ and hence, φ belongs to M.

THEOREM 3.7. $S_A^z \subset \mathscr{D}_A$.

Proof. Suppose f belongs to s_A^z. Then for $\varphi \in c_0(\Sigma) \subset l_\infty(\Sigma)$, $\varphi\overset{\circ}{f}$ belongs to $(S_A^z)^0$ and hence to $(A^z)^0$ by (3.6). Therefore $f \in \mathscr{D}_A$ by (3.4).

We now restrict our attention to the case of $A = L_p(G)$ for $1 \leqq p < \infty$. As before we write $D_A = D_p$; we also write $S_A = S_p$ and $\mathscr{D}_A = \mathscr{D}_p$. To compare D_p and S_p we use the following.

LEMMA 3.8. *Let* $1 \leqq p < \infty$. *If* $f \in L_p(G)$ *and* $\|f\|_{S_p} < \infty$, *then* $f \in S_p$.

Proof. Let f belong to $L_p(G)$ with $\|f\|_{S_p} < \infty$. Suppose f has Fourier series

$$f \sim \sum_{j=1}^{\infty} d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)}) .$$

For $\varphi \in L_p(G)^*$ and any nonvoid finite $F \subset Z^+$, we have

$$\left| \sum_{j \in F} \varphi(d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)})) \right| \leqq \|f\|_{S_p} \|\varphi\|_{op} .$$

Hence, we see

$$\sup_{F \text{finite} \subset Z^+} \left| \sum_{j \in F} \varphi(d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)})) \right| < \infty ,$$

which implies

$$\sum_{j=1}^{\infty} | \varphi(d_{\sigma_j} tr(A_{\sigma_j} U^{(\sigma_j)})) | < \infty .$$

Thus the Fourier series of f is weakly subseries Cauchy and, since $L_p(G)$ is weakly complete, the series is weakly subseries convergent. Therefore, by the Orlicz-Pettis theorem ([2], p. 60, or [6], p. 19) it is norm convergent and unconditionally convergent to some $g \in L_p(G)$. Comparing transforms, we see that $f = g$ and consequently, f belongs to S_p.

Finally, we state the main result of this section, generalizing the abelian result of Bachelis.

THEOREM 3.9. *Let* $1 \leqq p < \infty$. *Then we have*

(i) $D_p \subset S_p$, *and*

(ii) $\mathscr{D}_p = S_p^z$.

Proof. Observe that $\|f\|_{S_p} \leqq \|f\|_{D_p}$ for every $f \in D_p$, and that $\|f\|_{S_p} \leqq \|f\|_{\mathscr{D}_p}$ for every $f \in \mathscr{D}_p$. The theorem now follows from (3.8).

4. $\mathscr{S}_3^\infty$ as a source of examples. Throughout this section G will denote $\mathscr{S}_3^\infty = \prod_{\aleph_0} \mathscr{S}_3$, where $\mathscr{S}_3$ is the symmetric group on three symbols. Using this group we demonstrate that Bachelis' result does not extend to the non-abelian case.

THEOREM 4.1. *Let* $G = \mathscr{S}_3^\infty$ *and* $1 \leqq p < \infty$. *Then*

(i) $D_p = S_p$ *if and only if* $p = 2$, *and*

(ii) $D_p = L_p$ *if and only* $p = 2$.

Proof. By (1.5, iii) and (3.9), we have

$$L_2(G) = D_2 \subset S_2 \subset L_2(G) .$$

Suppose $p \neq 2$. Observe that (ii) follows from (i) because

$$D_p \subset S_p \subset L_p .$$

Note also that $\|f\|_{S_p} \leqq \|f\|_{D_p}$ for every $f \in D_p$. Hence to prove that $D_p \neq S_p$ it is enough to find sequences $\{f^{(n)}\}$ in D_p and $\{g^{(n)}\}$ in $T(G)$ such that

$$(1) \qquad \frac{\|f^{(n)} * g^{(n)}\|_p}{\|\widehat{g^{(n)}}\|_\infty \|f^{(n)}\|_{S_p}} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty .$$

We select these sequences as follows. Let σ be the representation class on $\mathscr{S}_3$ of dimension 2 (see [4], 27.61). For f and g in $T_\sigma(\mathscr{S}_3)$ which will be specified later, form

$$f^{(n)}(\underline{\underline{x}}) = \prod_{k=1}^{n} f(x_k)$$

and

$$g^{(n)}(\underline{\underline{x}}) = \prod_{k=1}^{n} g(x_k) ,$$

where $\underline{\underline{x}} \in G$ is given by $\underline{\underline{x}} = (x_1, x_2, \cdots)$. Then $f^{(n)}$ and $g^{(n)}$ are elements of $T_{\sigma^{(n)}}(G)$ where $\sigma^{(n)}$ is the element of Σ_G given by

$$U_{\underline{\underline{x}}}^{(\sigma^{(n)})} = U_{x_1}^{(\sigma)} \otimes \cdots \otimes U_{x_n}^{(\sigma)} \quad \text{for every } \underline{\underline{x}} \in G .$$

It is easily verified that

$$\|f^{(n)}\|_{S_p} = \|f^{(n)}\|_p = \|f\|_p^n ,$$

$$\|f^{(n)} * g^{(n)}\|_p = \|f * g\|_p^n ,$$

and

$$\|\widehat{g^{(n)}}\|_\infty = \|\hat{g}\|_\infty^n .$$

Hence, to show (1) it suffices to find f and g in $T_\sigma(\mathscr{S}_3)$ such that

$$\frac{\|f * g\|_p}{\|\hat{g}\|_\infty \|f\|_p} > 1 .$$

Let $g = 2u_{11}^{(\sigma)} + 2iu_{22}^{(\sigma)}$ and note that $\|\hat{g}\|_\infty = 1$. The rest of the argument divides into two cases.

Case 1. $1 \leqq p < 2$. In this case we let $f = 2\chi_\sigma$ so that $f * g = g$, and we compute

$$\| f \|_p = 2\left[\frac{2^p + 2}{6}\right]^{1/p} \qquad \text{(see [4], 27.61)} .$$

Also, we have

$$\| g \|_p = 2\left[\frac{(1 + 2^{1-p})\, 2\sqrt{2}^{\,p}}{6}\right]^{1/p} ,$$

and therefore we conclude

$$\frac{\| f * g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = 2^{1/p-1/2} > 1 .$$

Case 2. $2 < p < \infty$. In this case we let $f = 2iu_{12}^{(\sigma)} + 2u_{21}^{(\sigma)}$. Then $f * g = -2u_{12}^{(\sigma)} + 2u_{21}^{(\sigma)}$ and so we have

$$\| f \|_p = \sqrt{6}\left(\frac{2}{3}\right)^{1/p} \quad \text{and} \quad \| f * g \|_p = 2\sqrt{3}\left(\frac{1}{3}\right)^{1/p} .$$

Therefore, we conclude

$$\frac{\| f * g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = 2^{1/2-1/p} > 1 .$$

The question naturally arises as to whether D_A^z is equal to $\mathscr{D}_A$. The next example shows that in some cases the answer is no.

THEOREM 4.2. *If $G = \mathscr{S}_3^\infty$ and $1 \leqq p < 4$, then $D_p^z = \mathscr{D}_p$ if and only if $p = 2$.*

Proof. By (1.5, iii) and (3.3, iv) we have

$$D_2^z = \mathscr{D}_2 = L_2^z(G) .$$

Suppose $p \neq 2$. Since $D_p^z \subset \mathscr{D}_p$ and $\| \ \|_{\mathscr{D}_p} \leqq \| \ \|_{D_p}$ on D_p^z, to show that $D_p^z \neq \mathscr{D}_p$, it suffices to find sequences $\{f^{(n)}\}$ in D_p^z and $\{g^{(n)}\}$ in $T(G)$ such that

$$\frac{\| f^{(n)} * g^{(n)} \|_p}{\| \widehat{g^{(n)}} \|_\infty \| f^{(n)} \|_{\mathscr{D}}} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty .$$

As in the proof of (4.1) we construct the sequences by choosing f and g on $\mathscr{S}_3$ as follows. First, let $f = 2\chi_\sigma$. Then $f * g = g$ for any $g \in T_\sigma(\mathscr{S}_3)$, and $\| f \|_p = 2\,[(2^p + 2)/6]^{1/p}$. Also we have $f^{(n)} = 2^n \chi_{\sigma^{(n)}}$ and

$\| f^{(n)} \|_{\mathscr{D}_p} = \| f^{(n)} \|_p = \| f \|_p^n$. As before, it suffices to find $g \in T_\sigma(\mathscr{S}_3)$ with the property that

$$\frac{\| g \|_p}{\| \hat{g} \|_\infty \| f \|_p} > 1 .$$

Again we consider two cases.

Case 1. $1 \leqq p < 2$. Let $g = 2u_{11}^{(\sigma)} + 2iu_{22}^{(\sigma)}$. Then as in (4.1), Case 1, we have

$$\frac{\| g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = 2^{1/p - 1/2} > 1 .$$

Case 2. $2 < p < 4$. Let $g = 2u_{12}^{(\sigma)} + 2u_{21}^{(\sigma)}$. Then $\| \hat{g} \|_\infty = 1$ and

$$\| g \|_p = 2\left[\frac{2\sqrt{3}^{\,p}}{6}\right]^{1/p} .$$

Therefore we see

$$\frac{\| g \|_p}{\| \hat{g} \|_\infty \| f \|_p} = \left[\frac{2 \cdot 3^{p/2}}{2^p + 2}\right]^{1/p} > 1 .$$

Finally, we observe that for $G = \mathscr{S}_3^\infty$ we have the following.

THEOREM 4.3. *$K(G) \subsetneqq S_{C(G)}$.*

Proof. Since $\| f \|_u \leqq \| f \|_{K(G)}$ for f is $K(G)$, it follows that

$$K(G) = S_{K(G)} \subset S_{C(G)} .$$

Also, since $\| f \|_{S_{C(G)}} \leqq \| f \|_{K(G)}$ for f in $K(G)$, to show that $K(G) \neq S_{C(G)}$, we need only find $f \in T_\sigma(\mathscr{S}_3)$ such that

$$\frac{\| f \|_{K(\mathscr{S}_3)}}{\| f \|_\infty} > 1 .$$

If we let $f = u_{12}^{(\sigma)} + u_{21}^{(\sigma)}$, then we have $\| f \|_\infty = \sqrt{3}$ and $\| f \|_{K(\mathscr{S}_3)} = 2$. Hence, the proof is complete.

The techniques used to prove (4.1) – (4.3) can also be applied to show the following.

THEOREM 4.4. *If $G = \mathscr{S}_3^\infty$ and $1 \leqq p < \infty$, then*

$$\mathscr{D}_p(G) = L_p^z(G) \textit{ if and only if } p = 2 .$$

5. Open questions.

(5.1) Is $T(G)$ dense in D_A? If so, then it can easily be shown that D_{D_A} is isometrically isomorphic to D_A. One easily shows that the density of $T(G)$ is equivalent to the condition that $S_{D_A} = D_A$.

(5.2) Another question of interest is whether or not D_A is self-adjoint (that is, closed under $f \to \tilde{f}$, where $\tilde{f}(x) = \overline{f(x^{-1})}$) whenever A is. Equivalently, is $\hat{D}_A$ a left ideal in $\mathscr{E}_0(\Sigma)$ when A is self-adjoint?

(5.3) Are there any conditions on a compact non-abelian group G sufficient to imply that $D_p = S_p$ for $p \neq 2$?

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UNIVERSITY OF OREGON
AND
SIMMON COLLEGE, BOSTON, MASS.

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