Pacific Journal of Mathematics

# UNICOHERENT COMPACTIFICATIONS

Michael Howard Clapp and Raymond Frank Dickman

Vol. 43, No. 1 March 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 1, 1972

# UNICOHERENT COMPACTIFICATIONS

M. H. CLAPP AND R. F. DICKMAN, JR.

**In this paper we give necessary and sufficient conditions for the Freudenthal compactification of a rimcompact, locally connected and connected Hausdorff space to be unicoherent. We give several necessary and sufficient conditions for a locally connected generalized continuum to have a unicoherent compactification and show that if such a space $X$ has a unicoherent compactification, then $\gamma X$ is the smallest unicoherent compactification of $X$ in the usual ordering of compactifications.**

A connected topological space $X$ is said to be *unicoherent* if, $H \cdot K$ is connected whenever $X = H + K$ where $H$ and $K$ are closed connected sets. A continuum is a compact connected metric space and a generalized continuum is a locally compact, connected, separable metric space. By a mapping we will always mean a continuous function. If $B$ is a subset of a space $X$, the closure of $B$ in $X$ will be denoted by $\mathrm{cl}_X B$ and the boundary of $B$ in $X$ will be denoted by $\mathrm{Fr}_X B$. An open set (respectively, a closed set) of a space $X$ will be called a $\gamma$-open (respectively, $\gamma$-closed) subset of $X$ provided it has a compact boundary in $X$. A space is rimcompact (or semicompact) provided every point has arbitrarily small neighborhoods with compact boundaries. All compactifications considered here are Hausdorff.

In [7] K. Morita showed that for any rimcompact Hausdorff space $X$ there exists a topologically unique compactification $\gamma X$ of $X$ satisfying:

(a) For every point $x$ of $\gamma X$ and every open set $R$ of $\gamma X$ containing $x$ there exists an open set $V$ of $\gamma X$ containing $x$ such that $V \subset R$ and $\mathrm{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint $\gamma$-closed subsets of $X$ have disjoint closures in $\gamma X$.

Furthermore if $C$ is any compactification of $X$ satisfying (a), there exists a mapping $h$ of $\gamma X$ onto $C$ such that $h|X$ is the identity map. The compactification $\gamma X$ of $X$ is called the Freudenthal compactification of $X$ after H. Freudenthal who first defined it [4].

DEFINITION. We say that a connected space $X$ is $\gamma$-unicoherent if whenever $X = H + K$, where $H$ and $K$ are $\gamma$-closed and connected sets, $H \cdot K$ is connected.

THEOREM 1. *If $X$ is a locally connected, connected, rimcompact Hausdorff space, then $\gamma X$, the Freudenthal compactification of $X$, is* *unicoherent iff $X$ is $\gamma$-unicoherent.*

*Proof.* Suppose that $X$ is $\gamma$-unicoherent and $\gamma X$ is not unicoherent. Then $\gamma X = H + K$ where $H$ and $K$ are closed and connected sets and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let $U$ and $V$ be open subsets of $\gamma X$ containing $A$ and $B$ respectively such that $\mathrm{cl}_{\gamma X} U \cdot \mathrm{cl}_{\gamma X} V = \Phi$ and $(\mathrm{Fr}_{\gamma X} V + \mathrm{Fr}_{\gamma X} U) \subset X$. By Propositions (2.8) and (4.1) of [1], $\gamma X$ is locally connected so if $C$ denotes the component of $U + V + H$ that contains $H$ and $D$ denotes the component of $U + V + K$ that contains $K$, $C$ and $D$ are open connected subsets of $\gamma X$ such that $(Fr_{\gamma X} C + \mathrm{Fr}_{\gamma X} D) \subset X$. By Lemma 5 of [6], $C \cdot X$ and $D \cdot X$ are connected so that $L = \mathrm{cl}_X (C \cdot X)$ and $M = \mathrm{cl}_X(D \cdot X)$ are $\gamma$-closed and connected subsets of $X$. Furthermore $X = L + M$ and $L \cdot M$ is not connected. This contradicts our hypothesis that $X$ is $\gamma$-unicoherent and thus $\gamma X$ must be unicoherent.

Now suppose that $\gamma X$ is unicoherent and $X$ is not $\gamma$-unicoherent. Then $X = H + K$ where $H$ and $K$ are $\gamma$-closed and connected subsets of $X$ and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let $H'$, $K'$, $A'$ and $B'$ denote the closures of $H$, $K$, $A$ and $B$, respectively, in $\gamma X$. Since the boundary of $H \cdot K$ in $X$ is a subset of the union of the boundaries of $H$ and $K$ in $X$, $H \cdot K$ and hence $A$ and $B$ are $\gamma$-closed subsets of $X$. Then by property (b) of Morita's characterization of $\gamma X$, $A'$ and $B'$ are disjoint closed subsets of $\gamma X$. We now argue that $H' \cdot K'$ is a subset of $A' + B'$. Suppose to the contrary that there exists a point $x$ in $H' \cdot K'$ that does not belong to $A' + B'$. Let $U$ be any open subsets of $\gamma X$ containing $x$ such that $U$ does not intersect $A' + B'$ and such that $\mathrm{Fr}_{\gamma X} \subset X$. Let $Q$ be the component of $U$ that contains $x$ and note that $\mathrm{Fr}_{\gamma X} Q$ is a subset of $X$ and $Q$ is an open subset of $\gamma X$. Then since $X$ is dense in $\gamma X$ and $x$ is a limit point of $H'$ and $K'$, $Q \cdot H$ and $Q \cdot K$ are nonempty sets. But by Lemma 5 of [6], $Q \cdot X$ is connected and since $Q$ misses $H \cdot K$, $Q \cdot X$ must lie entirely in $H$ or $K$. Of course this implies that either $Q \cdot H$ or $Q \cdot K$ is empty and this is a contradiction. Thus $H' \cdot K' = A' + B'$ and this contradicts the unicoherence of $\gamma X$. Therefore $X$ is $\gamma$-unicoherent.

We need the following notation and definitions. Let $S^1$ denote the unit circle in the complex plane, let $I_1 = \{z = e^{i\theta}: 0 \leqq \theta \leqq \Pi\}$ and let $I_2 = \{z = e^{i\theta}: \Pi \leqq \theta \leqq 2\Pi\}$. For any space $W$ let $\mathscr{A}(W)$ denote the set of mappings of $W$ into $S^1$ and let $\mathscr{A}_j(W)$ be the set of all mappings of $W$ into $I_j$, $j = 1, 2$. For each $f \in \mathscr{A}_j(W)$, $j = 1, 2$, let $B_j(f)$ denote the set of all points $t \in I_j$ such that $\mathrm{Fr}\, f^{-1}(t)$ contains a compact set $K$ that separates $W$ into two disjoint open sets $M$ and $N$ where $f$ maps $M$ into the arc from 1 to $t$ on $I_j$ and $f$ maps $N$ into the arc from $t$ to $-1$ on $I_j$. Finally let $E(W) = \{f \in \mathscr{A}(W): B_1(f \mid f^{-1}(I_1)) +$

$B_2(f\,|\,f^{-1}(I_2))$ is dense in $S^1\}$.

THEOREM 2. *Suppose that $X$ is a locally connected, rimcompact Hausdorff space. A necessary and sufficient condition that $\gamma X$ be unicoherent is that every element of $E(X)$ be nullhomotopic.*

*Proof of the necessity.* Suppose that $\gamma X$ is unicoherent and let $f$ be an element of $E(X)$. For $j = 1, 2$, there exists a point $t_j \in I_j$ such that $\mathrm{Fr}_X\, f^{-1}(t_j)$ contains a compact set $K_j$ that separates $f^{-1}(I_j)$ into two disjoint open sets $M_j$ and $N_j$ where $f$ maps $M_j$ into the arc from 1 to $t_j$ on $I_j$ and $f$ maps $N_j$ into the arc from $t_j$ to $-1$ on $I_j$. Then if we let $M$ denote $K_1 + K_2 + M_1 + M_2$ and let $N$ denote $K_1 + K_2$ $N_1 + N_2$, $X = M + N$ and the boundaries (relative to $X$) of $M$ and $N$ are subsets of $K = K_1 + K_2$. We assert that the boundaries of $M_0 = \mathrm{cl}_{\gamma X} M$ and $N_0 = \mathrm{cl}_{\gamma X} N$ relative to $\gamma X$ are also subsets of $K$. In order to see this suppose that $x$ is an element of the boundary of $M_0$ and $x \notin K$. Then since $\gamma X$ is locally connected, there exists an open connected set $R$ of $\gamma X$ containing $x$ such that $R \cdot K = \Phi$ and $\mathrm{Fr}_{\gamma X} R \subset X$. Then $R \cdot M \neq \Phi$ and $R \cdot (X \backslash M) \neq \Phi$ since $X$ is dense in $\gamma X$. Furthermore $R \cdot X$ is connected by Lemma 5 of [6] and so $R \cdot X$ is a connected subset of $X$ that meets $M$ and $X \backslash M$. This implies that $R$ meets $K$ and this contradicts our selection of $x$. Hence the boundaries of $M_0$ and $N_0$ in $\gamma X$ are subsets of $K$. Also by Theorem 3 of [7], $M_0$ and $N_0$ are topologically equivalent to $\gamma M$ and $\gamma N$ respectively. Then by Lemma 1 of [3], $f\,|\,M$ has a continuous extension $f_M$ to $M_0$ and $f\,|\,N$ has a continuous extension $f_N$ to $N_0$. Then since $N_0 \cdot M_0 \subset K$, the function $h$ of $\gamma X$ into $S^1$ defined by $h\,|\,M_0 = f_M$ and $h\,|\,N_0 = f_N$ is continuous. By Lemma (7.4) of [9, p. 228], $h$ is exponentially representable on $\gamma X$, i.e. there exists a real valued function $\theta$ on $\gamma X$ such that $h(x) = e^{i\theta(x)}$ for all $x \in X$. It is evident that this implies that $f = h\,|\,X$ is exponentially representable an $X$ and by Theorem (6.2) of [9, p. 226], $f$ is nullhomotopic.

*Proof of the sufficiency.* Suppose that every element of $E(X)$ is nullhomotopic and suppose that $\gamma X$ is not unicoherent. Then by the proof of Theorem 1 there exists closed and connected sets $H$ and $K$ of $\gamma X$ such that $H \cdot K$ is not connected, $\mathrm{Fr}\, H$ and $\mathrm{Fr}\, K$ are subsets of $X$ and $L = H \cdot X$ and $M = K \cdot X$ are connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$. We note that $L$ and $M$ are $\gamma$-closed subsets of and thus by Theorem 3 of [7], $\gamma L$ is homeomorphic to $H$ and $\gamma M$ is homeomorphic to $K$. It then follows from Lemma 2 of [3] that there exists a mapping $f$ of $H$ into $I_1$ such that $f(A) = 1$, $f(B) = -1$ and $B_1(f\,|\,H \cdot X)$ is dense in $I_1$. Similarly there exists a mapping $g$ of $K$ into $I_2$ such that $g(A) = 1$, $g(B) = -1$ and $B_2(g\,|\,K \cdot X)$ is dense in $I_2$. Then if we define $h: \gamma X \to S^1$ by $h|H = f$ and $h|K = g$ we have that $h$ is continuous and $k = h|X$ is an element of $E(X)$. Then by our hypothesis and Proposition 6.2 of [9, p. 226], $k$ is exponentially representable, i.e. there exists a real-valued mapping $\theta$ on $X$ such that for each $x \in X$, $k(x) = e^{i\theta(x)}$. But then $\theta(A) \subset \{0, \pm 2\Pi, \pm 4\Pi, \cdots\}$ and $\theta(B) \subset \{\pm\Pi, \pm 3\Pi, \cdots\}$ and so if $a \in \theta(A)$ and $b \in \theta(B)$, the interval $[a, b]$ lies in $\theta(A) \cdot \theta(B)$ since $L$ and $M$ are connected. This is a contradiction since then $k(L) \cdot k(M)$ would then contain a semicircle whereas it consists of the points $-1$ and $1$. Hence $\gamma X$ is unicoherent.

DEFINITION. A connected space $X$ is said to be weakly unicoherent if whenever $X = H + K$ where $H$ and $K$ are closed and connected sets and $K$ is compact, $H \cdot K$ is connected.

THEOREM 3. *Let $X$ be a locally connected generalized continuum. A necessary and sufficient condition for $\gamma X$ to be unicoherent is that $X$ be weakly-unicoherent.*

*Proof of the necessity.* Suppose that $\gamma X$ is unicoherent. Since $X$ is locally compact, $X$ is open in $\gamma X$ and $X^* = \gamma X \backslash X$ is closed. Then by Theorem (2.3) of [2], $X = \gamma X \backslash X^*$ is weakly-unicoherent.

*Proof of the sufficiency.* Suppose that $\gamma X$ is not unicoherent. Then as in the proof of Theorem 1, $\gamma X$ has a representation $\gamma X = P + Q$ where $P$ and $Q$ are open connected subsets of $\gamma X$, the boundaries of $P$ and $Q$ in $\gamma X$ are subsets of $X$, $\mathrm{cl}_{\gamma x} P \cdot \mathrm{cl}_{\gamma x} Q = A + B$ where $A$ and $B$ are disjoint nonempty closed sets and $P$ has a nonempty intersection with both the boundary of $A$ and the boundary of $B$. By Lemma 5 of [6], $P' = P \cdot X$ is a connected open subset of $X$ and thus is arcwise connected. Furthermore since the boundaries of $A$ and $B$ are subsets of $X$ there exists an arc $\alpha\beta$ in $P'$ such that $\alpha\beta \cdot A = \alpha$ and $\alpha\beta \cdot B = \beta$. Let $R$ be the component of $P' \backslash (A + B)$ that contains $\alpha\beta \backslash (\alpha + \beta)$ and let $W$ be an open subset of $\gamma X$ containing $A$ such that $B \cdot \mathrm{cl}\, W = \phi$ and the boundary of $W$ is a subset of $X$. Then $H = R \cdot \mathrm{Fr}_{\gamma x} W$ is a nonempty compact subset of $R$ and there exists a continuum $K_0$ of $X$ such that $H \subset K_0 \subset R$. Let $K$ be the union of $K_0$ together with all the components of $R \backslash K_0$ with boundary entirely in $K_0$, i.e. having no boundary points in $X \cdot (A + B)$. Then $K$ separates $R$ since $W \cdot R$ contains a subarc $\alpha b \backslash \alpha$ from some point $b \in \alpha\beta$ and $X \backslash \mathrm{cl}_x W$ contains a subarc $a\beta$ of $\alpha\beta$. But $X \backslash K$ is connected since $X \backslash K$ is the union of the closure of $Q$ in $X$ plus all of the components of $X \backslash (A \cdot B)$ except $R$ plus all of the components of $R - K_0$ having a boundary point in $X \cdot (A + B)$. This contradicts Whyburn's characterization of weak-unicoherence in [8, p. 185].

COROLLARY 3.1. *Let $X$ be a locally connected generalized continuum. Then $X$ is weakly-unicoherent iff $X$ is $\gamma$-unicoherent.*

This corollary follows immediately from Theorems 1 and 3.

REMARK. The authors have been unable to discover a direct proof of Corollary (3.1). In general the two types of unicoherency are not equivalent and in the absence of local compactness, Theorem 3 is not valid.

EXAMPLE. Let $Y = \{z \text{ complex } | 1/2 \leqq |z| \leqq 1\}$,

$$\begin{aligned} S &= \{z|\ |z| = 1\},\ A \text{ a countable dense subset of } S\,, \\ L_z &= Y\cdot\{\text{ray from origin thru } z\} \\ C_r &= \{z|\ |z| = r\},\ r \in [1/2, 1]; \\ Z &= \{C_r\cdot L_a | r \text{ is rational, } a \in A\}\,. \end{aligned}$$

The set $Z$ is countable and dense in $Y$. Let $X = Y - Z$. The set $X$ is evidently $T_2$, connected and locally connected (in fact, path connected and locally path connected), rim compact but not locally compact. Moreover:

1. $X$ is weakly-unichoherent. To see this, note that any continuum $K \subset X$ has empty interior in $X$. If therefore $X = H + K$, $H$ closed and connected and $K$ compact and connected, then necessarily the open set $X - H$ is a subset of $K$, and thus empty. It follows that $H\cdot K = K$, which is connected.

2. $X$ is not $\gamma$-unicoherent. For let $p, q \in S - A$ be two distinct points. Then $L_p$ and $L_q$ are compact and disjoint subsets of $X$. Assume $0 \leqq ARGp < ARGq$. Then

$$\begin{aligned} H &= \{z \in X | ARGp \leqq ARGz \leqq ARG_q\} \text{ and} \\ K &= \{z \in X | ARGq \leqq ARG_z \leqq ARGp + 2\pi\} \end{aligned}$$

are closed, connected subsets of $X$ such that $X = H + K$, $H\cdot K = L_p + L_q$ is compact but not connected.

3. $\gamma X$ is not unicoherent. To show this it is sufficient to show that $\gamma X$ is just the set $Y$. To this end we use the characterization of $\gamma X$ obtained by Morita [6]. We show that

(a) For any point $x \in \gamma X$ and open set $R$ of $\gamma X$ containing $x$, there is an open set $V$ of $rX$ containing $x$ such that $V \subset R$ and $\mathrm{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint $\gamma$-closed subsets of $X$ have disjoint closures in $\gamma X$.

That (a) holds is evident from the definition of $X$. To see that (b) holds, let $A$ and $B$ be disjoint $\gamma$-closed subsets of $X$ and suppose that $p \in \mathrm{cl}_{\gamma X} A \cdot \mathrm{cl}_{\gamma X} B$. First of all we note that $p$ cannot belong to $X$ for then it would lie in $A \cdot B$ which is empty. In particular $p$ does not lie in the compact set $(\mathrm{Fr}_X A + \mathrm{Fr}_X B)$. By our construction of $X$ there exists an open subset $V$ of $Y$ containing $p$ such that $V \cdot (\mathrm{Fr}_X A + \mathrm{Fr}_X B) = \Phi$ and $V \cdot X$ is connected. Since $p$ belongs to the closure of $A$ in $Y$, $V \cdot X \cdot A$ is not empty and since $V \cdot X$ misses $\mathrm{Fr}_X A$, $V \cdot X$ must lie entirely in $A$. But this is a contradiction since $V \cdot X$ must meet $B$. Therefore $A$ and $B$ have disjoint closures in $Y$.

DEFINITION. A mapping $f: X \in Y$ is *monotone* provided for every $y \in Y$, $f^{-1}(y)$ is compact and connected.

THEOREM 4. *If $X$ is a locally connected generalized continuum and $Y$ is any unicoherent compactification of $X$, then there exists a monotone mapping $g$ of $Y$ onto $\gamma X$ such that $g | X$ is the identity.*

*Proof.* Let $Z$ denote the quotient space of $Y$ obtained from the decomposition whose only nondegenerate elements are the components of $Y \backslash X$ and let $p$ denote the natural map of $Y$ onto $Z$. Then since $X$ is open in $Y$, $Z$ is a Hausdorff compactification of $X$. Furthermore since point inverses of $p$ are connected, it follows from Proposition (2.2.1) of [9], that $Z$ is unicoherent. Also $Z \backslash X$ is totally disconnected and by the maximality of $\gamma X$ there exists a mapping $h$ of $\gamma X$ onto $Z$ such that $h | X$ is the identity and $h(\gamma X \backslash X) = Z \backslash X$. We assert that $h$ is a homeomorphism. In order to prove this we need only show that $h$ is one-to-one on $\gamma X \backslash X$. To this end let $x, y \in \gamma X$, $x \neq y$ and suppose that $h(x) = h(y)$. There exists a connected and open set $R$ of $\gamma X$ containing $x$ such that $y \notin \mathrm{cl}_\gamma R = K$ and $\mathrm{Fr}_\gamma R \subset X$. Then $Z = h(K) + h(\gamma X \backslash R)$ and $h(K) \cdot h(\gamma X \backslash R) = h(x) + h(\mathrm{Fr}\, R)$ is not connected. This contradicts the unicoherence of $Z$ and hence $h$ must be a homeomorphism. Then $g = h^{-1} \circ p$ is the desired monotone mapping.

COROLLARY 4.1. *Suppose that $X$ is a locally connected generalized continuum. Then $X$ has a unicoherent compactification if and only if $\gamma X$ is unicoherent.*

*Proof.* This result follows immediately from Theorem 4 and the fact that monotone images of compact unicoherent continua are unicoherent.

THEOREM 5. *Suppose that $X$ is a locally connected generalized continuum. Then the following are equivalent*

( i ) *$X$ is weakly-unicoherent*
( ii ) *$\gamma X$ is unicoherent*
(iii) *$X$ is $\gamma$-unicoherent*
(iv) *$X$ has a unicoherent compactification*
( v ) *every mapping of $X$ into $S^1$ with compact boundaries of point inverses is null-homotopic.*

*Proof.* The equivalence of (i)—(iv) has been established in Theorems (1) – (4). As an immediate consequence of Theorem (3.3) of [2], we have that (v) implies (i) and (ii) implies (v) follows from Theorem 1 of this paper.

DEFINITION. A connected space $X$ is said to have the *complementation property* provided whenever $K$ is a compact set in $X$, $X/K$ has at most one component with a non-compact closure. See [2] for some characterizations of this property.

THEOREM 6. *Let $X$ be a locally connected generalized continuum and let $Y$ be any unicoherent, locally connected continuum. There exists a unicoherent compactification $Z$ of $X$ with $Z\backslash X$ homeomorphic to $Y$ if and only if $X$ is weakly-unicoherent and has the complementation property.*

*Proof of the necessity.* Suppose that $Z$ is a unicoherent compactification of $X$ and $Z\backslash X$ is homeomorphic to $Y$. Then by Theorem (4.2) of [2], $X$ is weakly-unicoherent and has the complementation property.

*Proof of the sufficiency.* Suppose that $X$ is weaklyunicoherent and has the complementation property. Then by Theorem (2.2) of [5] there exists a compactification $Z$ of $X$ with $Z\backslash X$ homeomorphic to $Y$ and by Theorem (4.2) of [2], $Z$ is unicoherent. This completes the proof.

REMARK. It appears to be difficult to establish results concerning the unicoherence of a compactification of an arbitrary completely regular space. We can show that the Freudenthal compactification of a rim-compact, locally connected $\gamma$-unicoherent space is the smallest unicoherent compactification of $X$ with $\gamma X\backslash X$ zero-dimensional.

## REFERENCES

1. J. DeGroot and R. H. McDowell, *Locally connected spaces and their compactifications*, Illinois J. Math., **11** (1967), 353-364.

2. R. F. Dickman, Jr., *Unicoherence and related properties*, Duke Math. J., **34** (1967), 343-352.
3. ———, *Some characterizations of the Freudenthal compactification of a semicompact space*, Proc. Amer. Math. Soc., **19** (1968), 631-633.
4. H. Freudenthal, *Neuaufbau der Endentheorie*, Ann. of Math., (2), **43** (1942), 261-279.
5. K. D. Magill, Jr., *A note on compactifications*, Math Zeitschr., **94** (1966), 322-325.
6. K. Morita, *On images of an open interval under closed continuous mappings*, Proc. Japan Acad., **32** (1959), 15-19.
7. ———, *On bicompactifications of semibicompact spaces*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A, **4** (1952), 222-229.
8. G. T. Whyburn, *Open mapings on 2-dimensional manifolds*, J. Math. Mech., **10** (1961), 181-198.
9. ———, *Analytic Topology*, Amer. Math. Soc. Colloq. Pub., Vol. **28** (1942).

Received April 27, 1971.

CALIFORNIA STATE COLLEGE AT FULLERTON
AND
VIRGINIA POLYTECHNIC INSTITUTE AND
STATE UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

# Pacific Journal of Mathematics

Vol. 43, No. 1 March, 1972