DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

Kenneth R. Goodearl
DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

K. R. GOODEARL

This paper is an investigation of conditions on a module \( A \) under which the natural map
\[
A \otimes \langle IIC_a \rangle \longrightarrow \Pi(A \otimes C_a)
\]
is an injection. The investigation leads to a theorem that a commutative von Neumann regular ring is self-injective if and only if the natural map
\[
(\Pi F_a) \otimes (\Pi G_\beta) \longrightarrow \Pi(F_a \otimes G_\beta)
\]
is an injection for all collections \( \{F_a\} \) and \( \{G_\beta\} \) of free modules. An example is constructed of a commutative ring \( R \) for which the natural map
\[
R([s]) \otimes R([t]) \longrightarrow R([s, t])
\]
is not an injection.

\( R \) denotes a ring with unit, and all \( R \)-modules are unital. All tensor products are taken over \( R \).

We state for reference the following theorem of H. Lenzing [2, Satz 1 and Satz 2]:

**Theorem L.** (a) A right \( R \)-module \( A \) is finitely generated if and only if for any collection \( \{C_a\} \) of left \( R \)-modules, the natural map \( A \otimes \Pi C_a \rightarrow \Pi(A \otimes C_a) \) is surjective.

(b) A right \( R \)-module \( A \) is finitely presented if and only if for any collection \( \{C_a\} \) of left \( R \)-modules, the natural map \( A \otimes \Pi C_a \rightarrow \Pi(A \otimes C_a) \) is an isomorphism.

**Theorem 1.** For any right \( R \)-module \( A \), the following conditions are equivalent:

(a) If \( \{C_a\} \) is any collection of flat left \( R \)-modules, then the natural map \( A \otimes \Pi C_a \rightarrow \Pi(A \otimes C_a) \) is an injection.

(b) There is a set \( X \) of cardinality at least \( \text{card}(R) \) such that the natural map \( A \otimes R^x \rightarrow A^x \) is an injection.

(c) If \( B \) is any finitely generated submodule of \( A \), then the inclusion \( B \rightarrow A \) factors through a finitely presented module.

Note that condition (c) always holds when \( R \) is right noetherian, for then all finitely generated submodules of \( A \) are finitely presented.

**Proof.** (b) \( \Rightarrow \) (c): If \( R \) is finite, then it is right noetherian and...
(c) holds. Thus we may assume that $R$ is infinite.

Let $f: F \to A$ be an epimorphism with $F_R$ free, and set $K = \ker f$. There is a finitely generated submodule $G$ of $F$ such that $fG = B$.

We have a commutative diagram with exact rows as follows (Diagram I):

$$
\begin{array}{ccccccccc}
K \otimes R^X & \rightarrow & F \otimes R^X & \rightarrow & A \otimes R^X & \rightarrow & 0 \\
\phi & & \phi' & & \phi'' & & \\
0 & \rightarrow & K^X & \rightarrow & F^X & \rightarrow & A^X & \rightarrow & 0 \\
\end{array}
$$

\textbf{Diagram I}

Since $G$ is finitely generated, $G^X \leq \phi'(F \otimes R^X)$. A short diagram chase (using the injectivity of $\phi''$) shows that $(G \cap K)^X \leq \phi(K \otimes R^X)$.

$\text{card } (G) \leq \text{card } (R)$ because $R$ is infinite, hence $\text{card } (G \cap K) \leq \text{card } (X)$. Thus there is a surjection $\alpha \mapsto g_\alpha$ of $X$ onto $G \cap K$. The element $g = \{g_\alpha\}$ in $(G \cap K)^X$ must be the image under $\phi$ of some element $h_1 \otimes r_1 + \cdots + h_n \otimes r_n$ in $K \otimes R^X$. It follows easily that $G \cap K$ is contained in the submodule $H$ of $K$ generated by $h_1, \cdots, h_n$. Note that $G \cap H = G \cap K$.

$G + H$ is contained in some finitely generated free submodule $F_0$ of $F$. The map $f$ induces a monomorphism of $G/(G \cap H)$ into $A$, and this monomorphism factors through the finitely presented module $F_0/H$. Since $fG = B$, the inclusion $B \to A$ also factors through $F_0/H$.

(c) $\Rightarrow$ (a): Consider any $x$ belonging to the kernel of the natural map $\phi: A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$. There is a finitely generated submodule
B of A such that x is in the image of the map $B \otimes \Pi C_a \to A \otimes \Pi C_a$. By (c), the inclusion $B \to A$ factors through some finitely presented module $E$.

We have a commutative diagram as follows (Diagram II):

$\phi'$ is an isomorphism by Theorem L, and $\varphi$ is a monomorphism because all the $C_a$'s are flat. Another diagram chase now shows that $x = 0$.

**Corollary.** Suppose that $R$ is (von Neumann) regular. For any right $R$-module $A$, the following conditions are equivalent:

(a) If $\{C_a\}$ is any collection of left $R$-modules, then the natural map $A \otimes \Pi C_a \to \Pi (A \otimes C_a)$ is an injection.

(b) There is a set $X$ of cardinality at least $\text{card} (R)$ such that the natural map $A \otimes R^x \to A^x$ is injective.

(c) All finitely generated submodules of $A$ are projective.

**Proof.** (b) $\Rightarrow$ (c): If $B$ is a finitely generated submodule of $A$, then Theorem 1 says that the inclusion $B \to A$ factors through a finitely presented module $E$. $E$ is flat (because $R$ is regular) and hence is projective. Thus $B$ can be embedded in a projective module. Since $R$ is semihereditary, $B$ must be projective.

(c) $\Rightarrow$ (a): All the $C_a$'s are flat (since $R$ is regular), and all finitely generated submodules of $A$ are finitely presented, so this follows directly from Theorem 1.

**Theorem 2.** Assume that $R$ is a commutative regular ring. Then the following conditions are equivalent:

(a) If $\{F_a\}$ and $\{G_\beta\}$ are any collections of free $R$-modules, then the natural map $(\Pi F_a) \otimes (\Pi G_\beta) \to \Pi (F_a \otimes G_\beta)$ is an injection.

(b) There is a set $X$ of cardinality at least $\text{card} (R)$ such that the natural map $R^X \otimes R^x \to R^{X \times x}$ is an injection.

(c) $R$ is injective as a module over itself.

**Proof.** (b) $\Rightarrow$ (c): By [1, Theorem 2.1], it suffices to show that any finitely generated nonsingular $R$-module $B$ is projective.

[1, Lemma 2.2] says that we can embed $B$ in a finite direct sum $Q_1 \oplus \cdots \oplus Q_n$, where each $Q_i$ is a copy of the maximal quotient ring $Q$ of $R$. Then $B$ can be embedded in a direct sum $B_1 \oplus \cdots \oplus B_n$, where $B_i$ is a finitely generated $R$-submodule of $Q_i$. Since $R$ is semihereditary, $B$ will be projective provided each $B_i$ is projective. Thus without loss of generality we may assume that $B$ is an $R$-submodule of $Q$.

Let $b_1, \cdots, b_n$ generate $B$. Since $R$ is an essential submodule of $Q$, there is an essential ideal $I$ of $R$ such that $b_i I \leq R$ for all $i$. 
Since \( R \) is commutative, the multiplications by the elements of \( I \) induce homomorphisms of \( B \) into \( R \). Together, these homomorphisms induce a homomorphism \( f: B \to R' \). \( Q \) is a nonsingular \( R \)-module because it has the nonsingular \( R \)-module \( R \) as an essential submodule. Thus no nonzero element of \( B \) is annihilated by \( I \); i.e., \( f: B \to R' \) is an injection. Since \( \text{card}(I) \leq \text{card}(R) \leq \text{card}(X) \), there must also be an embedding of \( B \) into \( R^x \).

Since the natural map \( R^x \otimes R^x \to (R^x)^x \) is injective by (b), the corollary to Theorem 1 says that all finitely generated submodules of \( R^x \) are projective. Thus \( B \) must be projective.

\((c) \Rightarrow (a)\): By [1, Theorem 2.1], all finitely generated nonsingular \( R \)-modules are projective. Since \( R_n \) is nonsingular, \( \Pi F_\alpha \) is nonsingular; thus all finitely generated submodules of \( \Pi F_\alpha \) are projective. By the corollary to Theorem 1, the natural map \( (\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi_a[(\Pi F_\alpha) \otimes G_\beta] \) is an injection. Likewise, each of the maps \( (\Pi F_\alpha) \otimes G_\beta \to \Pi_a(F_\alpha \otimes G_\beta) \) is injective. Thus the map \( (\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi(F_\alpha \otimes G_\beta) \) must be injective.

In particular, Theorem 2 asserts that if \( R \) is a countable commutative regular ring which is not self-injective, then the natural map \( R^x \otimes R^x \to R^{x \times x} \) is not an injection for any infinite set \( X \). For example, let \( F_1, F_2, \ldots \) be a countable sequence of copies of some countable field \( F \); let \( R \) be the subalgebra of \( \Pi F_\alpha \) generated by 1 and \( \bigoplus F_\alpha \). \( R \) is obviously a countable commutative regular ring. Since \( \Pi F_\alpha \) is a proper essential extension of \( R_n \), \( R_R \) is not injective.

If \( N \) is the set of natural numbers, then the natural map \( R^N \otimes R^N \to R^{N \times N} \) is not an injection. Thus the tensor product of two one-variable power series rings, \( R[[s]] \otimes R[[t]] \), is not embedded in \( R[[s, t]] \) by the natural map.

REFERENCES


Received July 28, 1971.

UNIVERSITY OF WASHINGTON

Author's current address: UNIVERSITY OF UTAH
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNJII
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

J. DUGUNJII
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA UNIVERSITY OF TOKYO
MONTANA STATE UNIVERSITY UNIVERSITY OF UTAH
UNIVERSITY OF NEVADA WASHINGTON STATE UNIVERSITY
NEW MEXICO STATE UNIVERSITY UNIVERSITY OF WASHINGTON
OREGON STATE UNIVERSITY *
UNIVERSITY OF OREGON *
OSAKA UNIVERSITY AMERICAN MATHEMATICAL SOCIETY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan
Alexander (Smbat) Abian, *The use of mitotic ordinals in cardinal arithmetic* ......................................................... 1

Helen Elizabeth. Adams, *Filtrations and valuations on rings* ..................... 7

Benno Artmann, *Geometric aspects of primary lattices* .......................... 15

Marilyn Breen, *Determining a polytope by Radon partitions* ................. 27

David S. Browder, *Derived algebras in L₁ of a compact group* ............... 39

Aiden A. Bruen, *Unimbeddable nets of small deficiency* ......................... 51

Michael Howard Clapp and Raymond Frank Dickman, *Unicoherent compactifications* .................................................. 55

Heron S. Collins and Robert A. Fontenot, *Approximate identities and the strict topology* ............................................. 63

R. J. Gazik, *Convergence in spaces of subsets* .................................. 81

Joan Geramita, *Automorphisms on cylindrical semigroups* ....................... 93

Kenneth R. Goodearl, *Distributing tensor product over direct product* .... 107

Julien O. Hennefeld, *The non-conjugacy of certain algebras of operators* ................................................................. 111

C. Ward Henson, *The nonstandard hulls of a uniform space* ................... 115

M. Jeanette Huebener, *Complementation in the lattice of regular topologies* ................................................................. 139

Dennis Lee Johnson, *The diophantine problem Y² − X³ = A in a polynomial ring* ......................................................... 151

Albert Joseph Karam, *Strong Lie ideals* ............................................. 157

Soon-Kyu Kim, *On low dimensional minimal sets* .................................. 171

Thomas Latimer Kriete, III and Marvin Rosenblum, *A Phragmén-Lindelöf theorem with applications to M(u, v) functions* .......... 175

William A. Lampe, *Notes on related structures of a universal algebra* .... 189

Theodore Windle Palmer, *The reducing ideal is a radical* ....................... 207

Kulumani M. Rangaswamy and N. Vanaja, *Quasi projectives in abelian and module categories* .......................................... 221

Ghulam M. Shah, *On the univalence of some analytic functions* ............... 239

Joseph Earl Valentine and Stanley G. Wayment, *Criteria for Banach spaces* .................................................................................. 251

Jerry Eugene Vaughan, *Linearly stratifiable spaces* ................................ 253

Zbigniew Zielezny, *On spaces of distributions strongly regular with respect to partial differential operators* ......................... 267