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UNICOHERENT COMPACTIFICATIONS

MICHAEL HOWARD CLAPP AND RAYMOND FRANK DICKMAN

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In this paper we give necessary and sufficient conditions for the Freudenthal compactification of a rimcompact, locally connected and connected Hausdorff space to be unicoherent. We give several necessary and sufficient conditions for a locally connected generalized continuum to have a unicoherent compactification and show that if such a space X has a unicoherent compactification, then γX is the smallest unicoherent compactification of X in the usual ordering of compactifications.

A connected topological space X is said to be *unicoherent* if, $H \cdot K$ is connected whenever $X = H + K$ where H and K are closed connected sets. A continuum is a compact connected metric space and a generalized continuum is a locally compact, connected, separable metric space. By a mapping we will always mean a continuous function. If B is a subset of a space X , the closure of B in X will be denoted by $\text{cl}_X B$ and the boundary of B in X will be denoted by $\text{Fr}_X B$. An open set (respectively, a closed set) of a space X will be called a γ -open (respectively, γ -closed) subset of X provided it has a compact boundary in X . A space is rimcompact (or semicompact) provided every point has arbitrarily small neighborhoods with compact boundaries. All compactifications considered here are Hausdorff.

In [7] K. Morita showed that for any rimcompact Hausdorff space X there exists a topologically unique compactification γX of X satisfying:

(a) For every point x of γX and every open set R of γX containing x there exists an open set V of γX containing x such that $V \subset R$ and $\text{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint γ -closed subsets of X have disjoint closures in γX .

Furthermore if C is any compactification of X satisfying (a), there exists a mapping h of γX onto C such that $h|_X$ is the identity map. The compactification γX of X is called the Freudenthal compactification of X after H. Freudenthal who first defined it [4].

DEFINITION. We say that a connected space X is γ -unicoherent if whenever $X = H + K$, where H and K are γ -closed and connected sets, $H \cdot K$ is connected.

THEOREM 1. *If X is a locally connected, connected, rimcompact Hausdorff space, then γX , the Freudenthal compactification of X , is*

unicoherent iff X is γ -unicoherent.

Proof. Suppose that X is γ -unicoherent and γX is not unicoherent. Then $\gamma X = H + K$ where H and K are closed and connected sets and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let U and V be open subsets of γX containing A and B respectively such that $\text{cl}_{\gamma X} U \cdot \text{cl}_{\gamma X} V = \emptyset$ and $(\text{Fr}_{\gamma X} V + \text{Fr}_{\gamma X} U) \subset X$. By Propositions (2.8) and (4.1) of [1], γX is locally connected so if C denotes the component of $U + V + H$ that contains H and D denotes the component of $U + V + K$ that contains K , C and D are open connected subsets of γX such that $(\text{Fr}_{\gamma X} C + \text{Fr}_{\gamma X} D) \subset X$. By Lemma 5 of [6], $C \cdot X$ and $D \cdot X$ are connected so that $L = \text{cl}_X(C \cdot X)$ and $M = \text{cl}_X(D \cdot X)$ are γ -closed and connected subsets of X . Furthermore $X = L + M$ and $L \cdot M$ is not connected. This contradicts our hypothesis that X is γ -unicoherent and thus γX must be unicoherent.

Now suppose that γX is unicoherent and X is not γ -unicoherent. Then $X = H + K$ where H and K are γ -closed and connected subsets of X and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let H', K', A' and B' denote the closures of H, K, A and B , respectively, in γX . Since the boundary of $H \cdot K$ in X is a subset of the union of the boundaries of H and K in X , $H \cdot K$ and hence A and B are γ -closed subsets of X . Then by property (b) of Morita's characterization of γX , A' and B' are disjoint closed subsets of γX . We now argue that $H' \cdot K'$ is a subset of $A' + B'$. Suppose to the contrary that there exists a point x in $H' \cdot K'$ that does not belong to $A' + B'$. Let U be any open subsets of γX containing x such that U does not intersect $A' + B'$ and such that $\text{Fr}_{\gamma X} U \subset X$. Let Q be the component of U that contains x and note that $\text{Fr}_{\gamma X} Q$ is a subset of X and Q is an open subset of γX . Then since X is dense in γX and x is a limit point of H' and K' , $Q \cdot H$ and $Q \cdot K$ are nonempty sets. But by Lemma 5 of [6], $Q \cdot X$ is connected and since Q misses $H \cdot K$, $Q \cdot X$ must lie entirely in H or K . Of course this implies that either $Q \cdot H$ or $Q \cdot K$ is empty and this is a contradiction. Thus $H' \cdot K' = A' + B'$ and this contradicts the unicoherence of γX . Therefore X is γ -unicoherent.

We need the following notation and definitions. Let S^1 denote the unit circle in the complex plane, let $I_1 = \{z = e^{i\theta}: 0 \leq \theta \leq \Pi\}$ and let $I_2 = \{z = e^{i\theta}: \Pi \leq \theta \leq 2\Pi\}$. For any space W let $\mathcal{A}(W)$ denote the set of mappings of W into S^1 and let $\mathcal{A}_j(W)$ be the set of all mappings of W into I_j , $j = 1, 2$. For each $f \in \mathcal{A}_j(W)$, $j = 1, 2$, let $B_j(f)$ denote the set of all points $t \in I_j$ such that $\text{Fr } f^{-1}(t)$ contains a compact set K that separates W into two disjoint open sets M and N where f maps M into the arc from 1 to t on I_j and f maps N into the arc from t to -1 on I_j . Finally let $E(W) = \{f \in \mathcal{A}(W): B_i(f | f^{-1}(I_i)) +$

$B_2(f|f^{-1}(I_2))$ is dense in S^1).

THEOREM 2. *Suppose that X is a locally connected, rimcompact Hausdorff space. A necessary and sufficient condition that γX be unicoherent is that every element of $E(X)$ be nullhomotopic.*

Proof of the necessity. Suppose that γX is unicoherent and let f be an element of $E(X)$. For $j = 1, 2$, there exists a point $t_j \in I_j$ such that $\text{Fr}_X f^{-1}(t_j)$ contains a compact set K_j that separates $f^{-1}(I_j)$ into two disjoint open sets M_j and N_j where f maps M_j into the arc from 1 to t_j on I_j and f maps N_j into the arc from t_j to -1 on I_j . Then if we let M denote $K_1 + K_2 + M_1 + M_2$ and let N denote $K_1 + K_2 + N_1 + N_2$, $X = M + N$ and the boundaries (relative to X) of M and N are subsets of $K = K_1 + K_2$. We assert that the boundaries of $M_0 = \text{cl}_{\gamma X} M$ and $N_0 = \text{cl}_{\gamma X} N$ relative to γX are also subsets of K . In order to see this suppose that x is an element of the boundary of M_0 and $x \notin K$. Then since γX is locally connected, there exists an open connected set R of γX containing x such that $R \cdot K = \emptyset$ and $\text{Fr}_{\gamma X} R \subset X$. Then $R \cdot M \neq \emptyset$ and $R \cdot (X \setminus M) \neq \emptyset$ since X is dense in γX . Furthermore $R \cdot X$ is connected by Lemma 5 of [6] and so $R \cdot X$ is a connected subset of X that meets M and $X \setminus M$. This implies that R meets K and this contradicts our selection of x . Hence the boundaries of M_0 and N_0 in γX are subsets of K . Also by Theorem 3 of [7], M_0 and N_0 are topologically equivalent to γM and γN respectively. Then by Lemma 1 of [3], $f|M$ has a continuous extension f_M to M_0 and $f|N$ has a continuous extension f_N to N_0 . Then since $N_0 \cdot M_0 \subset K$, the function h of γX into S^1 defined by $h|M_0 = f_M$ and $h|N_0 = f_N$ is continuous. By Lemma (7.4) of [9, p. 228], h is exponentially representable on γX , i.e. there exists a real valued function θ on γX such that $h(x) = e^{i\theta(x)}$ for all $x \in X$. It is evident that this implies that $f = h|X$ is exponentially representable on X and by Theorem (6.2) of [9, p. 226], f is nullhomotopic.

Proof of the sufficiency. Suppose that every element of $E(X)$ is nullhomotopic and suppose that γX is not unicoherent. Then by the proof of Theorem 1 there exists closed and connected sets H and K of γX such that $H \cdot K$ is not connected, $\text{Fr} H$ and $\text{Fr} K$ are subsets of X and $L = H \cdot X$ and $M = K \cdot X$ are connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$. We note that L and M are γ -closed subsets of and thus by Theorem 3 of [7], γL is homeomorphic to H and γM is homeomorphic to K . It then follows from Lemma 2 of [3] that there exists a mapping f of H into I_1 such that $f(A) = 1$, $f(B) = -1$ and $B_1(f|H \cdot X)$ is dense in I_1 . Similarly there exists a mapping g of K into I_2 such that $g(A) = 1$, $g(B) = -1$ and $B_2(g|K \cdot X)$ is dense

in I_2 . Then if we define $h: \gamma X \rightarrow S^1$ by $h|H = f$ and $h|K = g$ we have that h is continuous and $k = h|X$ is an element of $E(X)$. Then by our hypothesis and Proposition 6.2 of [9, p. 226], k is exponentially representable, i.e. there exists a real-valued mapping θ on X such that for each $x \in X$, $k(x) = e^{i\theta(x)}$. But then $\theta(A) \subset \{0, \pm 2\pi, \pm 4\pi, \dots\}$ and $\theta(B) \subset \{\pm \pi, \pm 3\pi, \dots\}$ and so if $a \in \theta(A)$ and $b \in \theta(B)$, the interval $[a, b]$ lies in $\theta(A) \cdot \theta(B)$ since L and M are connected. This is a contradiction since then $k(L) \cdot k(M)$ would then contain a semicircle whereas it consists of the points -1 and 1 . Hence γX is unicoherent.

DEFINITION. A connected space X is said to be weakly unicoherent if whenever $X = H + K$ where H and K are closed and connected sets and K is compact, $H \cdot K$ is connected.

THEOREM 3. *Let X be a locally connected generalized continuum. A necessary and sufficient condition for γX to be unicoherent is that X be weakly-unicoherent.*

Proof of the necessity. Suppose that γX is unicoherent. Since X is locally compact, X is open in γX and $X^* = \gamma X \setminus X$ is closed. Then by Theorem (2.3) of [2], $X = \gamma X \setminus X^*$ is weakly-unicoherent.

Proof of the sufficiency. Suppose that γX is not unicoherent. Then as in the proof of Theorem 1, γX has a representation $\gamma X = P + Q$ where P and Q are open connected subsets of γX , the boundaries of P and Q in γX are subsets of X , $\text{cl}_{\gamma X} P \cdot \text{cl}_{\gamma X} Q = A + B$ where A and B are disjoint nonempty closed sets and P has a nonempty intersection with both the boundary of A and the boundary of B . By Lemma 5 of [6], $P' = P \cdot X$ is a connected open subset of X and thus is arcwise connected. Furthermore since the boundaries of A and B are subsets of X there exists an arc $\alpha\beta$ in P' such that $\alpha\beta \cdot A = \alpha$ and $\alpha\beta \cdot B = \beta$. Let R be the component of $P' \setminus (A + B)$ that contains $\alpha\beta \setminus (\alpha + \beta)$ and let W be an open subset of γX containing A such that $B \cdot \text{cl} W = \phi$ and the boundary of W is a subset of X . Then $H = R \cdot \text{Fr}_{\gamma X} W$ is a nonempty compact subset of R and there exists a continuum K_0 of X such that $H \subset K_0 \subset R$. Let K be the union of K_0 together with all the components of $R \setminus K_0$ with boundary entirely in K_0 , i.e. having no boundary points in $X \cdot (A + B)$. Then K separates R since $W \cdot R$ contains a subarc $\alpha b \setminus \alpha$ from some point $b \in \alpha\beta$ and $X \setminus \text{cl}_x W$ contains a subarc $\alpha\beta$ of $\alpha\beta$. But $X \setminus K$ is connected since $X \setminus K$ is the union of the closure of Q in X plus all of the components of $X \setminus (A \cdot B)$ except R plus all of the components of $R - K_0$ having a boundary point in $X \cdot (A + B)$. This contradicts Whyburn's characterization of weak-unicoherence in [8, p. 185].

COROLLARY 3.1. *Let X be a locally connected generalized continuum. Then X is weakly-unicoherent iff X is γ -unicoherent.*

This corollary follows immediately from Theorems 1 and 3.

REMARK. The authors have been unable to discover a direct proof of Corollary (3.1). In general the two types of unicoherency are not equivalent and in the absence of local compactness, Theorem 3 is not valid.

EXAMPLE. Let $Y = \{z \text{ complex } | 1/2 \leq |z| \leq 1\}$,

$$S = \{z \mid |z| = 1\}, A \text{ a countable dense subset of } S,$$

$$L_z = Y \cdot \{\text{ray from origin thru } z\}$$

$$C_r = \{z \mid |z| = r\}, r \in [1/2, 1];$$

$$Z = \{C_r \cdot L_a \mid r \text{ is rational, } a \in A\}.$$

The set Z is countable and dense in Y . Let $X = Y - Z$. The set X is evidently T_2 , connected and locally connected (in fact, path connected and locally path connected), rim compact but not locally compact. Moreover:

1. X is weakly-unicoherent. To see this, note that any continuum $K \subset X$ has empty interior in X . If therefore $X = H + K$, H closed and connected and K compact and connected, then necessarily the open set $X - H$ is a subset of K , and thus empty. It follows that $H \cdot K = K$, which is connected.

2. X is not γ -unicoherent. For let $p, q \in S - A$ be two distinct points. Then L_p and L_q are compact and disjoint subsets of X . Assume $0 \leq ARGp < ARGq$. Then

$$H = \{z \in X \mid ARGp \leq ARGz \leq ARGq\} \text{ and}$$

$$K = \{z \in X \mid ARGq \leq ARGz \leq ARGp + 2\pi\}$$

are closed, connected subsets of X such that $X = H + K$, $H \cdot K = L_p + L_q$ is compact but not connected.

3. γX is not unicoherent. To show this it is sufficient to show that γX is just the set Y . To this end we use the characterization of γX obtained by Morita [6]. We show that

(a) For any point $x \in \gamma X$ and open set R of γX containing x , there is an open set V of rX containing x such that $V \subset R$ and $\text{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint γ -closed subsets of X have disjoint closures in γX .

That (a) holds is evident from the definition of X . To see that (b) holds, let A and B be disjoint γ -closed subsets of X and suppose that $p \in \text{cl}_{\gamma X} A \cdot \text{cl}_{\gamma X} B$. First of all we note that p cannot belong to X for then it would lie in $A \cdot B$ which is empty. In particular p does not lie in the compact set $(\text{Fr}_X A + \text{Fr}_X B)$. By our construction of X there exists an open subset V of Y containing p such that $V \cdot (\text{Fr}_X A + \text{Fr}_X B) = \emptyset$ and $V \cdot X$ is connected. Since p belongs to the closure of A in Y , $V \cdot X \cdot A$ is not empty and since $V \cdot X$ misses $\text{Fr}_X A$, $V \cdot X$ must lie entirely in A . But this is a contradiction since $V \cdot X$ must meet B . Therefore A and B have disjoint closures in Y .

DEFINITION. A mapping $f: X \in Y$ is *monotone* provided for every $y \in Y$, $f^{-1}(y)$ is compact and connected.

THEOREM 4. *If X is a locally connected generalized continuum and Y is any unicoherent compactification of X , then there exists a monotone mapping g of Y onto γX such that $g|X$ is the identity.*

Proof. Let Z denote the quotient space of Y obtained from the decomposition whose only nondegenerate elements are the components of $Y \setminus X$ and let p denote the natural map of Y onto Z . Then since X is open in Y , Z is a Hausdorff compactification of X . Furthermore since point inverses of p are connected, it follows from Proposition (2.2.1) of [9], that Z is unicoherent. Also $Z \setminus X$ is totally disconnected and by the maximality of γX there exists a mapping h of γX onto Z such that $h|X$ is the identity and $h(\gamma X \setminus X) = Z \setminus X$. We assert that h is a homeomorphism. In order to prove this we need only show that h is one-to-one on $\gamma X \setminus X$. To this end let $x, y \in \gamma X$, $x \neq y$ and suppose that $h(x) = h(y)$. There exists a connected and open set R of γX containing x such that $y \notin \text{cl}_\gamma R = K$ and $\text{Fr}_\gamma R \subset X$. Then $Z = h(K) + h(\gamma X \setminus R)$ and $h(K) \cdot h(\gamma X \setminus R) = h(x) + h(\text{Fr}_\gamma R)$ is not connected. This contradicts the unicoherence of Z and hence h must be a homeomorphism. Then $g = h^{-1} \circ p$ is the desired monotone mapping.

COROLLARY 4.1. *Suppose that X is a locally connected generalized continuum. Then X has a unicoherent compactification if and only if γX is unicoherent.*

Proof. This result follows immediately from Theorem 4 and the fact that monotone images of compact unicoherent continua are unicoherent.

THEOREM 5. *Suppose that X is a locally connected generalized continuum. Then the following are equivalent*

- (i) X is weakly-unicoherent
- (ii) γX is unicoherent
- (iii) X is γ -unicoherent
- (iv) X has a unicoherent compactification
- (v) every mapping of X into S^1 with compact boundaries of point inverses is null-homotopic.

Proof. The equivalence of (i)—(iv) has been established in Theorems (1) — (4). As an immediate consequence of Theorem (3.3) of [2], we have that (v) implies (i) and (ii) implies (v) follows from Theorem 1 of this paper.

DEFINITION. A connected space X is said to have the *complementation property* provided whenever K is a compact set in X , X/K has at most one component with a non-compact closure. See [2] for some characterizations of this property.

THEOREM 6. *Let X be a locally connected generalized continuum and let Y be any unicoherent, locally connected continuum. There exists a unicoherent compactification Z of X with $Z \setminus X$ homeomorphic to Y if and only if X is weakly-unicoherent and has the complementation property.*

Proof of the necessity. Suppose that Z is a unicoherent compactification of X and $Z \setminus X$ is homeomorphic to Y . Then by Theorem (4.2) of [2], X is weakly-unicoherent and has the complementation property.

Proof of the sufficiency. Suppose that X is weakly-unicoherent and has the complementation property. Then by Theorem (2.2) of [5] there exists a compactification Z of X with $Z \setminus X$ homeomorphic to Y and by Theorem (4.2) of [2], Z is unicoherent. This completes the proof.

REMARK. It appears to be difficult to establish results concerning the unicoherence of a compactification of an arbitrary completely regular space. We can show that the Freudenthal compactification of a rim-compact, locally connected γ -unicoherent space is the smallest unicoherent compactification of X with $\gamma X \setminus X$ zero-dimensional.

REFERENCES

1. J. DeGroot and R. H. McDowell, *Locally connected spaces and their compactifications*, Illinois J. Math., **11** (1967), 353-364.

2. R. F. Dickman, Jr., *Unicoherence and related properties*, Duke Math. J., **34** (1967), 343-352.
3. ———, *Some characterizations of the Freudenthal compactification of a semicom-
pact space*, Proc. Amer. Math. Soc., **19** (1968), 631-633.
4. H. Freudenthal, *Neuaufbau der Endentheorie*, Ann. of Math., (2), **43** (1942), 261-279.
5. K. D. Magill, Jr., *A note on compactifications*, Math Zeitschr., **94** (1966), 322-325.
6. K. Morita, *On images of an open interval under closed continuous mappings*, Proc.
Japan Acad., **32** (1959), 15-19.
7. ———, *On bicompatifications of semibicompact spaces*, Sci. Rep. Tokyo Bunrika
Daigaku Sect. A, **4** (1952), 222-229.
8. G. T. Whyburn, *Open mappings on 2-dimensional manifolds*, J. Math. Mech., **10**
(1961), 181-198.
9. ———, *Analytic Topology*, Amer. Math. Soc. Colloq. Pub., Vol. **28** (1942).

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