

Pacific Journal of Mathematics

DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

KENNETH R. GOODEARL

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This paper is an investigation of conditions on a module A under which the natural map

$$A \otimes (\prod C_\alpha) \longrightarrow \prod(A \otimes C_\alpha)$$

is an injection. The investigation leads to a theorem that a commutative von Neumann regular ring is self-injective if and only if the natural map

$$(\prod F_\alpha) \otimes (\prod G_\beta) \longrightarrow \prod(F_\alpha \otimes G_\beta)$$

is an injection for all collections $\{F_\alpha\}$ and $\{G_\beta\}$ of free modules. An example is constructed of a commutative ring R for which the natural map

$$R[[s]] \otimes R[[t]] \longrightarrow R[[s, t]]$$

is not an injection.

R denotes a ring with unit, and all R -modules are unital. All tensor products are taken over R .

We state for reference the following theorem of H. Lenzing [2, Satz 1 and Satz 2]:

THEOREM L. (a) *A right R -module A is finitely generated if and only if for any collection $\{C_\alpha\}$ of left R -modules, the natural map $A \otimes \prod C_\alpha \rightarrow \prod(A \otimes C_\alpha)$ is surjective.*

(b) *A right R -module A is finitely presented if and only if for any collection $\{C_\alpha\}$ of left R -modules, the natural map $A \otimes \prod C_\alpha \rightarrow \prod(A \otimes C_\alpha)$ is an isomorphism.*

THEOREM 1. *For any right R -module A , the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of flat left R -modules, then the natural map $A \otimes \prod C_\alpha \rightarrow \prod(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set X of cardinality at least $\text{card}(R)$ such that the natural map $A \otimes R^X \rightarrow A^X$ is an injection.*

(c) *If B is any finitely generated submodule of A , then the inclusion $B \rightarrow A$ factors through a finitely presented module.*

Note that condition (c) always holds when R is right noetherian, for then all finitely generated submodules of A are finitely presented.

Proof. (b) \Rightarrow (c): If R is finite, then it is right noetherian and

(c) holds. Thus we may assume that R is infinite.

Let $f: F \rightarrow A$ be an epimorphism with F_R free, and set $K = \ker f$. There is a finitely generated submodule G of F such that $fG = B$.

We have a commutative diagram with exact rows as follows (Diagram I):

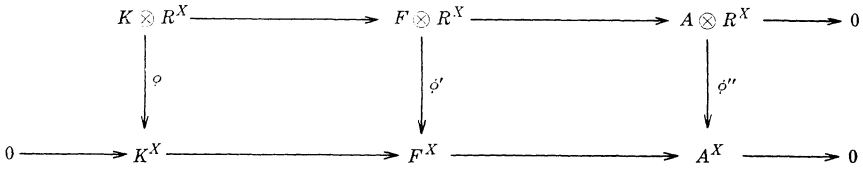


DIAGRAM I

Since G is finitely generated, $G^X \leq \phi'(F \otimes R^X)$. A short diagram chase (using the injectivity of ϕ'') shows that $(G \cap K)^X \leq \phi(K \otimes R^X)$.

$\text{card}(G) \leq \text{card}(R)$ because R is infinite, hence $\text{card}(G \cap K) \leq \text{card}(X)$. Thus there is a surjection $\alpha \mapsto g_\alpha$ of X onto $G \cap K$. The element $g = \{g_\alpha\}$ in $(G \cap K)^X$ must be the image under ϕ of some element $h_1 \otimes r_1 + \dots + h_n \otimes r_n$ in $K \otimes R^X$. It follows easily that $G \cap K$ is contained in the submodule H of K generated by h_1, \dots, h_n . Note that $G \cap H = G \cap K$.

$G + H$ is contained in some finitely generated free submodule F_0 of F . The map f induces a monomorphism of $G/(G \cap H)$ into A , and this monomorphism factors through the finitely presented module F_0/H . Since $fG = B$, the inclusion $B \rightarrow A$ also factors through F_0/H .

(c) \Rightarrow (a): Consider any x belonging to the kernel of the natural map $\phi: A \otimes \Pi C_\alpha \rightarrow \Pi(A \otimes C_\alpha)$. There is a finitely generated submodule

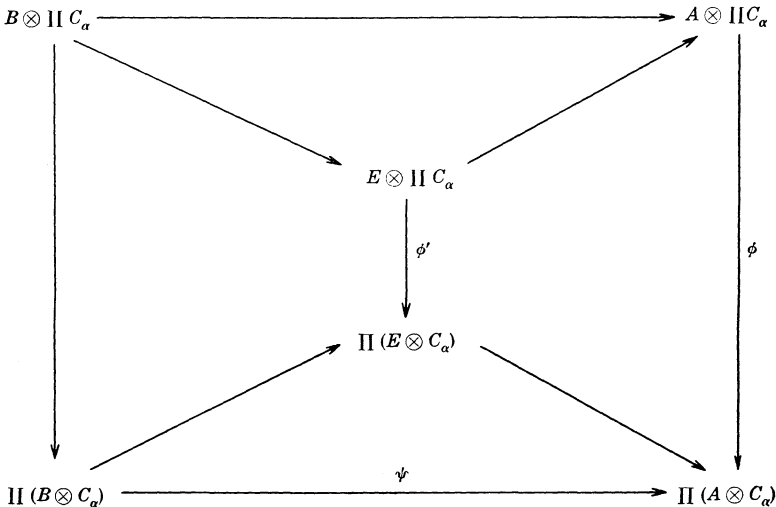


DIAGRAM II

B of A such that x is in the image of the map $B \otimes \Pi C_\alpha \rightarrow A \otimes \Pi C_\alpha$. By (c), the inclusion $B \rightarrow A$ factors through some finitely presented module E .

We have a commutative diagram as follows (Diagram II):

ϕ' is an isomorphism by Theorem L, and ψ is a monomorphism because all the C_α 's are flat. Another diagram chase now shows that $x = 0$.

COROLLARY. *Suppose that R is (von Neumann) regular. For any right R -module A , the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of left R -modules, then the natural map $A \otimes \Pi C_\alpha \rightarrow \Pi(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set X of cardinality at least $\text{card}(R)$ such that the natural map $A \otimes R^X \rightarrow A^X$ is injective.*

(c) *All finitely generated submodules of A are projective.*

Proof. (b) \Rightarrow (c): If B is a finitely generated submodule of A , then Theorem 1 says that the inclusion $B \rightarrow A$ factors through a finitely presented module E . E is flat (because R is regular) and hence is projective. Thus B can be embedded in a projective module. Since R is semihereditary, B must be projective.

(c) \Rightarrow (a): All the C_α 's are flat (since R is regular), and all finitely generated submodules of A are finitely presented, so this follows directly from Theorem 1.

THEOREM 2. *Assume that R is a commutative regular ring. Then the following conditions are equivalent:*

(a) *If $\{F_\alpha\}$ and $\{G_\beta\}$ are any collections of free R -modules, then the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \rightarrow \Pi(F_\alpha \otimes G_\beta)$ is an injection.*

(b) *There is a set X of cardinality at least $\text{card}(R)$ such that the natural map $R^X \otimes R^X \rightarrow R^{X \times X}$ is an injection.*

(c) *R is injective as a module over itself.*

Proof. (b) \Rightarrow (c): By [1, Theorem 2.1], it suffices to show that any finitely generated nonsingular R -module B is projective.

[1, Lemma 2.2] says that we can embed B in a finite direct sum $Q_1 \oplus \cdots \oplus Q_n$, where each Q_i is a copy of the maximal quotient ring Q of R . Then B can be embedded in a direct sum $B_1 \oplus \cdots \oplus B_n$, where B_i is a finitely generated R -submodule of Q_i . Since R is semihereditary, B will be projective provided each B_i is projective. Thus without loss of generality we may assume that B is an R -submodule of Q .

Let b_1, \dots, b_n generate B . Since R is an essential submodule of Q , there is an essential ideal I of R such that $b_i I \leq R$ for all i .

Since R is commutative, the multiplications by the elements of I induce homomorphisms of B into R . Together, these homomorphisms induce a homomorphism $f: B \rightarrow R^I$. Q is a nonsingular R -module because it has the nonsingular R -module R as an essential submodule. Thus no nonzero element of B is annihilated by I ; i.e., $f: B \rightarrow R^I$ is an injection. Since $\text{card}(I) \leq \text{card}(R) \leq \text{card}(X)$, there must also be an embedding of B into R^X .

Since the natural map $R^X \otimes R^X \rightarrow (R^X)^X$ is injective by (b), the corollary to Theorem 1 says that all finitely generated submodules of R^X are projective. Thus B must be projective.

(c) \Rightarrow (a): By [1, Theorem 2.1], all finitely generated nonsingular R -modules are projective. Since R_R is nonsingular, ΠF_α is nonsingular; thus all finitely generated submodules of ΠF_α are projective. By the corollary to Theorem 1, the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \rightarrow \Pi_\beta[(\Pi F_\alpha) \otimes G_\beta]$ is an injection. Likewise, each of the maps $(\Pi F_\alpha) \otimes G_\beta \rightarrow \Pi_\alpha(F_\alpha \otimes G_\beta)$ is injective. Thus the map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \rightarrow \Pi(F_\alpha \otimes G_\beta)$ must be injective.

In particular, Theorem 2 asserts that if R is a countable commutative regular ring which is not self-injective, then the natural map $R^X \otimes R^X \rightarrow R^{X \times X}$ is not an injection for any infinite set X . For example, let F_1, F_2, \dots be a countable sequence of copies of some countable field F ; let R be the subalgebra of ΠF_n generated by 1 and $\bigoplus F_n$. R is obviously a countable commutative regular ring. Since ΠF_n is a proper essential extension of R_R , R_R is not injective.

If N is the set of natural numbers, then the natural map $R^N \otimes R^N \rightarrow R^{N \times N}$ is not an injection. Thus the tensor product of two one-variable power series rings, $R[[s]] \otimes R[[t]]$, is not embedded in $R[[s, t]]$ by the natural map.

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Received July 28, 1971.

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