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This paper characterizes the automorphisms of a cylindrical semigroup S in terms of the automorphisms of the defining subgroups and subsemigroups. The following theorem is representative of the type of information given in this paper.

Let $F: \boldsymbol{R} \rightarrow A$ be a dense homomorphism of the additive real numbers to the compact abelian group A. Let λ be a positive real number. Multiplication by λ shall also denote the automorphism of A whose restriction to $F(\boldsymbol{R})$ is given by $F\lambda F^{-1}$. The set of all such λ for a given F is called Λ_F.

Theorem. Let f and λ be as above. Let G be a compact group. Let

$$S = \{(p, f(p)\, g): p \in \boldsymbol{H} \text{ and } g \in G\} \cup \alpha \times A \times G\ .$$

Then $\alpha: S \rightarrow S$ is an automorphism if and only if $\alpha(p, f(p), g) = (\lambda p, f(\lambda p), \tau(f(p))\xi(g))$; $\alpha(\infty, a, g) = (\infty, \lambda a, \tau(a)\xi(g))$, where $\tau: A \rightarrow G$ is a homomorphism into the centre of G and, $\xi: G \rightarrow G$ is an automorphism. Theorem. Let S be as in theorem above. Let $\mathscr{A}(G)$ be the automorphism group of G, and $Z(G)$, the center of G. The automorphism group of S is isomorphic as an abstract group to $\mathscr{A}(G) \times (\Lambda_F \times \mathrm{Hom}(A, Z(G)))$ with the following multiplication

$$(\xi, (\lambda, \tau))(\bar{\xi}, (\bar{\lambda}, \bar{\tau})) = (\xi \circ \bar{\xi}, (\lambda\bar{\lambda}, (\tau \circ \bar{\lambda})(\xi \circ \bar{\tau})))\ .$$

Cylindrical semigroups play an important role Mislove's description of $Irr(X)$ and are the building blocks used in the construction of a hormos. Hofmann and Mostert [3] have shown that every compact irreducible semigroup is a hormos. The definition and description of a cylindrical semigroup, given in § I, is from their book.

I. Definitions and notation. All spaces are Hausdorff. All homomorphisms are continuous unless otherwise stated. A homomorphism will be called abstract if it is not assumed continuous. A group considered with the discrete topology will be called abstract. A topological semigroup is a topological space, S, together with a continuous associative multiplication $m: S \times S \rightarrow S$; $m(s, t) = st$. All semigroups are topological with identity 1. A topological group is a semigroup with the map $\phi: S \rightarrow S$, $\phi(s) = s^{-1}$, continuous also. An *ideal*, I, in a semigroup, S, is a subset of S such that: if $x \in S$ then $(xI \cup Ix) \subset I$. If S is compact and abelian then S has an ideal $M(S)$ which is minimal with respect to set inclusion, is unique, and is a group. An *idempotent* $x \in S$ has the property $x^2 = x$. The maximal subgroup of S containing an idempotent e is called the *group of units of* e and denoted $H(e)$. The group of units of 1 is also denoted $H(S)$ and called *the group of units of* S. If $\alpha: S \to S$ is an automorphism then $\alpha(H(S)) = H(S)$ and $\alpha(M(S)) = M(S)$.

NOTATION. The following notation is standard throughout the paper.

$[a, b]$—In a totally ordered set, the closed interval from a to b.

$]a, b[$—The open interval from a to b.

$\boldsymbol{H}$—The semigroup of nonnegative real numbers under addition with the usual topology.

$\boldsymbol{H}^*$—The one point compactification of $\boldsymbol{H}$, written $[0, \infty]$.

$\boldsymbol{H}_r^*$—$\boldsymbol{H}^*/[r, \infty]$.

$\varLambda$—The abstract group of positive real numbers under multiplication.

$\boldsymbol{R}$—The group of real numbers under addition with the usual topology.

$Z(G)$—The center of a group G.

$[p]$—The image of p under the quotient map $\boldsymbol{H}^* \to \boldsymbol{H}_r^*$.

$*$—As in B^*, the closure of $B \subset X$, except as noted above for $\boldsymbol{H}$.

$X \backslash A$—For $A \subset X$, the complement of A in X.

1. DEFINITION. Let A and G be compact groups. Let A be an abelian and $f: \boldsymbol{H} \to A$ a homomorphism such that $f(\boldsymbol{H})^* = A$. Consider $\boldsymbol{H}^* \times A \times G$ with coordinate-wise multiplication, and let S be that subsemigroup defined by:

$$S = \{(p, f(p), g): p \in \boldsymbol{H}, g \in G\} \cup \infty \times A \times G .$$

Any homomorphic image of S is called a cylindrical semigroup.

The following theorem which describes cylindrical semigroups is from [3, p. 85, Prop. 2.2].

THEOREM A (Hofmann and Mostert). *Let S be a cylindrical semigroup as defined above. Let e be the identity of G and*

$$S' = \{(p, f(p), e): p \in \boldsymbol{H}\} \cup \infty \times A \times e .$$

Let $\phi: \to T$ be a surmorphism onto a compact semigroup T. Then there are:

(i) *compact semigroups T_1, T_1', X and a compact group B,*

(ii) *surmorphisms h_1, h_2, h_3, h_4, ϕ_1, ϕ_2*

(iii) *monomorphisms i_1, i_2*

such that the following diagram commutes:

$$
\begin{array}{ccccc}
\boldsymbol{H}^* & \xrightarrow{\quad h_1 \quad} & \boldsymbol{H}_r^* & \xrightarrow{id} & \boldsymbol{H}_r^* \\
{\scriptstyle \pi}\uparrow & & \uparrow{\scriptstyle \pi'} & & \uparrow{\scriptstyle h_4} \\
\boldsymbol{H}^* \times A \times G & \xrightarrow{h_1 \times h_2 \times id} & \boldsymbol{H}_r^* \times B \times G & \xrightarrow{h_3} & X \\
{\scriptstyle \cup}\uparrow & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} \\
S & \xrightarrow{\quad \phi_1 \quad} & T_1 & \xrightarrow{\phi_2} & T \\
{\scriptstyle \cup}\uparrow & & {\scriptstyle \cup}\uparrow & & \uparrow{\scriptstyle \cup} \\
S' & \xrightarrow{\qquad\quad} & T_1' & \longrightarrow & \phi(S')
\end{array}
$$

(π, π' are projections; $\phi_2 \circ \phi_1 = \phi$).

Moreover, $h_3|_{H^* \times B \times e}$ *is a monomorphism and* $h_4 \circ i_2$ *is a surmorphism.*

From this theorem it is possible to describe T in terms of equivalence classes of elements in $\boldsymbol{H}_r^* \times B \times G$.

$f(0)$ is the identity of A. r, if it exists, is the least real number such that $\phi(r, f(r), e) = \phi(\infty, a, g)$ for some $a \in A$, $g \in G$.

$$B = \phi(\infty \times A \times e). \quad T_1' = \phi(S') \times e .$$

Let $\bar{f}\colon \boldsymbol{H} \to B$ be given by $\bar{f}(p) = \phi(\infty, f(p), e)$ then

$$i_1(T_1') = \{([p], f(p), e)\colon p \in \boldsymbol{H}\} \cup [r] \times B \times e .$$

If there is no such r, then $i_1(T_1') \subset \boldsymbol{H}^* \times B \times G$.

Let

$$G_{[p]} = \{g \in G\colon \phi(p, f(p), g) = \phi(p, f(p), e)\}$$

and

$$G_{[r]} = \{g \in G\colon \phi([r], f(0), g) = \phi([r], f(0), e)\}$$

where $r \leqq \infty$. $\{G_{[p]}\colon p \in \boldsymbol{H}^*\}$ has the following two properties:

(1) $$G_{[p]} \subseteqq G_{[q]} \quad \text{for} \quad p \leqq q ;$$

(2) $$G_{[p]} = \bigcap_{q>p} G_{[q]} .$$

Each $G_{[p]}$ is a normal subgroup of G. Denote $G/G_{[r]}$ by $\bar{G}$ and assume $G_{[0]} = \{e\}$.

$$i_2\phi(\{(p, f(p), g)\colon p \in \boldsymbol{H}, g \in G\}) = \{([p], f(p), gG_{[p]})\colon p \in \boldsymbol{H}, g \in G\}$$

where

$$(gG_{[p]})(\bar{g}G_{[\bar{p}]}) = g\bar{g}G_{[p+\bar{p}]} .$$

$i_2\phi(\infty \times A \times G) = ([r] \times B \times G)/K$ where K is a normal subgroup of

$[r] \times B \times G$. K has the property: if $([r], b, g) \in K$ and $([r], \bar{b}, \bar{g}) \in K$ then $b = \bar{b}$ if and only if $g = \bar{g}$.

We shall identify T with its image $i_2(T)$ and refer to $i_1(T_1')$ as T'. Since B is a compact abelian group and $\bar{f}: \boldsymbol{H} \to B$ is onto a dense subset of B, we may as well consider them as f and A to avoid extra notation. We say

$$T = \{([p], f(p), gG_{[p]}): p \in \boldsymbol{H}, g \in G\} \cup ([r] \times B \times G)/K .$$

II. Automorphisms on semigroups of the form of S. We first consider automorphisms of the cylindrical semigroup S given in Definition 1. $M(S)$, the minimal ideal of S, is $\infty \times A \times G$. $H(S)$, the group of units, is $\{(0, f(0), g): g \in G\}$. From Theorem A we have that an automorphism $\alpha: S \to S$ can be thought of as an automorphism on $S' \times H(S)$.

Consider the situation where $G = \{e\}$. We have $S = S'$, $M(S') = \infty \times A \times e$ and $S' \backslash M(S')$ is isomorphic to $\boldsymbol{H}$ by $(p, f(p), e) \leftrightarrow p$. For an automorphism $\alpha: S' \to S'$, $\alpha(M(S')) = M(S')$; and, α restricted to $S' \backslash M(S')$ corresponds to an automorphism of $\boldsymbol{H}$. Since the only automorphisms of $\boldsymbol{H}$ are multiplication by a positive real number λ, we have $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$.

How shall α behave on $M(S')$? Let $\boldsymbol{R}$ be the additive group of real numbers, then $f: \boldsymbol{H} \to A$ can be extended to $F: \boldsymbol{R} \to A$ (for $x \notin \boldsymbol{H}$, $F(x) = f(-x)^{-1}$) and $F(\boldsymbol{R})$ will be dense in A. Let $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$. Then:

$$\begin{aligned}
\alpha(\infty, f(p), e) &= \alpha((p, f(p), e)(\infty, f(0), e)) \\
&= \alpha(p, f(p), e)\alpha(\infty, f(0), e) \\
&= (\lambda p, f(\lambda p), e)(\infty, f(0), e) \\
&= (\infty, f(\lambda p), e) .
\end{aligned}$$

Define $\bar{\lambda}: F(\boldsymbol{R}) \to F(\boldsymbol{R})$ by $\bar{\lambda}(F(x)) = F(\lambda x)$. $\alpha|_{M(S')}: M(S') \to M(S')$ must be an extension of $\bar{\lambda}$. This extension will be called λ.

Any homomorphism between dense subgroups of compact groups can be extended to a unique homomorphism between the groups. If original map is an automorphism then the extension is also. The existence and uniqueness of the extension, as a function, follow from the fact that the subgroups are uniform spaces and the groups are completions of them [1]. That the extension is a homomorphism is an easy consequence of the definition of the extension.

2. LEMMA. *Let $S' = \{(p, f(p), e): p \in \boldsymbol{H}\} \cup \infty \times A \times e$. If f is neither one-to-one nor constant then the only automorphism of S' is*

the identity. Otherwise, $\alpha: S' \to S'$ *is an automorphism iff* $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$, $\alpha(\infty, a, e) = (\infty, \lambda a, e)$ *where* $F\lambda F^{-1}$ *is open and continuous or* F *is constant.*

Proof. If $\alpha: S' \to S'$ is an automorphism the discussion above shows that $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$ and $\alpha(\infty, a, e) = (\infty, \lambda a, e)$. If f is constant then $A = \{e\}$; S' is isomorphic to $\boldsymbol{H}^*$; and multiplication by any λ is an automorphism.

Suppose f is not constant. Consider the map $\bar{\lambda}: F(\boldsymbol{R}) \to F(\boldsymbol{R})$ given by $\bar{\lambda}(F(x)) = F(\lambda x)$. If F is not one-to-one then the kernel of F in $\boldsymbol{R}$ is cyclic and $\lambda: \boldsymbol{R} \to \boldsymbol{R}$ must preserve this kernel. This implies λ is an integer. Since λ^{-1} must also be an integer, we have $\lambda = 1$.

If F is one-to-one then $\bar{\lambda}$ is an automorphism of the abstract group $F(\boldsymbol{R})$. To be an automorphism of $F(\boldsymbol{R})$ with the induced topology from A, $\bar{\lambda}(= F\lambda F^{-1})$ must be open and continuous. The remark immediately preceding this lemma guarantees that $\bar{\lambda}$ can be extended to A when it is open and continuous.

Let $\Lambda_F = \{\lambda \in \Lambda: F\lambda F^{-1}$ is open and continuous$\}$.

When $G \neq \{e\}$ we have $\alpha: S' \times H(S) \to S' \times H(S)$ where $H(S)$ is isomorphic to G and $M(S) = \infty \times A \times G$. Since $\alpha(H(S)) = H(S)$, $\alpha(0, f(0), g) = (0, f(0), \xi(g))$ for some automorphism $\xi: G \to G$. Hence, the only possibility for $\alpha(\infty, f(0), g) = (\infty, a, h)$ is when $a = f(0)$. α restricted to $M(S)$ must therefore have the form $\alpha(\infty, a, g) = (\infty, \lambda a, \tau(a)\xi(g))$ with $\lambda \in \Lambda$, ξ as above and $\tau: A \to Z(G)$ (center of G), a homomorphism. τ must be continuous since $\tau = \pi_G \circ \alpha \circ i$ where π_G is the projection onto G, and i is the map $A \to \infty \times A \times G$ given by $i(a) = (\infty, a, e)$. Similarly τ must be a homomorphism. Since elements in $\infty \times A \times e$ commute with elements of $\infty \times f(0) \times G$, τ maps A into $Z(G)$.

3. THEOREM. *Let S be as in Definition* 1. $\alpha: S \to S$ *is an automorphism iff* $\alpha(x, a, g) = (\lambda x, \lambda a, \tau(a)\xi(g))$ *where* $\lambda \in \Lambda_F$; $\tau: A \to Z(G)$ *is a homomorphism and* $\xi: G \to G$ *is an automorphism.*

Proof. The above discussion establishes the only if part. Let λ, τ, ξ be given as described in the theorem. $\hat{\alpha}: \boldsymbol{H}^* \times A \times G \to \boldsymbol{H}^* \times A \times G$ can be defined by $\hat{\alpha}(x, a, g) = (\lambda x, \lambda a, \tau(a)\xi(g))$. It is immediate that $\hat{\alpha}$ is an abstract automorphism. Since $\boldsymbol{H}^* \times A \times G$ is compact, we need only that $\hat{\alpha}$ is continuous. Let $U \times V \times W$ be a basis open set. $\hat{\alpha}^{-1}(U \times V \times W) = \lambda^{-1}U \times \lambda^{-1}V \times \xi^{-1}(\tau(\lambda^{-1}V)^{-1})\xi^{-1}(W)$. Since λ and ξ are continuous, $\lambda^{-1}U$, $\lambda^{-1}V$ and $\xi^{-1}(W)$ are open. Since G is a topological group, for any set X, $X\xi^{-1}(W)$ is open. Hence $\hat{\alpha}^{-1}(U \times V \times W)$ is open. Let $\alpha = \hat{\alpha}|_S$.

III. Automorphisms on semigroups of the form of T. Recall $T = \{([p], f(p), gG_{[p]})\colon p \in \boldsymbol{H}, g \in G\} \cup ([r] \times A \times \bar{G})/_K$. It is easier to keep track of the situation by considering cases determined by r, G, and K.

Case (a). Let $r < \infty$ and $G = \{e\}$. Then $K = \{([r], f(0), e)\}$.

4. LEMMA. *Let T be given by Case* (a). *The only automorphism on T is the identity.*

Proof. Let α be an automorphism of T.

Suppose $p < r$. $\alpha([p], f(p), e) = ([q], f(q), e)$ for some $q < r$ since $\alpha(M(T)) = M(T)$. First, let us take the case where $p = r/n$ for some integer n. If $p < q$ then there exists $p' < p$ such that $\alpha([p'], f(p'), e) = ([p], f(p), e)$ and $\alpha([np'], f(np'), e) = ([np], f(np), e) = ([r], f(r), e) \in M(T)$. But $np' < r$ since $np = r$ and $p' < p$. This means $\alpha([np'], f(np'), e) \notin M(T)$. We have a contradiction; so $p \geqq q$. If we assume $p > q$, a similar contradiction arises from $nq < r$. So, if $p < r$ and $p = r/n$ then $\alpha([p], f(p), e) = ([p], f(p), e)$.

For $p < r$, if $p \neq r/n$ then there exists a sequence, possibly finite, of integers $\{n_i\}$ such that $p = \sum r/n_i$. α is continuous so, again, $\alpha([p], f(p), e) = ([p], f(p), e)$.

$$\begin{aligned}\alpha([r], f(r), e) &= \lim_{\bar{p}<r} \alpha([\bar{p}], f(\bar{p}), e)\\ &= \lim_{\bar{p}<r} ([\bar{p}], f(\bar{p}), e) = ([r], f(r), e)\ .\end{aligned}$$

For $p > r$, $p = nr + p'$ where $p' < r$. We have:

$$\begin{aligned}\alpha([p], f(p), e) &= \alpha([nr], f(nr), e)\alpha([p'], f(p'), e)\\ &= (\alpha([r], f(r), e))^n([p'], f(p'), e)\\ &= ([r], f(r), e)^n([p'], f(p'), e) = ([p], f(p), e)\ .\end{aligned}$$

So α is the identity map.

Case (b). Let $r = \infty$, $G_p = G_\infty$ for all p and $K = \{(\infty, f(0), e)\}$.

In this case, T is of the form of S where $\bar{G} = G/G_\infty$.

Case (c). Let $r < \infty$, $G_{[p]} = G_{[r]}$ for all p and $K = \{([r], f(0), e)\}$. Let $G/G_{[r]} = \bar{G}$.

5. THEOREM. *Let T be as in Case* (c). *$\alpha\colon T \to T$ is an automorphism iff $\alpha(x, a, g) = (x, a, \tau(a)\xi(g))$ where $\tau\colon A \to Z(\bar{G})$ is a homomor-*

phism and $\xi\colon \bar{G} \to \bar{G}$ *is an automorphism.*

Proof. From Lemma 4 we have $\lambda = 1$ and the precise arguments in the proof of Theorem 3 concerning τ and ξ hold here.

Case (d). Let $r \leqq \infty$, $G_{[p]} \neq G_{[q]}$ for $[p] \neq [q]$ and $K = \{([r], f(0), e)\}$.

In this case, the description becomes more complicated but is in fact, no more difficult to prove. The previous cases allowed $\tau\colon A \to Z(\bar{G})$ to be defined in $M(T)$ and then used in $T\backslash M(T)$. Here, since $\bar{G} = G/G_{[r]} \neq G/G_{[p]}$ for $[p] \neq [r]$, it is not possible to start by taking τ defined in $M(T)$ to be any homomorphism in $\mathrm{Hom}\,(A, Z(\bar{G}))$. Rather, we start with a homomorphism $h\colon \boldsymbol{H} \to T\backslash M(T)$ which must also determine a homomorphism $f(\boldsymbol{H}) \to Z(\bar{G})$. The latter homomorphism can then be extended to define τ. Without loss of generality, we may assume $G_{[0]} = \{e\}$.

6. THEOREM. *Let T be as described for Case* (d). *Let* $\xi\colon G \to G$ *be an automorphism. If* $r < \infty$, *let* $\xi(G_{[p]}) = G_{[p]}$ *for all* $p \in \boldsymbol{H}$. *If* $r = \infty$, *let there exist a* $\lambda \in \Lambda_F$ *such that* $\xi(G_{[p]}) = G_{[\lambda p]}$ *for all* $p \in \boldsymbol{H}$.

Let $h\colon \boldsymbol{H} \to T$ *be a homomorphism such that* $h(p) = ([p], f(p), gG_{[p]})$ *and*

$$\{h(p)([r], f(0), G_{[r]})\} \subseteqq [r] \times A \times (G/G_{[r]})$$

represents the graph of a homomorphism $f(\boldsymbol{H}) \to Z(G/G_{[r]})$.

$\alpha\colon T \to T$ *is an automorphism iff* $\alpha([p], f(p), gG_{[p]}) = h(\lambda p)(0, f(0), \xi(g))$, *and* $\alpha([r], a, gG_{[r]}) = ([r], \lambda a, \tau(a)\xi(g)G_{[r]})$ *where* $\tau\colon A \to Z(G/G_{[r]})$ *is a homomorphism.*

Proof. Let us assume $r = \infty$. The proof for $r < \infty$ follows this one replacing λ by 1 and p by $[p]$. Let α be given.

Define $\xi\colon G \to G$ in the usual way by considering $\alpha|_{H(T)}$. It is still the case that $(p, f(p), G_p) \to (\lambda p, f(\lambda p), gG_p)$. This follows directly from the top level of the diagram in Theorem A. One can show that $\xi(G_p) = G_{\lambda p}$ by considering $(p, f(p), G_p)$ written as $(p, f(p), gG_p)$ for $g \in G_p$. $\lambda \in \Lambda_F$ since once again λ must be extended to an automorphism of A in $M(T)$ (see Theorem 3).

Define

$$h\colon \boldsymbol{H} \to T \text{ by } h(p) = \alpha(\lambda^{-1}p, f(\lambda^{-1}p), G_{\lambda^{-1}p})\ .$$

h is the composition of three homomorphisms

$$\boldsymbol{H} \xrightarrow{\lambda^{-1}} \boldsymbol{H} \xrightarrow{\hat{f}} T \xrightarrow{\alpha} T \quad \text{where} \quad \hat{f}(p) = (p, f(p), G_p)\ .$$

Define $\lambda h(p) = h(\lambda p)$. λh is also a homomorphism but not of the type specified by the theorem.

Define $\tau\colon A \to Z(G/G_\infty)$, as was done in Theorem 3, by considering $\alpha|_{\infty \times A \times e}$.

Note:

$$h(p)(\infty, f(0), G_\infty) = \alpha(\infty, f(\lambda^{-1}p), G_\infty)$$
$$= (\infty, f(p), gG_\infty) = (\infty, f(p), \tau(f(p))) .$$

So $\{h(p)(\infty, f(0), G_\infty)\}$ represents the graph of a homomorphism from $f(\boldsymbol{H}) \to Z(G/G_\infty)$. We shall sometimes write $\tau(a)$ as $\tau(a)G_\infty$. We observe that $\{h(p)(\infty, f(0), G_\infty)\} = \{\lambda h(p)(\infty, f(0), G_\infty)\}$, so h and λh can be made to determine the same τ.

For the converse let ξ, and h be given. ξ determines $\lambda \in \Lambda_F$. λh determines the graph of a homomorphism since h does. Define $\tau(f(p)) = \pi_\infty(h(\lambda p)(\infty, f(0), G_\infty))$ where π_∞ is the projection. τ can be extended in the usual way to A.

Define $\alpha\colon T \to T$ by

$$\alpha(p, f(p), gG_p) = \lambda h(p)(0, f(0), \xi(g))$$
$$\alpha(\infty, a, gG_\infty) = (\infty, \lambda a, \tau(a)\xi(g)) .$$

Showing α is an abstract homomorphism is straightforward. One can prove α is continuous by writing T as the image of S and considering open sets. This proof is omitted because it is uninteresting and requires complicated notation.

Case (e). Let $r = \infty$, $G_p = G_q \neq G_\infty$ and $K = \{(\infty, f(0), e)\}$.

This situation is a simple version of Case (d). Since $G_p = G_{\lambda p}$ for all λ, we no longer have λ determined by $\xi\colon G \to G$. Any choice of $\lambda \in \Lambda_F$ will give an automorphism.

Case (f). Let $K \neq \{([r], f(0), e)\}$ and $K \neq [r] \times A \times \bar{G}$. Let $\hat{T} = \{([p], f(p), gG_{[p]})\colon p \in \boldsymbol{H}, g \in G\} \cup [r] \times A \times \bar{G}$ and let $T = \{([p], f(p), gG_{[p]}\} \cup ([r] \times A \times \bar{G})/K$. Let $k\colon \hat{T} \to T$ be the map which is the identity on $\hat{T} \backslash M(\hat{T})$ and the quotient map on $M(\hat{T})$. Recall: if $([r], a, g) \in K$ and $([r], \bar{a}, \bar{g}) \in K$ then $a = \bar{a}$ iff $g = \bar{g}$. When $r < \infty$, if $k(t_\gamma)$ is a convergent net in T such that $k(t_\gamma) \notin M(T)$ and $\lim k(t_\gamma) \in M(T)$, then $t\gamma$ is a convergent net in $\hat{T}$. Let $\pi_A(K) = \{a \in A\colon ([r], a, g) \in K$ for some $g \in \bar{G}\}$. Let β be the abstract isomorphism $\beta\colon \pi_A(K) \to \bar{G}$ given by $g = \beta(a)$ if $([r], a, g) \in K$.

7. LEMMA. *Let T and $\hat{T}$ be as above. Let $\hat{\alpha}\colon \hat{T} \to \hat{T}$ be charac-*

terized by (λ, τ, ξ) *or by* (λ, h, ξ) *as given in* 3, 5, 6. *Let* $\pi_A(K)$ *and* β *be as above. There exists an automorphism* $\alpha: T \to T$ *such that* $\alpha k = k\hat{\alpha}$ *iff* $\lambda|_{\pi_A(K)}$ *is an automorphism and* $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$ *for* $a \in \pi_A(K)$.

Proof. Suppose $\hat{\alpha}$ induces an automorphism α such that $\alpha k = k\hat{\alpha}$. Consider $\hat{\alpha}|_{M(\hat{T})}$ as an automorphism on the group $M(\hat{T})$. This induces $\alpha|_{M(T)}$ on $M(T)$ and for $\alpha|_{M(T)}$ to be well defined and one-to-one we must have $\hat{\alpha}(K) = K$. For $([r], a, \beta(a)) \in K$ we have $\hat{\alpha}([r], a, \beta(a)) = ([r], \lambda a, \tau(a)\xi(\beta(a))) \in K$. Hence, $\lambda a \in \pi_A(K)$ and $\beta(\lambda a) = \tau(a)\xi(\beta(a))$. Since $\hat{\alpha}^{-1}$ is also an automorphism $\lambda^{-1}a \in \pi_A(K)$ and λ is onto. $\beta(\lambda a) = \tau(a)\xi(\beta(a))$ implies $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$.

The proof of the converse is straightforward. It is convenient to consider the continuity of α on $T\backslash M(T)$ and $M(T)$ separately and then consider a net converging to $M(T)$.

8. THEOREM. *Let* $\hat{T}$, T *and* k *be as in Lemma* 7. $\alpha: T \to T$ *is an automorphism iff there exists an automorphism* $\hat{\alpha}: \hat{T} \to \hat{T}$ *such that* $\alpha k = k\hat{\alpha}$.

Proof. Let $\alpha : T \to T$ be an automorphism. We consider two cases: $r < \infty$ and $r = \infty$. Let $r < \infty$. We know from Theorems 5 and 6 that $\hat{\alpha}$ is determined by (ξ, h) or (ξ, τ). Constructing h is the more general situation. An argument similar to that of Theorem 4 establishes that

$$\alpha k([p], f(p), G_{[p]}) = k([p], f(p), \bar{g}G_{[p]}) .$$

Let $G_{[0]} = \{e\}$ and $\bar{G} = G/G_{[r]}$.

Define $\xi: G \to G$ by $\xi(g) = \pi_G \alpha k([0], 1, g)$. Clearly ξ is an automorphism.

Define $h: \boldsymbol{H} \to \hat{T}$ by:

$$h(p) = k^{-1}\alpha k([p], f(p), G_{[p]}) \quad \text{when } p < r ;$$
$$h(r) = \lim_{p<r} h(p) ;$$
$$h(p) = (h(r))^n h(q) \quad \text{when} \quad p = nr + q, q < r .$$

It is immediate that h is a homomorphism. Since $\alpha k([p], f(p), G_{[p]}) = k([p], f(p), gG_{[p]})$, we have also $\alpha k([r], a, G_{[r]}) = k([r], a, gG_{[r]})$.

Define $\tau: A \to Z(\bar{G})$ by $\tau(a) = gG_{[r]}$ such that $\alpha k([r], a, G_{[r]}) = k([r], a, gG_{[r]})$. τ is well-defined since if $([r], a, y) \in ([r], a, g)K$ then $([r], f(0), yg^{-1}) \in K$ and $y = g$. It is also immediate that τ is an abstract homomorphism. $\tau(f(p)) = \pi_{\bar{G}}(h(p)([r], f(0), e))$ so τ is continuous on $f(\boldsymbol{H})$ and hence on A. Even if $\hat{\alpha}$ is more efficiently given by (ξ, τ), h

can be defined and the above will show τ continuous.

Define $\hat{\alpha}$: $\hat{T} \to \hat{T}$ by (ξ, h) or (ξ, τ).

$$\alpha k([p], a, gG_{[p]}) = k([p], a, \tau(a)\xi(g)G_{[p]}) = k\hat{\alpha}([p], a, gG_{[p]}) .$$

So $\alpha k = k\hat{\alpha}$.

Now, let $r = \infty$ and $\bar{G} = G/G_\infty$. Define ξ as before. Either ξ determines λ (as in 6); or, define λ by checking $\alpha k(p, f(p), G_p)$. If f is not one-to-one then, $\lambda = 1$ or $A = \{1\}$. If f is one-to-one then λ is one-to-one on $f(\boldsymbol{H}) \subset A$ and can be extended to λ continuous on A. Since α^{-1} is also an automorphism the above process can be done for λ^{-1} which means λ is open on A and hence $\lambda \in \Lambda_F$.

Define h: $\boldsymbol{H} \to \hat{T}$ by $h(p) = k^{-1}\alpha k(\lambda^{-1}p, f(\lambda^{-1}p), G_{\lambda^{-1}p})$. h is a homomorphism since k is an isomorphism.

Define $\tau(f(p)) = \pi_{\bar{G}}(h(p)(\infty, f(0), G_\infty))$. τ is continuous since h and $\pi_{\bar{G}}$ are, and can be extended to A.

We define $\hat{\alpha}$: $\hat{T} \to \hat{T}$ by (λ, ξ, h) or (λ, ξ, τ). Again, $\alpha k = k\hat{\alpha}$.

So, for each case, $\hat{\alpha}$, an automorphism of $\hat{T}$ inducing $\hat{\alpha}$, can be constructed.

IV. Automorphism groups. This section describes the group structure of the groups of automorphisms given in II and III. All groups discussed here are discrete. Bowman [2] has described the topology of these groups. Since in each case the group is described as a semidirect product of groups of homomorphisms; we give the definition of semidirect product below.

Let A and B be two groups. Let g: $A \to \mathscr{A}(B)$, the group of automorphisms of B, be a function such that:

(i) $g(a_2)(g(a_1)b) = g(a_2a_1)(b)$;

or

(ii) $g(a_2)(g(a_1)b) = g(a_1a_2)(b)$.

$A \times B$ is a group with the following multiplication: $(a, b)(\bar{a}, \bar{b}) = (a\bar{a}, b(g(a)\bar{b}))$ when g is of type i; $(a, b)(\bar{a}, \bar{b}) = (a\bar{a}, (g(\bar{a})b)\bar{b})$ when g is of type ii. The semidirect product will be denoted $A \times_g B$.

Recall, the operation in $\mathscr{A}(G)$ is composition of functions; in Hom $(A, Z(G))$, multiplication of functions; in Λ_F, multiplication of real numbers.

We begin with $\mathscr{A}(S)$ where S is as in Definition 1. We have from Theorem 3 the correspondence $\alpha \leftrightarrow (\lambda, \tau, \bar{\xi})$ for $\alpha \in \mathscr{A}(S)$. It is immediate that this correspondence is one-to-one.

9. THEOREM. *Let S be as in Definition* 1. *The automorphism group of S is isomorphic to*

$$\mathscr{A}(G) \times_{g_2} (\Lambda_F \times_{g_1} \mathrm{Hom}\,(A, Z(G)))$$

where

$$g_1(\lambda)(\tau) = \tau \circ \lambda \qquad (\textit{of type ii})$$
$$g_2(\bar{\xi})(\lambda, \tau) = (\lambda, \bar{\xi} \circ \tau) \qquad (\textit{of type i}) .$$

Proof. Showing that the correspondence given by Theorem 3 is a homomorphism is only a matter of computing $\alpha \circ \bar{\alpha}$ where $\alpha, \bar{\alpha}$ are in $\mathscr{A}(S)$. The multiplication given by g_1 and g_2 is as follows:

$$(\bar{\xi}, (\lambda, \tau))(\bar{\bar{\xi}}, \bar{\lambda}, \bar{\tau})) = (\bar{\xi} \circ \bar{\bar{\xi}}, (\lambda\bar{\lambda}, (\tau \circ \bar{\lambda})(\bar{\xi} \circ \bar{\tau}))) .$$

Proceeding to the various forms of T discussed in §III, we have, in Case (a), $\mathscr{A}(T) = \{1_T\}$. In Case (b), T is really of the form of S so Theorem 9 applies. For Case (c) we have the following.

10. THEOREM. *Let T be as in Theorem 5. $\mathscr{A}(T)$ is isomorphic to $\mathscr{A}(G) \times_g \operatorname{Hom}(A, Z(G))$ where $g(\bar{\xi})(\tau) = \bar{\xi} \circ \tau$ (of type* i).

Proof. In this case T is almost like S. λ is forced to be 1. g here corresponds to g_2 in Theorem 9. $(\xi, \tau)(\bar{\xi}, \bar{\tau}) = (\xi \circ \bar{\xi}, \tau(\xi \circ \bar{\tau}))$.

For T described by Case (d), we construct a group isomorphic to the desired subgroup of $\operatorname{Hom}(\boldsymbol{H}, T)$. Let $H = \{h \in \operatorname{Hom}(\boldsymbol{H}, T): h$ is as in Theorem 6$\}$. H is a group under the following operation $*$. Let $h_i(p) = ([p], f(p), g_i G_{[p]})$. Define $h_1 * h_2$ by $h_1 * h_2(p) = ([p], f(p), g_1 g_2 G_{[p]})$. This group can be mapped isomorphically into $\prod_{p \in \boldsymbol{H}} (G/G_{[p]})$ and $\hat{h}$ is given by $h(p) = ([p], f(p), \hat{h}(p))$. Let $\mathscr{H}$ be the image of H in $\prod_{p \in \boldsymbol{H}} (G/G_{[p]})$. $\mathscr{H}$ is an abelian group under coordinate multiplication.

11. THEOREM. *Let T and $\mathscr{H}$ be as above. Let Ξ_F be the subgroup of $\mathscr{A}(G)$ satisfying Theorem 6, $(\xi(G_{[p]} = G_{[\lambda p]})$. Consider $\xi \in \Xi_F$ inducing a map called $\bar{\xi}: G/G_{[p]} \rightarrow G/G_{[\lambda p]}$ $\mathscr{A}(T)$ is isomorphic to $\Xi_F \times_g \mathscr{H}$ where $g(\xi)\hat{h} = \xi \circ \hat{h} \circ \lambda^{-1}$ (of type* i).

Proof. There are several things to check in this theorem. Again we will consider $r = \infty$ as in the proof of Theorem 6. $\bar{\xi}\hat{h}\lambda^{-1}: \boldsymbol{H} \rightarrow G/G_p$ since $\bar{\xi}$ is the induced map $G/G_{\lambda^{-1}p} \rightarrow G/G_p$.

From Theorem 6, we note if α is given

$$h(p) = \alpha(\lambda^{-1}p, f(\lambda^{-1}p), G_p)$$

and

$$\begin{aligned} \tau(f(p)) &= \pi_\infty(\alpha(p, f(p), G_p)(\infty, f(0), G_\infty)) \\ &= \pi_\infty((h(\lambda p))(\infty, f(0), G_\infty)) . \end{aligned}$$

If h is given $\alpha(p, f(p), G_p) = \lambda h(p) = h(\lambda p)$ and $\tau(f(p)) = \pi_\infty(h(\lambda p)(\infty, f(0), G_\infty))$. From this we see the correspondence between α and (ξ, h) is one-to-one and that the construction of τ does not depend on which representation is used.

The multiplication in $\Xi_F \times {}_g\mathscr{H}$ is

$$(\xi_1, \hat{h}_1)(\xi_2, \hat{h}_2) = (\xi_1 \circ \xi_2, (\hat{h}_1)(\bar{\xi}_1 \circ \hat{h}_2 \circ \lambda_1^{-1})) .$$

We note that $\hat{h}_1(\bar{\xi}_1 \hat{h}_2 \lambda_1^{-1})$ determines τ where $\tau = (\tau_1 \circ \lambda_2)(\xi_1 \circ \tau_2)$ which is exactly the product we expect to see in $\alpha_1 \circ \alpha_2$. From here it is immediate that the correspondence is an isomorphism.

In Case (e) we replace Ξ_F in Theorem 11 by $\Xi_0 \times \Lambda_F$ where $\xi \in \Xi_0$ if $\xi(G_\infty) = G_\infty$. The automorphism group of T is isomorphic to $(\Xi_0 \times \Lambda_F) \times {}_g\mathscr{H}$ where $g((\xi, \lambda))\hat{h} = \bar{\xi}\hat{h}\lambda^{-1}$ and g is of type i.

In Case (f) the isomorphism group of T is a subgroup of $\mathscr{A}(\hat{T})$.

V. Examples.

The following semigroups can be found in Chapter D of [3].

12. Example. Let $\boldsymbol{Z}$ be the integers under addition. Let $A = G = \hat{\alpha}/\boldsymbol{Z}$. Let $f: \boldsymbol{H} \to A$ be given by $f(p) = p + \boldsymbol{Z}$. Then

$$S = \{(p, p + \boldsymbol{Z}, q + \boldsymbol{Z}): p \in \boldsymbol{H}, q \in \boldsymbol{R}\} \cup \infty \times \boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z} .$$

$\mathscr{A}(S)$ is given by 9. Since f is not one-to-one $\Lambda_F = \{1\}$. $\mathscr{A}(\boldsymbol{R}/\boldsymbol{Z}) = \{-1, 1\}$ and $\operatorname{Hom}(R/\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}) = \boldsymbol{Z}$.

$\mathscr{A}(S) = \{-1, 1\} \times_{g_2} \boldsymbol{Z}$ and the multiplication is given by $(x, k)(y, n) = (xy, k + xn)$.

13. Example. Let S be as in 12. Let T be the homomorphic image of S obtained by letting $r = 1$ and not changing A or G. $\mathscr{A}(T)$ is given by 10 and $\mathscr{A}(T) = \mathscr{A}(S)$.

14. Example. Let S be as in 12. Let T be the homomorphic image of S obtained by letting $G_p = \boldsymbol{Z}$ for $p < \infty$ and $G_\infty = \boldsymbol{R}/\boldsymbol{Z}$. T is described in §II, Case (e). $\mathscr{A}(T)$ is given by Theorem 11 and the comment following it. This is a particularly simple example where $\Lambda_F = \{1\}$ and $\Xi_0 = \Xi = \mathscr{A}(G)$. $\mathscr{H} = \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{R}/\boldsymbol{Z}) = \boldsymbol{R}$. $\mathscr{H}$ must represent homomorphisms $h: \boldsymbol{H} \to T$. It does in this way: $h_r(p) = (p, p + \boldsymbol{Z}, rp + \boldsymbol{Z})$.

$\mathscr{A}(T) = \{-1, 1\} \times_g R$ where multiplication is given by $(x, r)(y, s) = (xy, r + xs)$.

15. Example. Let S be as in 12. Let T be the homomorphic image obtained from S by letting $K = \{(\infty, p + \boldsymbol{Z}, p + \boldsymbol{Z}): p \in \boldsymbol{R}\}$. The automorphisms of T are given by 7 and 8. They are a subgroup of $\mathscr{A}(S)$.

We examine $\mathscr{A}(S) = \{-1, 1\} \times_{g_2} \boldsymbol{Z}$ to see which automorphisms satisfy 7. Let $(x, k) \in \mathscr{A}(S)$. $\pi_A(K) = \boldsymbol{R}/\boldsymbol{Z}$ and $\beta(p + \boldsymbol{Z}) = p + \boldsymbol{Z}$. k is the homomorphism called τ in 7 and $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$. We have $k(p + \boldsymbol{Z}) = kp + \boldsymbol{Z} = p + \boldsymbol{Z} - xp + \boldsymbol{Z}$. If $x = 1, kp + \boldsymbol{Z} = Z$; if $x = -1, kp + \boldsymbol{Z} = 2p + \boldsymbol{Z}$. $\mathscr{A}(T) = \{(1, 0), (-1, 2)\}$ considered as a subgroup of $\mathscr{A}(S)$.

References

1. Bourbaki, *General Topology*, Addison Wesley, Pub. Co., Reading Mass.
2. T. Bowman, *On the Automorphism Groups of Some Types of Compact Semigroups*, Ph. D. Thesis, Tulane University, 1970.
3. K. H. Hofmann, and P. S. Mostert, *Elements of Compact Semigroups*, Charles E. Merrill Books, Inc., Columbus, Ohio. 1966.
4. M. W. Mislove, *The Existence of Irr(X), Semigroup Forum*, **1** (1970), 243.

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