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# DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

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# DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

K. R. GOODEARL

**This paper is an investigation of conditions on a module $A$ under which the natural map**

$$A \otimes (\Pi C_\alpha) \longrightarrow \Pi(A \otimes C_\alpha)$$

**is an injection. The investigation leads to a theorem that a commutative von Neumann regular ring is self-injective if and only if the natural map**

$$(\Pi F_\alpha) \otimes (\Pi G_\beta) \longrightarrow \Pi(F_\alpha \otimes G_\beta)$$

**is an injection for all collections $\{F_\alpha\}$ and $\{G_\beta\}$ of free modules. An example is constructed of a commutative ring $R$ for which the natural map**

$$R[[s]] \otimes R[[t]] \longrightarrow R[[s, t]]$$

**is not an injection.**

$R$ denotes a ring with unit, and all $R$-modules are unital. All tensor products are taken over $R$.

We state for reference the following theorem of H. Lenzing [2, Satz 1 and Satz 2]:

THEOREM L. (a) *A right $R$-module $A$ is finitely generated if and only if for any collection $\{C_\alpha\}$ of left $R$-modules, the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is surjective.*

(b) *A right $R$-module $A$ is finitely presented if and only if for any collection $\{C_\alpha\}$ of left $R$-modules, the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is an isomorphism.*

THEOREM 1. *For any right $R$-module $A$, the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of flat left $R$-modules, then the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set $X$ of cardinality at least card $(R)$ such that the natural map $A \otimes R^X \to A^X$ is an injection.*

(c) *If $B$ is any finitely generated submodule of $A$, then the inclusion $B \to A$ factors through a finitely presented module.*

Note that condition (c) always holds when $R$ is right noetherian, for then all finitely generated submodules of $A$ are finitely presented.

*Proof.* (b) $\Rightarrow$ (c): If $R$ is finite, then it is right noetherian and

(c) holds. Thus we may assume that $R$ is infinite.

Let $f\colon F \to A$ be an epimorphism with $F_R$ free, and set $K = \ker f$. There is a finitely generated submodule $G$ of $F$ such that $fG = B$.

We have a commutative diagram with exact rows as follows (Diagram I):

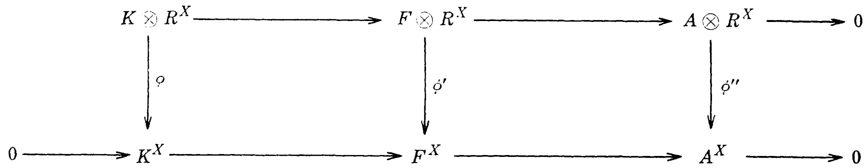


Diagram I

Since $G$ is finitely generated, $G^X \leqq \phi'(F \otimes R^X)$. A short diagram chase (using the injectivity of $\phi''$) shows that $(G \cap K)^X \leqq \phi(K \otimes R^X)$.

card $(G) \leqq$ card $(R)$ because $R$ is infinite, hence card $(G \cap K) \leqq$ card $(X)$. Thus there is a surjection $\alpha \mapsto g_\alpha$ of $X$ onto $G \cap K$. The element $g = \{g_\alpha\}$ in $(G \cap K)^X$ must be the image under $\phi$ of some element $h_1 \otimes r_1 + \cdots + h_n \otimes r_n$ in $K \otimes R^X$. It follows easily that $G \cap K$ is contained in the submodule $H$ of $K$ generated by $h_1, \cdots, h_n$. Note that $G \cap H = G \cap K$.

$G + H$ is contained in some finitely generated free submodule $F_0$ of $F$. The map $f$ induces a monomorphism of $G/(G \cap H)$ into $A$, and this monomorphism factors through the finitely presented module $F_0/H$. Since $fG = B$, the inclusion $B \to A$ also factors through $F_0/H$.

(c) $\Rightarrow$ (a): Consider any $x$ belonging to the kernel of the natural map $\phi\colon A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$. There is a finitely generated submodule

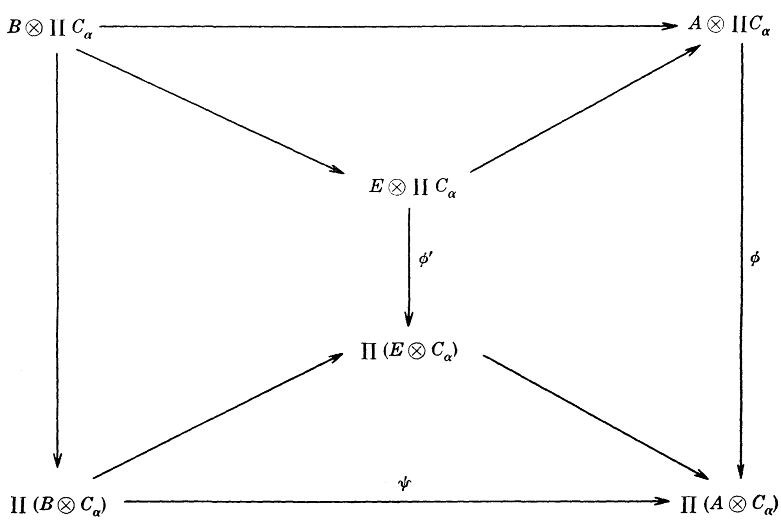


Diagram II

$B$ of $A$ such that $x$ is in the image of the map $B \otimes \Pi C_\alpha \to A \otimes \Pi C_\alpha$. By (c), the inclusion $B \to A$ factors through some finitely presented module $E$.

We have a commutative diagram as follows (Diagram II):

$\phi'$ is an isomorphism by Theorem L, and $\psi$ is a monomorphism because all the $C_\alpha$'s are flat. Another diagram chase now shows that $x = 0$.

COROLLARY. *Suppose that $R$ is (von Neumann) regular. For any right $R$-module $A$, the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of left $R$-modules, then the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set $X$ of cardinality at least card $(R)$ such that the natural map $A \otimes R^X \to A^X$ is injective.*

(c) *All finitely generated submodules of $A$ are projective.*

*Proof.* (b) $\Rightarrow$ (c): If $B$ is a finitely generated submodule of $A$, then Theorem 1 says that the inclusion $B \to A$ factors through a finitely presented module $E$. $E$ is flat (because $R$ is regular) and hence is projective. Thus $B$ can be embedded in a projective module. Since $R$ is semihereditary, $B$ must be projective.

(c) $\Rightarrow$ (a): All the $C_\alpha$'s are flat (since $R$ is regular), and all finitely generated submodules of $A$ are finitely presented, so this follows directly from Theorem 1.

THEOREM 2. *Assume that $R$ is a commutative regular ring. Then the following conditions are equivalent:*

(a) *If $\{F_\alpha\}$ and $\{G_\beta\}$ are any collections of free $R$-modules, then the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi(F_\alpha \otimes G_\beta)$ is an injection.*

(b) *There is a set $X$ of cardinality at least card $(R)$ such that the natural map $R^X \otimes R^X \to R^{X \times X}$ is an injection.*

(c) *$R$ is injective as a module over itself.*

*Proof.* (b) $\Rightarrow$ (c): By [1, Theorem 2.1], it suffices to show that any finitely generated nonsingular $R$-module $B$ is projective.

[1, Lemma 2.2] says that we can embed $B$ in a finite direct sum $Q_1 \oplus \cdots \oplus Q_n$, where each $Q_i$ is a copy of the maximal quotient ring $Q$ of $R$. Then $B$ can be embedded in a direct sum $B_1 \oplus \cdots \oplus B_n$, where $B_i$ is a finitely generated $R$-submodule of $Q_i$. Since $R$ is semihereditary, $B$ will be projective provided each $B_i$ is projective. Thus without loss of generality we may assume that $B$ is an $R$-submodule of $Q$.

Let $b_1, \cdots, b_n$ generate $B$. Since $R$ is an essential submodule of $Q$, there is an essential ideal $I$ of $R$ such that $b_i I \leqq R$ for all $i$.

Since $R$ is commutative, the multiplications by the elements of $I$ induce homomorphisms of $B$ into $R$. Together, these homomorphisms induce a homomorphism $f: B \to R^I$. $Q$ is a nonsingular $R$-module because it has the nonsingular $R$-module $R$ as an essential submodule. Thus no nonzero element of $B$ is annihilated by $I$; i.e., $f: B \to R^I$ is an injection. Since $\operatorname{card}(I) \leqq \operatorname{card}(R) \leqq \operatorname{card}(X)$, there must also be an embedding of $B$ into $R^X$.

Since the natural map $R^X \otimes R^X \to (R^X)^X$ is injective by (b), the corollary to Theorem 1 says that all finitely generated submodules of $R^X$ are projective. Thus $B$ must be projective.

(c) $\Rightarrow$ (a): By [1, Theorem 2.1], all finitely generated nonsingular $R$-modules are projective. Since $R_R$ is nonsingular, $\Pi F_\alpha$ is nonsingular; thus all finitely generated submodules of $\Pi F_\alpha$ are projective. By the corollary to Theorem 1, the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi_\beta[(\Pi F_\alpha) \otimes G_\beta]$ is an injection. Likewise, each of the maps $(\Pi F_\alpha) \otimes G_\beta \to \Pi_\alpha(F_\alpha \otimes G_\beta)$ is injective. Thus the map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi(F_\alpha \otimes G_\beta)$ must be injective.

In particular, Theorem 2 asserts that if $R$ is a countable commutative regular ring which is not self-injective, then the natural map $R^X \otimes R^X \to R^{X \times X}$ is not an injection for any infinite set $X$. For example, let $F_1, F_2, \cdots$ be a countable sequence of copies of some countable field $F$; let $R$ be the subalgebra of $\Pi F_n$ generated by 1 and $\oplus F_n$. $R$ is obviously a countable commutative regular ring. Since $\Pi F_n$ is a proper essential extension of $R_R$, $R_R$ is not injective.

If $N$ is the set of natural numbers, then the natural map $R^N \otimes R^N \to R^{N \times N}$ is not an injection. Thus the tensor product of two one-variable power series rings, $R[[s]] \otimes R[[t]]$, is not embedded in $R[[s, t]]$ by the natural map.

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