Pacific Journal of Mathematics

AUTOMORPHISMS ON CYLINDRICAL SEMIGROUPS

JOAN GERAMITA

Vol. 43, No. 1 March 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 1, 1972

AUTOMORPHISMS ON CYLINDRICAL SEMIGROUPS

JOAN MURPHY GERAMITA

This paper characterizes the automorphisms of a cylindrical semigroup S in terms of the automorphisms of the defining subgroups and subsemigroups. The following theorem is representative of the type of information given in this paper.

Let $F: \boldsymbol{R} \rightarrow A$ be a dense homomorphism of the additive real numbers to the compact abelian group A. Let λ be a positive real number. Multiplication by λ shall also denote the automorphism of A whose restriction to $F(\boldsymbol{R})$ is given by $F\lambda F^{-1}$. The set of all such λ for a given F is called Λ_F.

Theorem. Let f and λ be as above. Let G be a compact group. Let

$$S = \{(p, f(p)\, g): p \in \boldsymbol{H} \text{ and } g \in G\} \cup \alpha \times A \times G\ .$$

Then $\alpha: S \rightarrow S$ is an automorphism if and only if $\alpha(p, f(p), g) = (\lambda p, f(\lambda p), \tau(f(p))\xi(g))$; $\alpha(\infty, a, g) = (\infty, \lambda a, \tau(a)\xi(g))$, where $\tau: A \rightarrow G$ is a homomorphism into the centre of G and, $\xi: G \rightarrow G$ is an automorphism. Theorem. Let S be as in theorem above. Let $\mathscr{A}(G)$ be the automorphism group of G, and $Z(G)$, the center of G. The automorphism group of S is isomorphic as an abstract group to $\mathscr{A}(G) \times (\Lambda_F \times \mathrm{Hom}(A, Z(G)))$ with the following multiplication

$$(\xi, (\lambda, \tau))(\bar{\xi}, (\bar{\lambda}, \bar{\tau})) = (\xi \circ \bar{\xi}, (\lambda\bar{\lambda}, (\tau \circ \bar{\lambda})(\xi \circ \bar{\tau})))\ .$$

Cylindrical semigroups play an important role Mislove's description of $Irr(X)$ and are the building blocks used in the construction of a hormos. Hofmann and Mostert [3] have shown that every compact irreducible semigroup is a hormos. The definition and description of a cylindrical semigroup, given in § I, is from their book.

I. Definitions and notation. All spaces are Hausdorff. All homomorphisms are continuous unless otherwise stated. A homomorphism will be called abstract if it is not assumed continuous. A group considered with the discrete topology will be called abstract. A topological semigroup is a topological space, S, together with a continuous associative multiplication $m: S \times S \rightarrow S$; $m(s, t) = st$. All semigroups are topological with identity 1. A topological group is a semigroup with the map $\phi: S \rightarrow S$, $\phi(s) = s^{-1}$, continuous also. An *ideal*, I, in a semigroup, S, is a subset of S such that: if $x \in S$ then $(xI \cup Ix) \subset I$. If S is compact and abelian then S has an ideal $M(S)$ which is minimal with respect to set inclusion, is unique, and is a group. An *idempotent* $x \in S$ has the property $x^2 = x$. The maximal subgroup of S containing an idempotent e is called the *group of units of* e and denoted $H(e)$. The group of units of 1 is also denoted $H(S)$ and called *the group of units of* S. If $\alpha: S \to S$ is an automorphism then $\alpha(H(S)) = H(S)$ and $\alpha(M(S)) = M(S)$.

NOTATION. The following notation is standard throughout the paper.

$[a, b]$—In a totally ordered set, the closed interval from a to b.

$]a, b[$—The open interval from a to b.

$\boldsymbol{H}$—The semigroup of nonnegative real numbers under addition with the usual topology.

$\boldsymbol{H}^*$—The one point compactification of $\boldsymbol{H}$, written $[0, \infty]$.

$\boldsymbol{H}_r^*$—$\boldsymbol{H}^*/[r, \infty]$.

Λ—The abstract group of positive real numbers under multiplication.

$\boldsymbol{R}$—The group of real numbers under addition with the usual topology.

$Z(G)$—The center of a group G.

$[p]$—The image of p under the quotient map $\boldsymbol{H}^* \to \boldsymbol{H}_r^*$.

$*$—As in B^*, the closure of $B \subset X$, except as noted above for $\boldsymbol{H}$.

$X \backslash A$—For $A \subset X$, the complement of A in X.

1. DEFINITION. Let A and G be compact groups. Let A be an abelian and $f: \boldsymbol{H} \to A$ a homomorphism such that $f(\boldsymbol{H})^* = A$. Consider $\boldsymbol{H}^* \times A \times G$ with coordinate-wise multiplication, and let S be that subsemigroup defined by:

$$S = \{(p, f(p), g): p \in \boldsymbol{H}, g \in G\} \cup \infty \times A \times G .$$

Any homomorphic image of S is called a cylindrical semigroup.

The following theorem which describes cylindrical semigroups is from [3, p. 85, Prop. 2.2].

THEOREM A (Hofmann and Mostert). *Let S be a cylindrical semigroup as defined above. Let e be the identity of G and*

$$S' = \{(p, f(p), e): p \in \boldsymbol{H}\} \cup \infty \times A \times e .$$

Let $\phi: \to T$ be a surmorphism onto a compact semigroup T. Then there are:

(i) *compact semigroups T_1, T_1', X and a compact group B,*

(ii) *surmorphisms $h_1, h_2, h_3, h_4, \phi_1, \phi_2$*

(iii) *monomorphisms i_1, i_2*

such that the following diagram commutes:

$$\begin{tikzcd}
H^* \arrow[rr, "h_1"] & & H_r^* \arrow[r, "id"] & H_r^* \\
H^* \times A \times G \arrow[u, "\pi"] \arrow[rr, "h_1 \times h_2 \times id"] & & H_r^* \times B \times G \arrow[u, "\pi'"] \arrow[r, "h_3"] & X \arrow[u, "h_4"] \\
S \arrow[u, "\cup\!\!|"] \arrow[rr, "\phi_1"] & & T_1 \arrow[u, "i_1"] \arrow[r, "\phi_2"] & T \arrow[u, "i_2"] \\
S' \arrow[u, "\cup\!\!|"] \arrow[rr] & & T_1' \arrow[u, "\cup\!\!|"] \arrow[r] & \phi(S') \arrow[u, "\cup\!\!|"]
\end{tikzcd}$$

(π, π' are projections; $\phi_2 \circ \phi_1 = \phi$).

Moreover, $h_3|_{H^* \times B \times e}$ *is a monomorphism and* $h_4 \circ i_2$ *is a surmorphism.*

From this theorem it is possible to describe T in terms of equivalence classes of elements in $H_r^* \times B \times G$.

$f(0)$ is the identity of A. r, if it exists, is the least real number such that $\phi(r, f(r), e) = \phi(\infty, a, g)$ for some $a \in A$, $g \in G$.

$$B = \phi(\infty \times A \times e). \quad T_1' = \phi(S') \times e .$$

Let $\bar{f}\colon H \to B$ be given by $\bar{f}(p) = \phi(\infty, f(p), e)$ then

$$i_1(T_1') = \{([p], f(p), e)\colon p \in H\} \cup [r] \times B \times e .$$

If there is no such r, then $i_1(T_1') \subset H^* \times B \times G$.

Let

$$G_{[p]} = \{g \in G\colon \phi(p, f(p), g) = \phi(p, f(p), e)\}$$

and

$$G_{[r]} = \{g \in G\colon \phi([r], f(0), g) = \phi([r], f(0), e)\}$$

where $r \leqq \infty$. $\{G_{[p]}\colon p \in H^*\}$ has the following two properties:

(1) $$G_{[p]} \subseteqq G_{[q]} \quad \text{for} \quad p \leqq q ;$$

(2) $$G_{[p]} = \bigcap_{q>p} G_{[q]} .$$

Each $G_{[p]}$ is a normal subgroup of G. Denote $G/G_{[r]}$ by $\bar{G}$ and assume $G_{[0]} = \{e\}$.

$$i_2\phi(\{(p, f(p), g)\colon p \in H, g \in G\}) = \{([p], f(p), gG_{[p]})\colon p \in H, g \in G\}$$

where

$$(gG_{[p]})(\bar{g}G_{[\bar{p}]}) = g\bar{g}G_{[p+\bar{p}]} .$$

$i_2\phi(\infty \times A \times G) = ([r] \times B \times G)/K$ where K is a normal subgroup of

$[r] \times B \times G$. K has the property: if $([r], b, g) \in K$ and $([r], \bar{b}, \bar{g}) \in K$ then $b = \bar{b}$ if and only if $g = \bar{g}$.

We shall identify T with its image $i_2(T)$ and refer to $i_1(T_1')$ as T'. Since B is a compact abelian group and $\bar{f}: \boldsymbol{H} \to B$ is onto a dense subset of B, we may as well consider them as f and A to avoid extra notation. We say

$$T = \{([p], f(p), gG_{[p]}): p \in \boldsymbol{H}, g \in G\} \cup ([r] \times B \times G)/K .$$

II. Automorphisms on semigroups of the form of S. We first consider automorphisms of the cylindrical semigroup S given in Definition 1. $M(S)$, the minimal ideal of S, is $\infty \times A \times G$. $H(S)$, the group of units, is $\{(0, f(0), g): g \in G\}$. From Theorem A we have that an automorphism $\alpha: S \to S$ can be thought of as an automorphism on $S' \times H(S)$.

Consider the situation where $G = \{e\}$. We have $S = S'$, $M(S') = \infty \times A \times e$ and $S' \backslash M(S')$ is isomorphic to $\boldsymbol{H}$ by $(p, f(p), e) \leftrightarrow p$. For an automorphism $\alpha: S' \to S'$, $\alpha(M(S')) = M(S')$; and, α restricted to $S' \backslash M(S')$ corresponds to an automorphism of $\boldsymbol{H}$. Since the only automorphisms of $\boldsymbol{H}$ are multiplication by a positive real number λ, we have $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$.

How shall α behave on $M(S')$? Let $\boldsymbol{R}$ be the additive group of real numbers, then $f: \boldsymbol{H} \to A$ can be extended to $F: \boldsymbol{R} \to A$ (for $x \notin \boldsymbol{H}$, $F(x) = f(-x)^{-1}$) and $F(\boldsymbol{R})$ will be dense in A. Let $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$. Then:

$$\begin{aligned} \alpha(\infty, f(p), e) &= \alpha((p, f(p), e)(\infty, f(0), e)) \\ &= \alpha(p, f(p), e)\alpha(\infty, f(0), e) \\ &= (\lambda p, f(\lambda p), e)(\infty, f(0), e) \\ &= (\infty, f(\lambda p), e) . \end{aligned}$$

Define $\bar{\lambda}: F(\boldsymbol{R}) \to F(\boldsymbol{R})$ by $\bar{\lambda}(F(x)) = F(\lambda x)$. $\alpha|_{M(S')}: M(S') \to M(S')$ must be an extension of $\bar{\lambda}$. This extension will be called λ.

Any homomorphism between dense subgroups of compact groups can be extended to a unique homomorphism between the groups. If original map is an automorphism then the extension is also. The existence and uniqueness of the extension, as a function, follow from the fact that the subgroups are uniform spaces and the groups are completions of them [1]. That the extension is a homomorphism is an easy consequence of the definition of the extension.

2. LEMMA. *Let $S' = \{(p, f(p), e): p \in \boldsymbol{H}\} \cup \infty \times A \times e$. If f is neither one-to-one nor constant then the only automorphism of S' is*

the identity. Otherwise, $\alpha: S' \to S'$ *is an automorphism iff* $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$, $\alpha(\infty, a, e) = (\infty, \lambda a, e)$ *where* $F\lambda F^{-1}$ *is open and continuous or* F *is constant.*

Proof. If $\alpha: S' \to S'$ is an automorphism the discussion above shows that $\alpha(p, f(p), e) = (\lambda p, f(\lambda p), e)$ and $\alpha(\infty, a, e) = (\infty, \lambda a, e)$. If f is constant then $A = \{e\}$; S' is isomorphic to $\boldsymbol{H}^*$; and multiplication by any λ is an automorphism.

Suppose f is not constant. Consider the map $\bar{\lambda}: F(\boldsymbol{R}) \to F(\boldsymbol{R})$ given by $\bar{\lambda}(F(x)) = F(\lambda x)$. If F is not one-to-one then the kernel of F in $\boldsymbol{R}$ is cyclic and $\lambda: \boldsymbol{R} \to \boldsymbol{R}$ must preserve this kernel. This implies λ is an integer. Since λ^{-1} must also be an integer, we have $\lambda = 1$.

If F is one-to-one then $\bar{\lambda}$ is an automorphism of the abstract group $F(\boldsymbol{R})$. To be an automorphism of $F(\boldsymbol{R})$ with the induced topology from A, $\bar{\lambda}(= F\lambda F^{-1})$ must be open and continuous. The remark immediately preceding this lemma guarantees that $\bar{\lambda}$ can be extended to A when it is open and continuous.

Let $\Lambda_F = \{\lambda \in \Lambda: F\lambda F^{-1}$ is open and continuous$\}$.

When $G \neq \{e\}$ we have $\alpha: S' \times H(S) \to S' \times H(S)$ where $H(S)$ is isomorphic to G and $M(S) = \infty \times A \times G$. Since $\alpha(H(S)) = H(S)$, $\alpha(0, f(0), g) = (0, f(0), \xi(g))$ for some automorphism $\xi: G \to G$. Hence, the only possibility for $\alpha(\infty, f(0), g) = (\infty, a, h)$ is when $a = f(0)$. α restricted to $M(S)$ must therefore have the form $\alpha(\infty, a, g) = (\infty, \lambda a, \tau(a)\xi(g))$ with $\lambda \in \Lambda$, ξ as above and $\tau: A \to Z(G)$ (center of G), a homomorphism. τ must be continuous since $\tau = \pi_G \circ \alpha \circ i$ where π_G is the projection onto G, and i is the map $A \to \infty \times A \times G$ given by $i(a) = (\infty, a, e)$. Similarly τ must be a homomorphism. Since elements in $\infty \times A \times e$ commute with elements of $\infty \times f(0) \times G$, τ maps A into $Z(G)$.

3. THEOREM. *Let* S *be as in Definition* 1. $\alpha: S \to S$ *is an automorphism iff* $\alpha(x, a, g) = (\lambda x, \lambda a, \tau(a)\xi(g))$ *where* $\lambda \in \Lambda_F$; $\tau: A \to Z(G)$ *is a homomorphism and* $\xi: G \to G$ *is an automorphism.*

Proof. The above discussion establishes the only if part. Let λ, τ, ξ be given as described in the theorem. $\hat{\alpha}: \boldsymbol{H}^* \times A \times G \to \boldsymbol{H}^* \times A \times G$ can be defined by $\hat{\alpha}(x, a, g) = (\lambda x, \lambda a, \tau(a)\xi(g))$. It is immediate that $\hat{\alpha}$ is an abstract automorphism. Since $\boldsymbol{H}^* \times A \times G$ is compact, we need only that $\hat{\alpha}$ is continuous. Let $U \times V \times W$ be a basis open set. $\hat{\alpha}^{-1}(U \times V \times W) = \lambda^{-1}U \times \lambda^{-1}V \times \xi^{-1}(\tau(\lambda^{-1}V)^{-1})\xi^{-1}(W)$. Since λ and ξ are continuous, $\lambda^{-1}U, \lambda^{-1}V$ and $\xi^{-1}(W)$ are open. Since G is a topological group, for any set X, $X\xi^{-1}(W)$ is open. Hence $\hat{\alpha}^{-1}(U \times V \times W)$ is open. Let $\alpha = \hat{\alpha}|_S$.

III. Automorphisms on semigroups of the form of T. Recall $T = \{([p], f(p), gG_{[p]}) \colon p \in \boldsymbol{H}, g \in G\} \cup ([r] \times A \times \bar{G})/_K$. It is easier to keep track of the situation by considering cases determined by r, G, and K.

Case (a). Let $r < \infty$ and $G = \{e\}$. Then $K = \{([r], f(0), e)\}$.

4. LEMMA. *Let T be given by Case* (a). *The only automorphism on T is the identity.*

Proof. Let α be an automorphism of T.

Suppose $p < r$. $\alpha([p], f(p), e) = ([q], f(q), e)$ for some $q < r$ since $\alpha(M(T)) = M(T)$. First, let us take the case where $p = r/n$ for some integer n. If $p < q$ then there exists $p' < p$ such that $\alpha([p'], f(p'), e) = ([p], f(p), e)$ and $\alpha([np'], f(np'), e) = ([np], f(np), e) = ([r], f(r), e) \in M(T)$. But $np' < r$ since $np = r$ and $p' < p$. This means $\alpha([np'], f(np'), e) \notin M(T)$. We have a contradiction; so $p \geqq q$. If we assume $p > q$, a similar contradiction arises from $nq < r$. So, if $p < r$ and $p = r/n$ then $\alpha([p], f(p), e) = ([p], f(p), e)$.

For $p < r$, if $p \neq r/n$ then there exists a sequence, possibly finite, of integers $\{n_i\}$ such that $p = \sum r/n_i$. α is continuous so, again, $\alpha([p], f(p), e) = ([p], f(p), e)$.

$$\begin{aligned} \alpha([r], f(r), e) &= \lim_{\bar{p}<r} \alpha([\bar{p}], f(\bar{p}), e) \\ &= \lim_{\bar{p}<r} ([\bar{p}], f(\bar{p}), e) = ([r], f(r), e) . \end{aligned}$$

For $p > r$, $p = nr + p'$ where $p' < r$. We have:

$$\begin{aligned} \alpha([p], f(p), e) &= \alpha([nr], f(nr), e)\alpha([p'], f(p'), e) \\ &= (\alpha([r], f(r), e))^n([p'], f(p'), e) \\ &= ([r], f(r), e)^n([p'], f(p'), e) = ([p], f(p), e) . \end{aligned}$$

So α is the identity map.

Case (b). Let $r = \infty$, $G_p = G_\infty$ for all p and $K = \{(\infty, f(0), e)\}$.

In this case, T is of the form of S where $\bar{G} = G/G_\infty$.

Case (c). Let $r < \infty$, $G_{[p]} = G_{[r]}$ for all p and $K = \{([r], f(0), e)\}$. Let $G/G_{[r]} = \bar{G}$.

5. THEOREM. *Let T be as in Case* (c). *$\alpha\colon T \to T$ is an automorphism iff $\alpha(x, a, g) = (x, a, \tau(a)\xi(g))$ where $\tau\colon A \to Z(\bar{G})$ is a homomor-*

phism and $\xi\colon \bar{G} \to \bar{G}$ *is an automorphism.*

Proof. From Lemma 4 we have $\lambda = 1$ and the precise arguments in the proof of Theorem 3 concerning τ and ξ hold here.

Case (d). Let $r \leqq \infty$, $G_{[p]} \neq G_{[q]}$ for $[p] \neq [q]$ and $K = \{([r], f(0), e)\}$.

In this case, the description becomes more complicated but is in fact, no more difficult to prove. The previous cases allowed $\tau\colon A \to Z(\bar{G})$ to be defined in $M(T)$ and then used in $T\backslash M(T)$. Here, since $\bar{G} = G/G_{[r]} \neq G/G_{[p]}$ for $[p] \neq [r]$, it is not possible to start by taking τ defined in $M(T)$ to be any homomorphism in $\operatorname{Hom}(A, Z(\bar{G}))$. Rather, we start with a homomorphism $h\colon \boldsymbol{H} \to T\backslash M(T)$ which must also determine a homomorphism $f(\boldsymbol{H}) \to Z(\bar{G})$. The latter homomorphism can then be extended to define τ. Without loss of generality, we may assume $G_{[0]} = \{e\}$.

6. THEOREM. *Let T be as described for Case* (d). *Let* $\xi\colon G \to G$ *be an automorphism. If* $r < \infty$, *let* $\xi(G_{[p]}) = G_{[p]}$ *for all* $p \in \boldsymbol{H}$. *If* $r = \infty$, *let there exist a* $\lambda \in \Lambda_F$ *such that* $\xi(G_{[p]}) = G_{[\lambda p]}$ *for all* $p \in \boldsymbol{H}$.

Let $h\colon \boldsymbol{H} \to T$ *be a homomorphism such that* $h(p) = ([p], f(p), gG_{[p]})$ *and*

$$\{h(p)([r], f(0), G_{[r]})\} \subseteqq [r] \times A \times (G/G_{[r]})$$

represents the graph of a homomorphism $f(\boldsymbol{H}) \to Z(G/G_{[r]})$.

$\alpha\colon T \to T$ *is an automorphism iff* $\alpha([p], f(p), gG_{[p]}) = h(\lambda p)(0, f(0), \xi(g))$, *and* $\alpha([r], a, gG_{[r]}) = ([r], \lambda a, \tau(a)\xi(g)G_{[r]})$ *where* $\tau\colon A \to Z(G/G_{[r]})$ *is a homomorphism.*

Proof. Let us assume $r = \infty$. The proof for $r < \infty$ follows this one replacing λ by 1 and p by $[p]$. Let α be given.

Define $\xi\colon G \to G$ in the usual way by considering $\alpha|_{H(T)}$. It is still the case that $(p, f(p), G_p) \to (\lambda p, f(\lambda p), gG_p)$. This follows directly from the top level of the diagram in Theorem A. One can show that $\xi(G_p) = G_{\lambda p}$ by considering $(p, f(p), G_p)$ written as $(p, f(p), gG_p)$ for $g \in G_p$. $\lambda \in \Lambda_F$ since once again λ must be extended to an automorphism of A in $M(T)$ (see Theorem 3).

Define

$$h\colon \boldsymbol{H} \to T \text{ by } h(p) = \alpha(\lambda^{-1}p, f(\lambda^{-1}p), G_{\lambda^{-1}p}) .$$

h is the composition of three homomorphisms

$$\boldsymbol{H} \xrightarrow{\lambda^{-1}} \boldsymbol{H} \xrightarrow{\hat{f}} T \xrightarrow{\alpha} T \quad \text{where} \quad \hat{f}(p) = (p, f(p), G_p) .$$

Define $\lambda h(p) = h(\lambda p)$. λh is also a homomorphism but not of the type specified by the theorem.

Define $\tau: A \to Z(G/G_\infty)$, as was done in Theorem 3, by considering $\alpha|_{\infty \times A \times e}$.

Note:

$$h(p)(\infty, f(0), G_\infty) = \alpha(\infty, f(\lambda^{-1}p), G_\infty)$$
$$= (\infty, f(p), gG_\infty) = (\infty, f(p), \tau(f(p)))\,.$$

So $\{h(p)(\infty, f(0), G_\infty)\}$ represents the graph of a homomorphism from $f(\boldsymbol{H}) \to Z(G/G_\infty)$. We shall sometimes write $\tau(a)$ as $\tau(a)G_\infty$. We observe that $\{h(p)(\infty, f(0), G_\infty)\} = \{\lambda h(p)(\infty, f(0), G_\infty)\}$, so h and λh can be made to determine the same τ.

For the converse let ξ, and h be given. ξ determines $\lambda \in \Lambda_F$. λh determines the graph of a homomorphism since h does. Define $\tau(f(p)) = \pi_\infty(h(\lambda p)(\infty, f(0), G_\infty))$ where π_∞ is the projection. τ can be extended in the usual way to A.

Define $\alpha: T \to T$ by

$$\alpha(p, f(p), gG_p) = \lambda h(p)(0, f(0), \xi(g))$$
$$\alpha(\infty, a, gG_\infty) = (\infty, \lambda a, \tau(a)\xi(g))\,.$$

Showing α is an abstract homomorphism is straightforward. One can prove α is continuous by writing T as the image of S and considering open sets. This proof is omitted because it is uninteresting and requires complicated notation.

Case (e). Let $r = \infty$, $G_p = G_q \neq G_\infty$ and $K = \{(\infty, f(0), e)\}$.

This situation is a simple version of Case (d). Since $G_p = G_{\lambda p}$ for all λ, we no longer have λ determined by $\xi: G \to G$. Any choice of $\lambda \in \Lambda_F$ will give an automorphism.

Case (f). Let $K \neq \{([r], f(0), e)\}$ and $K \neq [r] \times A \times \bar{G}$. Let $\hat{T} = \{([p], f(p), gG_{[p]}) \colon p \in \boldsymbol{H}, g \in G\} \cup [r] \times A \times \bar{G}$ and let $T = \{([p], f(p), gG_{[p]}\} \cup ([r] \times A \times \bar{G})/K$. Let $k: \hat{T} \to T$ be the map which is the identity on $\hat{T} \backslash M(\hat{T})$ and the quotient map on $M(\hat{T})$. Recall: if $([r], a, g) \in K$ and $([r], \bar{a}, \bar{g}) \in K$ then $a = \bar{a}$ iff $g = \bar{g}$. When $r < \infty$, if $k(t_\gamma)$ is a convergent net in T such that $k(t_\gamma) \notin M(T)$ and $\lim k(t_\gamma) \in M(T)$, then $t\gamma$ is a convergent net in $\hat{T}$. Let $\pi_A(K) = \{a \in A \colon ([r], a, g) \in K$ for some $g \in \bar{G}\}$. Let β be the abstract isomorphism $\beta: \pi_A(K) \to \bar{G}$ given by $g = \beta(a)$ if $([r], a, g) \in K$.

7. Lemma. *Let T and $\hat{T}$ be as above. Let $\hat{\alpha}: \hat{T} \to \hat{T}$ be charac-*

terized by (λ, τ, ξ) *or by* (λ, h, ξ) *as given in* 3, 5, 6. *Let* $\pi_A(K)$ *and* β *be as above. There exists an automorphism* $\alpha\colon T \to T$ *such that* $\alpha k = k\hat{\alpha}$ *iff* $\lambda|_{\pi_A(K)}$ *is an automorphism and* $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$ *for* $a \in \pi_A(K)$.

Proof. Suppose $\hat{\alpha}$ induces an automorphism α such that $\alpha k = k\hat{\alpha}$. Consider $\hat{\alpha}|_{M(\hat{T})}$ as an automorphism on the group $M(\hat{T})$. This induces $\alpha|_{M(T)}$ on $M(T)$ and for $\alpha|_{M(T)}$ to be well defined and one-to-one we must have $\hat{\alpha}(K) = K$. For $([r], a, \beta(a)) \in K$ we have $\hat{\alpha}([r], a, \beta(a)) = ([r], \lambda a, \tau(a)\xi(\beta(a))) \in K$. Hence, $\lambda a \in \pi_A(K)$ and $\beta(\lambda a) = \tau(a)\xi(\beta(a))$. Since $\hat{\alpha}^{-1}$ is also an automorphism $\lambda^{-1}a \in \pi_A(K)$ and λ is onto. $\beta(\lambda a) = \tau(a)\xi(\beta(a))$ implies $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$.

The proof of the converse is straightforward. It is convenient to consider the continuity of α on $T\backslash M(T)$ and $M(T)$ separately and then consider a net converging to $M(T)$.

8. THEOREM. *Let* $\hat{T}$, T *and* k *be as in Lemma* 7. $\alpha\colon T \to T$ *is an automorphism iff there exists an automorphism* $\hat{\alpha}\colon \hat{T} \to \hat{T}$ *such that* $\alpha k = k\hat{\alpha}$.

Proof. Let $\alpha : T \to T$ be an automorphism. We consider two cases: $r < \infty$ and $r = \infty$. Let $r < \infty$. We know from Theorems 5 and 6 that $\hat{\alpha}$ is determined by (ξ, h) or (ξ, τ). Constructing h is the more general situation. An argument similar to that of Theorem 4 establishes that

$$\alpha k([p], f(p), G_{[p]}) = k([p], f(p), \bar{g}G_{[p]}) .$$

Let $G_{[0]} = \{e\}$ and $\bar{G} = G/G_{[r]}$.

Define $\xi\colon G \to G$ by $\xi(g) = \pi_G \alpha k([0], 1, g)$. Clearly ξ is an automorphism.

Define $h\colon \boldsymbol{H} \to \hat{T}$ by:

$$\begin{aligned} h(p) &= k^{-1}\alpha k([p], f(p), G_{[p]}) \quad \text{when } p < r ; \\ h(r) &= \lim_{p<r} h(p) ; \\ h(p) &= (h(r))^n h(q) \quad \text{when} \quad p = nr + q, q < r . \end{aligned}$$

It is immediate that h is a homomorphism. Since $\alpha k([p], f(p), G_{[p]}) = k([p], f(p), gG_{[p]})$, we have also $\alpha k([r], a, G_{[r]}) = k([r], a, gG_{[r]})$.

Define $\tau\colon A \to Z(\bar{G})$ by $\tau(a) = gG_{[r]}$ such that $\alpha k([r], a, G_{[r]}) = k([r], a, gG_{[r]})$. τ is well-defined since if $([r], a, y) \in ([r], a, g)K$ then $([r], f(0), yg^{-1}) \in K$ and $y = g$. It is also immediate that τ is an abstract homomorphism. $\tau(f(p)) = \pi_{\bar{G}}(h(p)([r], f(0), e))$ so τ is continuous on $f(\boldsymbol{H})$ and hence on A. Even if $\hat{\alpha}$ is more efficiently given by (ξ, τ), h

can be defined and the above will show τ continuous.

Define $\hat{\alpha}$: $\hat{T} \to \hat{T}$ by (ξ, h) or (ξ, τ).

$$\alpha k([p], a, gG_{[p]}) = k([p], a, \tau(a)\xi(g)G_{[p]}) = k\hat{\alpha}([p], a, gG_{[p]}) .$$

So $\alpha k = k\hat{\alpha}$.

Now, let $r = \infty$ and $\bar{G} = G/G_\infty$. Define ξ as before. Either ξ determines λ (as in 6); or, define λ by checking $\alpha k(p, f(p), G_p)$. If f is not one-to-one then, $\lambda = 1$ or $A = \{1\}$. If f is one-to-one then λ is one-to-one on $f(\boldsymbol{H}) \subset A$ and can be extended to λ continuous on A. Since α^{-1} is also an automorphism the above process can be done for λ^{-1} which means λ is open on A and hence $\lambda \in \Lambda_F$.

Define h: $\boldsymbol{H} \to \hat{T}$ by $h(p) = k^{-1}\alpha k(\lambda^{-1}p, f(\lambda^{-1}p), G_{\lambda^{-1}p})$. h is a homomorphism since k is an isomorphism.

Define $\tau(f(p)) = \pi_{\bar{G}}(h(p)(\infty, f(0), G_\infty))$. τ is continuous since h and $\pi_{\bar{G}}$ are, and can be extended to A.

We define $\hat{\alpha}$: $\hat{T} \to \hat{T}$ by (λ, ξ, h) or (λ, ξ, τ). Again, $\alpha k = k\hat{\alpha}$.

So, for each case, $\hat{\alpha}$, an automorphism of $\hat{T}$ inducing $\hat{\alpha}$, can be constructed.

IV. **Automorphism groups.** This section describes the group structure of the groups of automorphisms given in II and III. All groups discussed here are discrete. Bowman [2] has described the topology of these groups. Since in each case the group is described as a semidirect product of groups of homomorphisms; we give the definition of semidirect product below.

Let A and B be two groups. Let $g: A \to \mathscr{A}(B)$, the group of automorphisms of B, be a function such that:

(i) $g(a_2)(g(a_1)b) = g(a_2a_1)(b)$;

or

(ii) $g(a_2)(g(a_1)b) = g(a_1a_2)(b)$.

$A \times B$ is a group with the following multiplication: $(a, b)(\bar{a}, \bar{b}) = (a\bar{a}, b(g(a)\bar{b}))$ when g is of type i; $(a, b)(\bar{a}, \bar{b}) = (a\bar{a}, (g(\bar{a})b)\bar{b})$ when g is of type ii. The semidirect product will be denoted $A \times_g B$.

Recall, the operation in $\mathscr{A}(G)$ is composition of functions; in Hom $(A, Z(G))$, multiplication of functions; in Λ_F, multiplication of real numbers.

We begin with $\mathscr{A}(S)$ where S is as in Definition 1. We have from Theorem 3 the correspondence $\alpha \leftrightarrow (\lambda, \tau, \bar{\xi})$ for $\alpha \in \mathscr{A}(S)$. It is immediate that this correspondence is one-to-one.

9. THEOREM. *Let S be as in Definition* 1. *The automorphism group of S is isomorphic to*

$$\mathscr{A}(G) \times_{g_2} (\Lambda_F \times_{g_1} \mathrm{Hom}\,(A, Z(G)))$$

where

$$g_1(\lambda)(\tau) = \tau \circ \lambda \qquad (\textit{of type } \mathrm{ii})$$
$$g_2(\bar{\xi})(\lambda, \tau) = (\lambda, \bar{\xi} \circ \tau) \qquad (\textit{of type } \mathrm{i}) .$$

Proof. Showing that the correspondence given by Theorem 3 is a homomorphism is only a matter of computing $\alpha \circ \bar{\alpha}$ where $\alpha, \bar{\alpha}$ are in $\mathscr{A}(S)$. The multiplication given by g_1 and g_2 is as follows:

$$(\bar{\xi}, (\lambda, \tau))(\bar{\xi}, \bar{\lambda}, \bar{\tau})) = (\bar{\xi} \circ \bar{\xi}, (\lambda\bar{\lambda}, (\tau \circ \bar{\lambda})(\bar{\xi} \circ \bar{\tau}))) .$$

Proceeding to the various forms of T discussed in §III, we have, in Case (a), $\mathscr{A}(T) = \{1_T\}$. In Case (b), T is really of the form of S so Theorem 9 applies. For Case (c) we have the following.

10. THEOREM. *Let T be as in Theorem* 5. *$\mathscr{A}(T)$ is isomorphic to $\mathscr{A}(G) \times_g \operatorname{Hom}(A, Z(G))$ where $g(\bar{\xi})(\tau) = \bar{\xi} \circ \tau$ (of type* i).

Proof. In this case T is almost like S. λ is forced to be 1. g here corresponds to g_2 in Theorem 9. $(\xi, \tau)(\bar{\xi}, \bar{\tau}) = (\xi \circ \bar{\xi}, \tau(\xi \circ \bar{\tau}))$.

For T described by Case (d), we construct a group isomorphic to the desired subgroup of $\operatorname{Hom}(\boldsymbol{H}, T)$. Let $H = \{h \in \operatorname{Hom}(\boldsymbol{H}, T)\colon h$ is as in Theorem 6$\}$. H is a group under the following operation$*$. Let $h_i(p) = ([p], f(p), g_iG_{[p]})$. Define $h_1 * h_2$ by $h_1 * h_2(p) = ([p], f(p), g_1g_2G_{[p]})$. This group can be mapped isomorphically into $\prod_{p \in \boldsymbol{H}} (G/G_{[p]})$ and $\hat{h}$ is given by $h(p) = ([p], f(p), \hat{h}(p))$. Let $\mathscr{H}$ be the image of H in $\prod_{p \in \boldsymbol{H}} (G/G_{[p]})$. $\mathscr{H}$ is an abelian group under coordinate multiplication.

11. THEOREM. *Let T and $\mathscr{H}$ be as above. Let Ξ_F be the subgroup of $\mathscr{A}(G)$ satisfying Theorem* 6, *$(\xi(G_{[p]} = G_{[\lambda p]})$. Consider $\xi \in \Xi_F$ inducing a map called $\bar{\xi}\colon G/G_{[p]} \to G/G_{[\lambda p]}$ $\mathscr{A}(T)$ is isomorphic to $\Xi_F \times_g \mathscr{H}$ where $g(\xi)\hat{h} = \bar{\xi} \circ \hat{h} \circ \lambda^{-1}$ (of type* i).

Proof. There are several things to check in this theorem. Again we will consider $r = \infty$ as in the proof of Theorem 6. $\bar{\xi}\hat{h}\lambda^{-1}\colon \boldsymbol{H} \to G/G_p$ since $\bar{\xi}$ is the induced map $G/G_{\lambda^{-1}p} \to G/G_p$.

From Theorem 6, we note if α is given

$$h(p) = \alpha(\lambda^{-1}p, f(\lambda^{-1}p), G_p)$$

and

$$\begin{aligned} \tau(f(p)) &= \pi_\infty(\alpha(p, f(p), G_p)(\infty, f(0), G_\infty)) \\ &= \pi_\infty((h(\lambda p))(\infty, f(0), G_\infty)) . \end{aligned}$$

If h is given $\alpha(p, f(p), G_p) = \lambda h(p) = h(\lambda p)$ and $\tau(f(p)) = \pi_\infty(h(\lambda p)(\infty, f(0), G_\infty))$. From this we see the correspondence between α and (ξ, h) is one-to-one and that the construction of τ does not depend on which representation is used.

The multiplication in $\Xi_F \times {}_g\mathscr{H}$ is

$$(\xi_1, \hat{h}_1)(\xi_2, \hat{h}_2) = (\xi_1 \circ \xi_2, (\hat{h}_1)(\bar{\xi}_1 \circ \hat{h}_2 \circ \lambda_1^{-1})) \; .$$

We note that $\hat{h}_1(\bar{\xi}_1 \hat{h}_2 \lambda_1^{-1})$ determines τ where $\tau = (\tau_1 \circ \lambda_2)(\xi_1 \circ \tau_2)$ which is exactly the product we expect to see in $\alpha_1 \circ \alpha_2$. From here it is immediate that the correspondence is an isomorphism.

In Case (e) we replace Ξ_F in Theorem 11 by $\Xi_0 \times \Lambda_F$ where $\xi \in \Xi_0$ if $\xi(G_\infty) = G_\infty$. The automorphism group of T is isomorphic to $(\Xi_0 \times \Lambda_F) \times_g \mathscr{H}$ where $g((\xi, \lambda))\hat{h} = \bar{\xi}\hat{h}\lambda^{-1}$ and g is of type i.

In Case (f) the isomorphism group of T is a subgroup of $\mathscr{A}(\hat{T})$.

V. **Examples.** The following semigroups can be found in Chapter D of [3].

12. **Example.** Let $\boldsymbol{Z}$ be the integers under addition. Let $A = G = \hat{\alpha}/\boldsymbol{Z}$. Let $f: \boldsymbol{H} \rightarrow A$ be given by $f(p) = p + \boldsymbol{Z}$. Then

$$S = \{(p, p + \boldsymbol{Z}, q + \boldsymbol{Z}) \colon p \in \boldsymbol{H}, q \in \boldsymbol{R}\} \cup \infty \times \boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z} \; .$$

$\mathscr{A}(S)$ is given by 9. Since f is not one-to-one $\Lambda_F = \{1\}$. $\mathscr{A}(\boldsymbol{R}/\boldsymbol{Z}) = \{-1, 1\}$ and $\mathrm{Hom}\,(\boldsymbol{R}/\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}) = \boldsymbol{Z}$.

$\mathscr{A}(S) = \{-1, 1\} \times_{g_2} \boldsymbol{Z}$ and the multiplication is given by $(x, k)(y, n) = (xy, k + xn)$.

13. **Example.** Let S be as in 12. Let T be the homomorphic image of S obtained by letting $r = 1$ and not changing A or G. $\mathscr{A}(T)$ is given by 10 and $\mathscr{A}(T) = \mathscr{A}(S)$.

14. **Example.** Let S be as in 12. Let T be the homomorphic image of S obtained by letting $G_p = \boldsymbol{Z}$ for $p < \infty$ and $G_\infty = \boldsymbol{R}/\boldsymbol{Z}$. T is described in § II, Case (e). $\mathscr{A}(T)$ is given by Theorem 11 and the comment following it. This is a particularly simple example where $\Lambda_F = \{1\}$ and $\Xi_0 = \Xi = \mathscr{A}(G)$. $\mathscr{H} = \mathrm{Hom}\,(\boldsymbol{H}, \boldsymbol{R}/\boldsymbol{Z}) = \boldsymbol{R}$. $\mathscr{H}$ must represent homomorphisms $h: \boldsymbol{H} \rightarrow T$. It does in this way: $h_r(p) = (p, p + \boldsymbol{Z}, rp + \boldsymbol{Z})$.

$\mathscr{A}(T) = \{-1, 1\} \times_g R$ where multiplication is given by $(x, r)(y, s) = (xy, r + xs)$.

15. **Example.** Let S be as in 12. Let T be the homomorphic image obtained from S by letting $K = \{(\infty, p + \boldsymbol{Z}, p + \boldsymbol{Z}) \colon p \in \boldsymbol{R}\}$. The automorphisms of T are given by 7 and 8. They are a subgroup of $\mathscr{A}(S)$.

We examine $\mathscr{A}(S) = \{-1, 1\} \times_{g_2} \boldsymbol{Z}$ to see which automorphisms satisfy 7. Let $(x, k) \in \mathscr{A}(S)$. $\pi_A(K) = \boldsymbol{R}/\boldsymbol{Z}$ and $\beta(p + \boldsymbol{Z}) = p + \boldsymbol{Z}$. k is the homomorphism called τ in 7 and $\tau(a) = \beta(\lambda a)\xi(\beta(a))^{-1}$. We have $k(p + \boldsymbol{Z}) = kp + \boldsymbol{Z} = p + \boldsymbol{Z} - xp + \boldsymbol{Z}$. If $x = 1, kp + \boldsymbol{Z} = \boldsymbol{Z}$; if $x = -1, kp + \boldsymbol{Z} = 2p + \boldsymbol{Z}$. $\mathscr{A}(T) = \{(1, 0), (-1, 2)\}$ considered as a subgroup of $\mathscr{A}(S)$.

REFERENCES

1. Bourbaki, *General Topology*, Addison Wesley, Pub. Co., Reading Mass.
2. T. Bowman, *On the Automorphism Groups of Some Types of Compact Semigroups*, Ph. D. Thesis, Tulane University, 1970.
3. K. H. Hofmann, and P. S. Mostert, *Elements of Compact Semigroups*, Charles E. Merrill Books, Inc., Columbus, Ohio. 1966.
4. M. W. Mislove, *The Existence of Irr(X), Semigroup Forum*, **1** (1970), 243.

Received July 15, 1971 and in revised form September 28, 1971. This work is part of the author's doctoral dissertation written at Syracuse University under the direction of Professor Anne Hudson. This paper was prepared with the support of grant A 8042 from the National Research Council of Canada.

QUEEN'S UNIVERSITY,
KINGSTON, ONTARIO

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 43, No. 1 March, 1972