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Let $(X, \mathscr{U})$ be a uniform space in some set theoretical structure $\mathscr{M}$ and let $^*X$ be the set corresponding to X in an enlargement $^*\mathscr{M}$ of $\mathscr{M}$. In this paper a set of $\mathscr{U}$-finite elements of $^*X$ is defined and this set is used to define a nonstandard hull of $(X, \mathscr{U})$. The main result is that, with some specific exceptions depending on the existence of measurable cardinal numbers, this nonstandard hull is the same as the smallest of the nonstandard hulls defined by Luxemburg. This result is used in giving a characterization of subsets of X on which every uniformly continuous, real valued function is bounded. Also, two examples are given to illustrate the possible structure of the nonstandard hulls.

The nonstandard hulls defined by Luxemburg [4] are obtained from sets F of "*finite*" elements of $^*X$ which may be written in the form

$$F = \{p \mid p \in {}^*X \text{ and } {}^*f(p) \text{ is finite for all } f \text{ in } \mathscr{F}\}$$

where $\mathscr{F}$ is a set of uniformly continuous, real valued functions on $(X, \mathscr{U})$. The concept of finiteness introduced in this paper is entirely different. An element p of $^*X$ is *$\mathscr{U}$-finite* if, for each A in $\mathscr{U}$ there is a sequence $q_0, \cdots, q_n$ in $^*X$ which satisfies the conditions (i) $q_0 = p$, (ii) $q_n = {}^*x$ for some x in X, and (iii) for each $j = 0, \cdots, n-1$ the pairs (q_j, q_{j+1}) and (q_{j+1}, q_j) are both in $^*A$.

Our main result is that the set of $\mathscr{U}$-finite elements of $^*X$ is equal to the set

$$\{p \mid p \in {}^*X \text{ and } {}^*f(p) \text{ is finite for every uniformly continuous, real valued function } f \text{ on } (X, \mathscr{U})\}$$

if and only if it is impossible to partition X into a measurable cardinal number of subsets $\{X_a \mid a \in I\}$ which are "uniformly open" in the sense that there is an A in $\mathscr{U}$ such that

$$x \in X_a \quad \text{implies} \quad \{y \mid (y, x) \in A\} \subset X_a$$

for every a in I. In particular, these two sets of finite elements of $^*X$ are equal whenever the number of topologically connected components of $(X, \mathscr{U})$ is smaller than every measurable cardinal number.

This result is used in giving a characterization of those subsets Y of X such that every uniformly continuous, real valued function on $(X, \mathscr{U})$ is bounded on Y, generalizing a Theorem of Atsuji [2].

Also, two examples are presented which illustrate the possible structure of the nonstandard hulls defined using the set of $\mathscr{U}$-finite elements of $^*X$. These examples are based on ideas due to L. C. Moore, to whom the author is grateful for many helpful conversations on the subject of this paper.

1. Throughout this paper $\mathscr{M}$ denotes a set theoretical structure and $^*\mathscr{M}$ denotes an enlargement of $\mathscr{M}$. (The image of an element x of $\mathscr{M}$ under the embedding into $^*\mathscr{M}$ is denoted by $^*x$.) Whether $\mathscr{M}$ and $^*\mathscr{M}$ are taken to be structures for type theory (as in [4] and [6]) or to be structures for the ε-language of ordinary set theory (as in [5] and [7]) is a matter of taste. Most references in this paper will be to [4], although the concepts and results in [4] can easily be set in the frameworks of nonstandard analysis described in [5] and [7].

As is usual, it is assumed here that the set N of positive integers and the set R of real number are elements of $\mathscr{M}$, and that the embedding $x \longmapsto {}^*x$ is the identity on R (and thus also on N.) The extensions to $^*R$ of the operations $+$ and $\cdot$ on R, as well as of the ordering $<$ on R, will be denoted by the same symbols. In general the embedding $x \longmapsto {}^*x$ is not the identity on sets in $\mathscr{M}$. Given an element A of $\mathscr{M}$ it is convenient to introduce the notation $^*[A]$ for the set of standard elements of $^*A$; that is,

$$^*[A] = \{^*a \mid a \in A\} .$$

In dealing with uniform spaces there are certain useful operations on subsets of a cartesian product $C \times C$. If A and B are subsets of $C \times C$, recall that $A \circ B$ and A^{-1} are defined by

$$A \circ B = \{(x, z) \mid \text{ for some } y, (x, y) \in A \text{ and } (y, z) \in B\}$$

$$A^{-1} = \{(x, y) \mid (y, x) \in A\} .$$

The set A^n is defined recursively for $n \geqq 1$ by:

$$A^1 = A, \quad A^{n+1} = A^n \circ A .$$

Also, given an element x of C, the set $A(x)$ is defined by

$$A(x) = \{y \mid (y, x) \in A\} .$$

Note that if A, B and C are elements of $\mathscr{M}$ then $^*A$ and $^*B$ are subsets of $^*C \times {}^*C$ $(= {}^*(C \times C).)$ Moreover, the following equalities hold:

$$^*(A \circ B) = (^*A) \circ (^*B)$$
$$^*(A^{-1}) = (^*A)^{-1}$$

$$^*(A^n) = (^*A)^n$$
$$^*(A(x)) = (^*A)(^*x)$$

(where $x \in C$ and $n \geqq 1$.)

Throughout this paper $(X, \mathscr{U})$ denotes a uniform space which is an element of $\mathscr{M}$. The set of all uniformly continuous, real valued functions on $(X, \mathscr{U})$ is denoted by $C(X, \mathscr{U})$. It is assumed that the reader is familiar with certain parts of the nonstandard theory of uniform spaces, as presented in [4] or [5]. In particular, recall that the monad of the filter $\mathscr{U}$ (that is, the intersection of the family $^*[\mathscr{U}]$ of subsets of $^*X \times {}^*X$) is an equivalence relation on $^*X$. The equivalence class of p is denoted by $\mu(p)$, for each p in $^*X$.

The collection $^*[\mathscr{U}]$ generates a filter on $^*X \times {}^*X$ which will be denoted by $\widetilde{\mathscr{U}}$. A simple, direct argument can be used to show that $\widetilde{\mathscr{U}}$ is a uniform structure on $^*X$ and that the mapping $x \longmapsto {}^*x$ is a uniform space embedding of $(X, \mathscr{U})$ into $(^*X, \widetilde{\mathscr{U}})$. Alternately, let $\mathscr{R}$ be any set of bounded semimetrics on X which defines $\mathscr{U}$. ($\rho(x, y)$ is a semimetric on X if ρ is nonnegative, symmetric, satisfies the triangle inequality and $\rho(x, x) = 0$ for any x in X.) For each ρ in $\mathscr{R}$ a function $\tilde{\rho}$ may be defined on $^*X \times {}^*X$ by

$$\tilde{\rho}(p, q) = \mathrm{st}\,(^*\rho(p, q)) .$$

(Here "st" is the standard part operation on finite elements of $^*R$.) Then $\tilde{\rho}$ is a semimetric on $^*X$. For each $\rho \in \mathscr{R}$ and $\delta > 0$ in R, let

$$A(\rho, \delta) = \{(x, y) \mid \rho(x, y) < \delta\} .$$

Then the collection $\{A(\rho, \delta) \mid \rho \in \mathscr{R}, \delta > 0\}$ generates $\mathscr{U}$ so that the collection $\{^*A(\rho, \delta) \mid \rho \in \mathscr{R}, \delta > 0\}$ generates $\widetilde{\mathscr{U}}$. But

$$^*A(\rho, \delta) \subset \{(p, q) \mid \tilde{\rho}(p, q) \leqq \delta\}$$

and

$$\{(p, q) \mid \tilde{\rho}(p, q) < \delta\} \subset {}^*A(\rho, \delta) .$$

Therefore $\widetilde{\mathscr{U}}$ is the uniformity on $^*X$ defined by the set $\{\tilde{\rho} \mid \rho \in \mathscr{R}\}$ of semimetrics on $^*X$.

Let $X_0 = \{\mu(p) \mid p \in {}^*X\}$ and let $\mathscr{U}_0$ be the quotient uniformity on X_0 induced by $\widetilde{\mathscr{U}}$. Denote the quotient mapping from $(^*X, \widetilde{\mathscr{U}})$ onto $(X_0, \mathscr{U}_0)$ by π. The previous remarks show that $(X_0, \mathscr{U}_0)$ is the nonstandard hull for $(X, \mathscr{U})$ constructed in [4] using any set $\mathscr{R}$ of bounded semimetrics which defines $\mathscr{U}$. (See also p. 56 of [5], where $(X_0, \mathscr{U}_0)$ is constructed and called $T_{\mathscr{U}}$.)

The definition of $\widetilde{\mathscr{U}}$ makes it clear that $\mu(p) = \mu(q)$ if and only if p and q have exactly the same neighborhoods in the $\widetilde{\mathscr{U}}$-topology on $^*X$. Thus π is not only uniformly continuous, but also $\pi(^*A)$ (which equals $\{(\mu(p), \mu(q)) \mid (p, q) \in {}^*A\}$ by definition) is in $\mathscr{U}_0$ whenever A is in $\mathscr{U}$. Therefore π is an open mapping. Moreover, any net in $^*X$ which is mapped by π onto a Cauchy net (convergent net) in $(X_0, \mathscr{U}_0)$ is a Cauchy net (convergent net) in $(^*X, \widetilde{\mathscr{U}})$. If the $\mathscr{U}$-topology on X is Hausdorff, then the map taking x to $\mu(x)$ is a uniform space embedding of $(X, \mathscr{U})$ into $(X_0, \mathscr{U}_0)$. (Otherwise it simply identifies those pairs of points which have exactly the same neighborhoods in the $\mathscr{U}$-topology.)

Constructing "nonstandard hulls" of $(X, \mathscr{U})$ in general involves two distinct steps: (i) the identification of a set F of "finite" elements of $^*X$, and (ii) the construction of a uniformity on F (and then on the set $\{\mu(p) \mid p \in F\}$ by a quotient operation.) There are many different useful concepts of "finiteness" for elements of $^*X$, each one motivated by considerations depending on the kind of mathematical structure which X is assumed to carry. However there seems to be only one natural way to carry out step (ii)—by putting on F the uniformity obtained by restriction from $\widetilde{\mathscr{U}}$. In that case, the nonstandard hull constructed using F is just the subspace $\pi(F)$ of $(X_0, \mathscr{U}_0)$.

For example, let $\mathscr{S}$ be any set of semimetrics which defines $\mathscr{U}$. In defining a nonstandard hull using $\mathscr{S}$, Luxemburg [4] takes F to be the set

$$\{p \mid {}^*\rho(p, {}^*x) \text{ is finite if } x \in X \text{ and } \rho \in \mathscr{S}\} .$$

The uniformity put on F is the one defined by a set $\{\rho' \mid \rho \in \mathscr{S}\}$ of semimetrics on F, where

$$\rho'(p, q) = \mathrm{st}\,({}^*\rho(p, q))$$

for each ρ in $\mathscr{S}$ and p, q in F. If $\mathscr{R}$ is the set $\{\min(\rho, 1) \mid \rho \in \mathscr{S}\}$, then $\mathscr{R}$ also defines $\mathscr{U}$. Moreover, the uniformity defined on F by the set $\{\tilde{\rho} \mid \rho \in \mathscr{R}\}$ is easily seen to be the same as the one defined on F by $\{\rho' \mid \rho \in \mathscr{S}\}$. That is, this uniformity is just the restriction of $\widetilde{\mathscr{U}}$ to F.

In this paper an entirely different concept of "finiteness" for elements of $^*X$ is introduced. It is based on the intuitive idea that a point is "finitely far away" from a set if there is a finite chain of small steps from the point to (some element of) the set, no matter how small the steps are required to be. Thus an element of $^*X$ is taken to be finite if it is "finitely far away" from $^*[X]$, relative to the uniform space $(^*X, \widetilde{\mathscr{U}})$. (See Definition 1.2)

DEFINITION 1.1. Let $(Y, \mathscr{V})$ be any uniform space.

(i) If $A \in \mathscr{V}$ and $x, y \in X$, then an *A-chain from x to y* is a finite sequence $x_0, \cdots, x_n$ in Y which satisfies: $x_0 = x$, $x_n = y$ and, for each $i = 0, \cdots, n-1$, $(x_i, x_{i+1}) \in A \cap A^{-1}$. (The *number of steps* for such an A-chain is n.)

(ii) If $x, y \in Y$, then $x \equiv_A y$ if and only if there is an A-chain from x to y.

(iii) If $x, y \in Y$, then $x \equiv_{\mathscr{V}} y$ if and only if $x \equiv_A y$ for every A in $\mathscr{V}$.

If A is in $\mathscr{V}$, then $A \cap A^{-1}$ is symmetric and contains the diagonal of $Y \times Y$, so that $\equiv_A$ is an equivalence relation on Y. Therefore $\equiv_{\mathscr{V}}$ is also an equivalence relation on Y. The latter relation can be calculated from any collection $\mathscr{B}$ which generates $\mathscr{V}$ as a filter on $Y \times Y$, in the sense that

$$x \equiv_{\mathscr{V}} y \longleftrightarrow x \equiv_A y \text{ for every } A \in \mathscr{B} .$$

Also, observe that if A is in $\mathscr{V}$, then the equivalence classes under $\equiv_A$ are both open and closed in the $\mathscr{V}$-topology on Y.

Definition 1.1 will be applied to both of the uniform spaces $(X, \mathscr{U})$ and $({}^*X, \widetilde{\mathscr{U}})$. Since ${}^*[\mathscr{U}]$ generates $\widetilde{\mathscr{U}}$ as a filter on ${}^*X \times {}^*X$, it follows that for each $p, q \in {}^*X$

$$p \equiv_{\widetilde{\mathscr{U}}} q \longleftrightarrow p \equiv_{{}^*A} q \text{ for every } A \in \mathscr{U} .$$

Note that for each $A \in \mathscr{U}$, ${}^*(\equiv_A)$ is also an equivalence relation on ${}^*X$, and in general it will not be the same relation as $\equiv_{{}^*A}$. Indeed, p and q are in the same ${}^*(\equiv_A)$ equivalence class if there is a *-finite sequence (hence an internal element of ${}^*\mathscr{M}$) $q_0, \cdots, q_\omega$ in ${}^*X$ which satisfies: $q_0 = p$, $q_\omega = q$ and $(q_i, q_{i+1}) \in {}^*(A \cap A^{-1})$ for every $i = 0, \cdots, \omega - 1$. Such a *-finite sequence may exist without any such finite sequence existing: in that case $p \equiv_{{}^*A} q$ would be false.

DEFINITION 1.2. An element p of ${}^*X$ is *$\mathscr{U}$-finite* if, for each $A \in \mathscr{U}$, there exists an x in X which satisfies $p \equiv_{{}^*A} {}^*x$.

The set of $\mathscr{U}$-finite elements of ${}^*X$ will be denoted by $\mathrm{fin}_{\mathscr{U}}({}^*X)$.

It is clear that if p is $\mathscr{U}$-finite, then every element of $\mu(p)$ is also $\mathscr{U}$-finite. In the language of [4], this says that $\mathrm{fin}_{\mathscr{U}}({}^*X)$ is μ-saturated. Also the condition $p \in \mathrm{fin}_{\mathscr{U}}({}^*X)$ is equivalent to a condition on the ultrafilter $\{Y \mid Y \subset X \text{ and } p \in {}^*Y\}$ determined by p. Namely, p is $\mathscr{U}$-finite if and only if for each $A \in \mathscr{U}$ there exist $x \in X$ and $n \geqq 1$ such that $p \in ({}^*A)^n({}^*x) = {}^*(A^n(x))$. Therefore, if p is $\mathscr{U}$-finite then each element of the monad of the ultrafilter $\{Y \mid Y \subset X$ and $p \in {}^*Y\}$ is also $\mathscr{U}$-finite. In the language of [4] this says that $\mathrm{fin}_{\mathscr{U}}({}^*X)$ is μ_d-saturated.

If ρ is any semimetric on X which defines a weaker uniformity than $\mathscr{U}$, and $a \in X$, then the function $f(x) = \rho(x, a)$ is $\mathscr{U}$-uniformly continuous (since $|\rho(x, a) - \rho(y, a)| \leqq \rho(x, y)$.) Thus the sets F of finite elements of $^*X$ considered in [4] are all of the form

$$F = \{p \mid {}^*f(p) \text{ is finite for every } f \in \mathscr{F}\}$$

where $\mathscr{F}$ is a set of $\mathscr{U}$-uniformly continuous, real valued functions on X. The next result shows that each of these sets has $\operatorname{fin}_{\mathscr{U}}({}^*X)$ as a subset.

THEOREM 1.3. *If* $f \in C(X, \mathscr{U})$ *and* $p \in \operatorname{fin}_{\mathscr{U}}({}^*X)$ *then* ${}^*f(p)$ *is finite.*

Proof. Since f is uniformly continuous, there exists A in $\mathscr{U}$ which satisfies

$$(x, y) \in A \longrightarrow |f(x) - f(y)| \leqq 1 .$$

Since p is $\mathscr{U}$-finite, there is a ${}^*A$-chain $q_0, \cdots, q_n$ from p to ${}^*x$, for some x in X. Therefore

$$\begin{aligned} |{}^*f(p) - {}^*f(x)| &\leqq \Sigma |{}^*f(q_i) - {}^*f(q_{i+1})| \\ &\leqq n . \end{aligned}$$

It follows that ${}^*f(p)$ must be finite.

THEOREM 1.4. $\operatorname{fin}_{\mathscr{U}}({}^*X)$ *is closed in the* $\widetilde{\mathscr{U}}$*-topology on* ${}^*X$, *and* $\operatorname{pns}_{\mathscr{U}}({}^*X) \subset \operatorname{fin}_{\mathscr{U}}({}^*X)$.

Proof. For each A in $\mathscr{U}$ the set

$$\{p \mid p \equiv_{{}^*A} {}^*x \text{ for some } x \in X\}$$

is a disjoint union of $\equiv_{{}^*A}$ equivalence classes, each of which is open and closed in the $\widetilde{\mathscr{U}}$-topology on ${}^*X$. It follows that this set is, itself, open and closed in that topology. Finally, $\operatorname{fin}_{\mathscr{U}}({}^*X)$ is an intersection of such sets, so that it must be a closed set.

That $\operatorname{pns}_{\mathscr{U}}({}^*X)$ is a subset of $\operatorname{fin}_{\mathscr{U}}({}^*X)$ follows immediately, using the obvious fact that ${}^*[X]$ is a subset of $\operatorname{fin}_{\mathscr{U}}({}^*X)$ and using Theorem 3.15.2 of [4]. (This Theorem implies that $\operatorname{pns}_{\mathscr{U}}({}^*X)$ is the closure of ${}^*[X]$ in the $\widetilde{\mathscr{U}}$-topology on ${}^*X$. The extra assumptions on ${}^*\mathscr{M}$ made in [4] are not needed for this result. See also Theorem 7.5.3 of [5].)

Let κ be an uncountable cardinal number which is strictly larger than the cardinality of some filter basis for $\mathscr{U}$. It is well known that there must be a set $\mathscr{R}$ of bounded semimetrics which defines $\mathscr{U}$ and which has cardinality less than κ. Theorem 3.15.1 of [4] implies that if ${}^*\mathscr{M}$ is κ-saturated, then $(X_0, \mathscr{U}_0)$ is a complete uniform space. (Theorem 3.15.1 has the added assumption that ${}^*\mathscr{M}$ is an ultrapower of $\mathscr{M}$, but this is not necessary. It may be removed by noting that the completeness of $(X_0, \mathscr{U}_0)$ can be proved by considering only Cauchy nets over index sets of cardinality less than κ, and then using Theorem 1.8.3.)

Therefore when ${}^*\mathscr{M}$ is κ-saturated the uniform space $({}^*X, \widetilde{\mathscr{U}})$ is also complete. By Theorem 1.4 this implies that the restriction of $\widetilde{\mathscr{U}}$ to $\mathrm{fin}_{\mathscr{U}}({}^*X)$ defines a complete uniform space. It should also be noted that each set of the form $\{p \mid {}^*f(p)$ is finite if $f \in \mathscr{F}\}$ is closed in the $\widetilde{\mathscr{U}}$-topology when $\mathscr{F}$ is a subset of $C(X, \mathscr{U})$. Therefore each of the nonstandard hulls of [4] is a complete uniform space when ${}^*\mathscr{M}$ is κ-saturated, even when $\mathscr{F}$ may have cardinality $\geqq \kappa$.

2. This section is concerned with the relationship between $\mathrm{fin}_{\mathscr{U}}({}^*X)$ and the set

$$F_0 = \{p \mid {}^*f(p) \text{ is finite for all } f \in C(X, \mathscr{U})\} .$$

As argued in §1, $\pi(F_0)$ is the smallest of the nonstandard hulls of $(X, \mathscr{U})$ constructed in [4]. By Theorem 1.3, $\mathrm{fin}_{\mathscr{U}}({}^*X)$ is a subset of F_0. In fact, the two sets are equal, except in certain circumstances depending on the existence of measurable cardinal numbers. (Corollary 2.5) The principal tool in proving this is the following result.

LEMMA 2.1. *If A is in $\mathscr{U}$ and $x \equiv_A y$ for all $x, y \in X$, then there is a semimetric ρ on X which satisfies*

(i) *the uniformity defined by ρ contains A and is weaker than $\mathscr{U}$, and*

(ii) *for each $p, q \in {}^*X$,*

$$p \equiv_{*A} q \longleftrightarrow {}^*\rho(p, q) \textit{ is finite} .$$

Proof. The proof uses a modification of a construction given in [3]. Let A be in $\mathscr{U}$ and suppose $x \equiv_A y$ holds for all $x, y \in X$. It may be assumed that A is symmetric (replacing A by $A \cap A^{-1}$ if necessary.) Let Z be the set of all the integers. Select a sequence $\{A_n \mid n \in Z\}$ of symmetric sets in $\mathscr{U}$ as follows: (i) $A_0 = A$, (ii) for $n > 0$ define A_n inductively by

$$A_n = (A_{n-1})^3 ,$$

(iii) for $n < 0$ select A_n inductively so that

$$(A_n)^3 \subset A_{n+1} .$$

Then $\{A_n \mid n \in Z\}$ is a chain of sets in $\mathscr{U}$, and it satisfies

$$(A_n)^3 \subset A_{n+1} \quad \text{for all } n \in Z . \tag{2.1}$$

Moreover, since $n \geqq 0$ implies $A_n = (A^3)^n$, it follows that

$$\cup \{A_n \mid n \in Z\} = \cup \{A^{3^n} \mid n \geqq 1\} . \tag{2.2}$$

The assumption that $x \equiv_A y$ holds for every $x, y \in X$ means that the right side of (2.2) is equal to $X \times X$. Therefore a function g on $X \times X$ may be defined by

$$g(x, y) = \begin{cases} 2^n & \text{if } (x, y) \in A_n \sim A_{n-1} \\ 0 & \text{if } (x, y) \in A_n \quad \text{for all } n \in Z . \end{cases}$$

In particular, for $n \geqq 0$

$$g(x, y) \leqq 2^n \longleftrightarrow (x, y) \in A^{3^n}(= A_n) .$$

Passing this to ${}^*\mathscr{M}$, it follows that for any $p, q \in {}^*X$ and $n \in N$

$${}^*g(p, q) \leqq 2^n \longleftrightarrow (p, q) \in ({}^*A)^{3^n} .$$

Therefore, if $p, q \in {}^*X$, then

$${}^*g(p, q) \text{ is finite} \longleftrightarrow p \equiv_{{}^*A} q .$$

The desired semimetric ρ is then defined from g by

$$\rho(x, y) = \inf \left\{ \sum_{i=0}^{n-1} g(x_i, x_{i+1}) \mid x_0, \cdots, x_n \text{ is a sequence in } X, x_0 = x \text{ and } x_n = y \right\} .$$

(That ρ is nonnegative, symmetric and satisfies $\rho(x, x) = 0$ for all x in X follows from the fact that the function g has the same properties. That ρ satisfies the triangle inequality is equally obvious.) The fundamental fact about ρ is the inequality

$$\rho(x, y) \leqq g(x, y) \leqq 2\rho(x, y) \tag{2.3}$$

which holds for all $x, y \in X$. The first inequality follows immediately from the definition. The second is proved by showing that if $x_0, \cdots, x_n$ is a sequence in X,

$$g(x_0, x_n) \leqq 2 \cdot \sum_{i=0}^{n-1} g(x_i, x_{i+1}) . \tag{2.4}$$

The proof of (2.4) is by induction on n, using (2.1). The details are like those in the proof of Theorem 6.7 in [3], and they will be omitted.

Passing the inequality (2.3) to ${}^*\mathscr{M}$, it follows that ${}^*\rho(p, q)$ is finite exactly when ${}^*g(x, y)$ is finite. Therefore, for any $p, q \in {}^*X$

$$p \equiv_{*A} q \leftrightarrow {}^*\rho(p, q) \text{ is finite .}$$

It thus remains only to show that ρ satisfies (i). The definition of g implies that $A = \{(x, y) \mid g(x, y) \leqq 1\}$, and by equation (2.3) it follows that A contains the set $\{(x, y) \mid \rho(x, y) \leqq 1/2\}$. This shows that A is in the uniformity defined by ρ. Finally, for each $n \in Z$

$$A_n = \{(x, y) \mid g(x, y) \leqq 2^n\} \subset \{(x, y) \mid \rho(x, y) \leqq 2^n\} .$$

This shows that the uniformity defined by ρ is weaker than $\mathscr{U}$, and completes the proof.

Throughout the rest of this section let $\mathscr{K}$ denote the set of all cardinal numbers κ which support ω-complete, free ultrafilters. It is well known that if $\mathscr{K}$ is nonempty, then the smallest member κ_0 of $\mathscr{K}$ is actually measurable. (In fact, every ω-complete ultrafilter on κ_0 is $< \kappa_0$-complete.) Moreover, in that case the class $\mathscr{K}$ consists exactly of the cardinal numbers $\geqq \kappa_0$. (There does not seem to be any accepted term designating the members of $\mathscr{K}$. Some authors call them "measurable" but this does not agree with current terminology in set theory.)

Given a set I in $\mathscr{M}$ and an element p of ${}^*I$, let $\mathrm{Fil}(p)$ denote the ultrafilter $\{J \mid J \subset I \text{ and } p \in {}^*J\}$ on I determined by p. ($\mathrm{Fil}_I(p)$ will be used for $\mathrm{Fil}(p)$ if necessary to avoid confusion.) Recall that $\mathrm{Fil}(p)$ is a free ultrafilter if and only if p is not standard.

LEMMA 2.2. *For each* $p \in {}^*I$, $\mathrm{Fil}(p)$ *is* ω*-complete if and only if* ${}^*f(p)$ *is finite for every real valued function* f *on* I.

Proof. Given any real valued function f on I and $n \geqq 1$, define

$$A_n(f) = \{x \mid x \in I \text{ and } |f(x)| \geqq n\} .$$

Then $\{A_n(f) \mid n \geqq 1\}$ is a decreasing chain of subsets of I and the intersection of the chain is empty.

If $p \in {}^*I$ and there exists a real valued function f on I such that ${}^*f(p)$ is infinite, then $p \in {}^*A_n(f)$ for every $n \geqq 1$. That is,

$$\{A_n(f) \mid n \geqq 1\}$$

is contained in $\mathrm{Fil}(p)$. This shows that $\mathrm{Fil}(p)$ is not ω-complete.

Conversely, suppose $\mathrm{Fil}(p)$ is not ω-complete. Then there exists a decreasing chain $\{A_n \mid n \geqq 1\}$ in Fil(p) whose intersection is empty. It may be assumed that $A_1 = I$. Thus a real valued function f may be defined on I by

$$f(x) = \max\{n \mid x \in A_n\} .$$

Evidently $A_n(f) = A_n$ for each $n \geqq 1$. The assumption that $p \in {}^*(A_n)$ for all $n \geqq 1$ implies that $|{}^*f(p)| \geqq n$ for all $n \geqq 1$. That is, ${}^*f(p)$ is infinite.

Let $\mathscr{D}$ be the discrete uniformity on X (that is, $\mathscr{D}$ is the principal filter on $X \times X$ generated by the diagonal set.) Clearly $C(X, \mathscr{D})$ is the set of all real valued functions on X and $\text{fin}_{\mathscr{D}}({}^*X) = {}^*[X]$. Thus Lemma 2.2 says that

$$\text{fin}_{\mathscr{D}}({}^*X) = \{p \mid {}^*f(p) \text{ is finite if } f \in C(X, \mathscr{D})\}$$

if and only if the cardinality of X is not in $\mathscr{K}$.

The next results describe completely the conditions under which an element of F_0 is not $\mathscr{U}$-finite.

THEOREM 2.3. *If $p \in {}^*X$ is not $\mathscr{U}$-finite but ${}^*f(p)$ is finite for every $f \in C(X, \mathscr{U})$, then there exists an element A of $\mathscr{U}$ which satisfies*

$$Y \in Fil_X(p) \rightarrow \textit{the number of } \equiv_A \textit{ equivalence classes which intersect } Y \textit{ is in } \mathscr{K}.$$

Proof. Assume that $p \in {}^*X$ is not $\mathscr{U}$-finite, and that ${}^*f(p)$ is finite whenever $f \in C(X, \mathscr{U})$. There exists a symmetric element A of $\mathscr{U}$ such that $p \equiv_{{}^*A} {}^*x$ is false for every $x \in X$. Let $\{X_a \mid a \in I\}$ be a one-to-one enumeration of the $\equiv_A$ equivalence classes, and let a function c from X to I be defined by

$$c(x) = a \longleftrightarrow x \in X_a .$$

It will be shown first that ${}^*c(p)$ is not a standard element of ${}^*I$. If otherwise, there exists $a \in I$ which satisfies ${}^*a = {}^*c(p)$, and hence $p \in {}^*(X_a)$. Let A_a equal $A \cap (X_a \times X_a)$ and let $\mathscr{U}_a$ be the uniformity obtained by restricting $\mathscr{U}$ to X_a. Since X_a is an $\equiv_A$ equivalence class, $x \equiv_{A_a} y$ holds for every $x, y \in X_a$. By Lemma 2.1 there exists a semimetric ρ on X_a which satisfies (i) the uniformity defined by ρ on X_a contains A_a and is weaker than $\mathscr{U}_a$, and (ii) for any $r, s \in {}^*(X_a)$,

$$r \equiv_{{}^*(A_a)} s \longleftrightarrow {}^*\rho(r, s) \text{ is finite} .$$

Since X_a is an $\equiv_A$ equivalence class, $r \equiv_{{}^*A} s$ is equivalent to $r \equiv_{{}^*(A_a)} s$, for elements r, s of $^*(X_a)$. Thus (ii) implies

(ii′) for any $r, s \in {}^*(X_a)$,

$$r \equiv_{*A} s \longleftrightarrow {}^*\rho(r, s) \text{ is finite .}$$

Let x_0 be a fixed element of X_a and define a function h on X by

$$h(x) = \begin{cases} 0 & \text{if } x \notin Y_a \\ \rho(x_0, x) & \text{if } x \in X_a . \end{cases}$$

Given $\delta > 0$, there exists an element B_a of $\mathscr{U}_a$ which satisfies

$$(x, y) \in B_a \longrightarrow \rho(x, y) < \delta$$

by (i) above. This implies that B_a contains a set of the form $B \cap (X_a \times X_a)$, where B is in $\mathscr{U}$, and it may be assumed that $B \subset A$. If $(x, y) \in B$, then either x and y are both outside X_a, and $h(x) = h(y) = 0$, or $(x, y) \in B_a$. In the latter case

$$|h(x) - h(y)| = |\rho(x_0, x) - \rho(x, y)| \leqq \rho(x, y) < \delta .$$

Therefore, h is an element of $C(X, \mathscr{U})$. This implies that $^*h(p)$ is finite. However, since $p \in {}^*(X_a)$, $^*h(p) = {}^*\rho({}^*x_0, p)$. Thus, by (ii′) above, $p \equiv_{*A} {}^*x_0$ which is a contradiction. This shows that $^*c(p)$ is not a standard element of $^*I$.

Now let Y be any subset of X which satisfies $p \in {}^*Y$, and let $J = c(Y)$. It must be shown that there exists an ω-complete, free ultrafilter on J. If not, then the ultrafilter Fil($^*c(p)$) is not ω-complete. (It is free since $^*c(p)$ is not standard.) In that case, by Lemma 2.2 there exists a real valued function f on J such that $^*f(^*c(p))$ is infinite. Define a function g on X by

$$g(x) = \begin{cases} 0 & \text{if } c(x) \notin J \\ f(c(x)) & \text{if } c(x) \in J . \end{cases}$$

If $(x, y) \in A$, then $x \equiv_A y$ and hence $c(x) = c(y)$. This implies that g is in $C(X, \mathscr{U})$. But $^*g(p) = {}^*f({}^*c(p))$, so that $^*g(p)$ is infinite. This contradiction shows that $\mathrm{Fil}_J({}^*c(p))$ is an ω-complete, free ultrafilter on J, and completes the proof.

THEOREM 2.4. *If $Y \subset X$ and the number of $\equiv_A$ equivalence classes which intersect Y is in $\mathscr{K}$, for some A in $\mathscr{U}$, then there exists an element p of $^*Y$ which is not $\mathscr{U}$-finite but which satisfies: $^*f(p)$ is finite for every $f \in C(X, \mathscr{U})$.*

Proof. Given $A \in \mathscr{U}$ and $Y \subset X$ as stated, there is a subset W of Y which has one element in common with each $\equiv_A$ equivalence class which intersects Y. Moreover, there exists an ω-complete, free ultrafilter on W. Since ${}^*\mathscr{M}$ is an enlargement of $\mathscr{M}$, this means that there is an element p of ${}^*W$ which is not standard and such that $\mathrm{Fil}_W(p)$ is ω-complete. By Lemma 2.2, ${}^*f(p)$ is finite for every real valued function f on W, hence for every f in $C(X, \mathscr{U})$. It thus suffices to show that $p \equiv_{{}^*A} {}^*x$ is false for every x in X. Otherwise, there exist $x \in X$ and $n \geqq 1$ which satisfy $(p, {}^*x) \in {}^*B^n$, where B is $A \cap A^{-1}$. Since $p \in {}^*W$ it follows that for some $w \in W$, $(w, x) \in B^n$. Therefore $(p, {}^*w) \in ({}^*B)^{2n}$. But since p is not standard, this implies that there exists $w' \in W$ such that w' is distinct from w and $(w', w) \in B^{2n}$. That is, $w' \equiv_A w$ and hence W has two elements from the same $\equiv_A$ equivalence class. This contradiction proves that p has the desired properties.

COROLLARY 2.5. *The equality*

$$\mathrm{fin}_{\mathscr{U}}({}^*X) = \{p \mid {}^*f(p) \text{ is finite for all } f \in C(X, \mathscr{U})\}$$

holds if and only if the number of $\equiv_A$ *equivalence classes is not in* $\mathscr{K}$ *for every* $A \in \mathscr{U}$.

In cases where the cardinality assumption of Corollary 2.5 holds (in particular, if there is no ω-complete, free ultrafilter on X) then the smallest nonstandard hull constructed in [4] is also the subspace $\pi(\mathrm{fin}_{\mathscr{U}}({}^*X))$ of $(X_0, \mathscr{U}_0)$. This fact is helpful in determining the elements of this nonstandard hull, since it is usually easier to show that $\mu(p)$ is an element by showing that p is $\mathscr{U}$-finite, and to show that $\mu(p)$ is not an element by exhibiting a function f in $C(X, \mathscr{U})$ such that ${}^*f(p)$ is infinite. (See the examples in §4.)

3. Atsuji [2] has given a condition on $(X, \mathscr{U})$ which is equivalent to the statement that every function in $C(X, \mathscr{U})$ is bounded, and which is closely related to the concepts discussed above. In this section a nonstandard proof is given of a natural generalization of Atsuji's Theorem. (The ideas used in proving this Theorem are also used in §4.)

DEFINITION 3.1. A subset Y of X is *finitely chainable in* $(X, \mathscr{U})$ if, for each $A \in \mathscr{U}$, there exist $y_1, \cdots, y_k$ in Y and $n \geqq 1$ which satisfy

$$Y \subset A^n(y_1) \cup \cdots \cup A^n(y_k) .$$

The uniform space $(X, \mathscr{U})$ is *finitely chainable* [2] if X is finitely chainable in $(X, \mathscr{U})$.

THEOREM 3.2. *For any subset* Y *of* X, Y *is finitely chainable in*

$(X, \mathscr{U})$ if and only if $^*Y \subset \mathrm{fin}_{\mathscr{U}}(^*X)$.

Proof. Suppose Y is finitely chainable in $(X, \mathscr{U})$. Given $A \in \mathscr{U}$, there exist $y_1, \cdots, y_k$ in Y and $n \geqq 1$ which satisfy

$$Y \subset A^n(y_1) \cup \cdots \cup A^n(y_k) .$$

If follows that $^*Y \subset (^*A)^n(^*y_1) \cup \cdots \cup (^*A)^n(^*y_k)$. If A is symmetric, this implies that each element of $^*Y$ is in the same $\equiv_{^*A}$ equivalence class with one of the elements $^*y_1, \cdots, ^*y_k$. Therefore $^*Y$ is contained in $\mathrm{fin}_{\mathscr{U}}(^*X)$.

Conversely, suppose Y is not finitely chainable in $(X, \mathscr{U})$. Thus there exists a symmetric set A in $\mathscr{U}$ such that for any $n \geqq 1$ and $y_1, \cdots, y_k \in Y$, the union $\mathrm{A}^n(y_1) \cup \cdots \cup A^n(y_k)$ does not contain Y. For each $y \in Y$ and $n \geqq 1$ define

$$S(n, y) = \{x \mid x \in Y \quad \text{and} \quad x \notin A^n(y)\} .$$

The assumptions on Y imply that the collection $\{S(n, y)\}$ has the finite intersection property. Since $^*\mathscr{M}$ is an enlargement, there exists $p \in {}^*Y$ which satisfies $p \in {}^*S(n, y)$ for every $y \in Y$ and $n \geqq 1$.

It will be shown that p is not $\mathscr{U}$-finite, thus showing that $^*Y$ is not contained in $\mathrm{fin}_{\mathscr{U}}(^*X)$. Otherwise there exist $x \in X$ and $n \geqq 1$ which satisfy $(p, {}^*x) \in (^*A)^n$. This implies that there exists y in $Y \cap A^n(x)$, and therefore $p \in A^{2n}(y)$. That is, $p \notin {}^*S(2n, y)$, which is a contradiction.

The following result generalizes the theorem due to Atsuji [2] which states that $(X, \mathscr{U})$ is finitely chainable if and only if every function in $C(X, \mathscr{U})$ is bounded.

THEOREM 3.3. *For any subset Y of X, Y is finitely chainable in $(X, \mathscr{U})$ if and only if every function in $C(X, \mathscr{U})$ is bounded on Y.*

Proof. If Y is finitely chainable in $(X, \mathscr{U})$, then by Theorem 3.2 $^*Y \subset \mathrm{fin}_{\mathscr{U}}(^*X)$. For any function f in $C(X, \mathscr{U})$, this implies that $^*Y \subset \{p \mid {}^*f(p)$ is finite$\}$ by Theorem 1.3. Therefore the set

$$\{|{}^*f(p)| \mid p \in {}^*Y\} ,$$

which is internal, has a finite upper bound M in R. But this implies that f is bounded by M on Y. That is, each member of $C(X, \mathscr{U})$ is bounded on Y.

Conversely, suppose each function in $C(X, \mathscr{U})$ is bounded on Y. To show that Y is finitely chainable in $(X, \mathscr{U})$ it suffices to prove $^*Y \subset \mathrm{fin}_{\mathscr{U}}(^*X)$, by Theorem 3.2. If not, then by Theorem 2.3 there must exist an element A of $\mathscr{U}$ such that the number of $\equiv_A$ equivalence classes which intersect Y is in $\mathscr{K}$. In particular there are countably many (distinct) $\equiv_A$ equivalence classes $X_1, \cdots, X_n, \cdots$, each of which intersects Y. The function f defined on X by

$$f(x) = \begin{cases} n & \text{if} \quad x \in X_n \\ 0 & \text{if} \quad x \notin X_n\,, \quad \text{all } n \geqq 1 \end{cases}$$

is therefore unbounded on Y. However, f is constant on $\equiv_A$ equivalence classes, and thus f is in $C(X, \mathscr{U})$. This is a contradiction, and completes the proof.

REMARK. Theorem 3.2 allows us to say exactly when there is a single function f in $C(X, \mathscr{U})$ which satisfies

$$\mathrm{fin}_{\mathscr{U}}({}^*X) = \{p \mid {}^*f(p) \text{ is finite}\}\ .$$

Namely, this equality holds if and only if the sets $\{x \mid |f(x)| \leqq n\}$ (for $n \geqq 1$) are all finitely chainable in $(X, \mathscr{U})$. (The equality holds if and only if $\{p \mid |{}^*f(p)| \leqq n\} \subset \mathrm{fin}_{\mathscr{U}}({}^*X)$ for all $n \geqq 1$ (by Theorem 1.3) if and only if $\{x \mid |f(x)| \leqq n\}$ is finitely chainable in $(X, \mathscr{U})$ for all $n \geqq 1$ (by Theorem 3.2).)

In particular, if $\mathscr{U}$ is the uniformity defined by some metric ρ on X, then the equality

$$\mathrm{fin}_{\mathscr{U}}({}^*X) = \{p \mid {}^*\rho(p, {}^*x) \text{ is finite}\}$$

holds for some (or, equivalently, every) x in X, if and only if

$$\{y \mid \rho(y, x) \leqq n\}$$

is finitely chainable in $(X, \mathscr{U})$ for every $n \geqq 1$.

4. Given a metric ρ on X, Robinson [6] says that p and q are in the same *galaxy* of ${}^*X$ if ${}^*\rho(p, q)$ is finite. Generalizing this idea Luxemburg [4] defines p and q to be in the same galaxy relative to a set $\mathscr{S}$ of semimetrics on X if ${}^*\rho(p, q)$ is finite for every ρ in $\mathscr{S}$. The following definition of the $\mathscr{U}$*-galaxies* of ${}^*X$ arises naturally from the considerations which led to Definition 1.2.

DEFINITION 4.1. If $p, q \in {}^*X$, then p and q are in the same $\mathscr{U}$-*galaxy* if $p \equiv_{\widetilde{\mathscr{U}}} q$.

THEOREM 4.2. *If p and q are in the same $\mathscr{U}$-galaxy and ρ is any semimetric on X which defines a uniformity weaker than $\mathscr{U}$, then ${}^*\rho(p, q)$ is finite.*

Proof. Since ρ defines a uniformity weaker than $\mathscr{U}$ there exists

$A \in \mathscr{U}$ which satisfies

$$(x, y) \in A \longrightarrow \rho(x, y) \leqq 1 .$$

Since p and q are in the same $\mathscr{U}$-galaxy, there is a $^*A$-chain $q_0, \cdots, q_n$ from p to q. Using the triangle inequality for $^*\rho$ yields

$$^*\rho(p, q) \leqq \sum {}^*\rho(q_i, q_{i+1}) \leqq n .$$

Therefore $^*\rho(p, q)$ is finite.

DEFINITION 4.3. A subset Y of X is *chain connected in* $(X, \mathscr{U})$ if $x \equiv_{\mathscr{U}} y$ for every $x, y \in Y$. The uniform space $(X, \mathscr{U})$ is *chain connected* if X is chain connected in $(X, \mathscr{U})$.

THEOREM 4.4. *Let $\mathscr{S}$ be the set of all semimetrics which define weaker uniformities than $\mathscr{U}$ and suppose that Y is chain connected in $(X, \mathscr{U})$. Then for every $p, q \in {}^*Y$: p and q are in the same $\mathscr{U}$-galaxy if and only if $^*\rho(p, q)$ is finite for every ρ in $\mathscr{S}$.*

Proof. Let Y and $\mathscr{S}$ be as stated and assume $p, q \in {}^*Y$. The implication in one direction is contained in Theorem 4.2. Conversely, suppose that $^*\rho(p, q)$ is finite for all ρ in $\mathscr{S}$. To prove that p and q are in the same $\mathscr{U}$-galaxy it is necessary to show that $p \equiv_{^*A} q$ for every symmetric set A in $\mathscr{U}$. Given such an A, the fact that Y is chain connected in $(X, \mathscr{U})$ means that there is an $\equiv_A$ equivalence class W which contains Y. Let $A_W = A \cap (W \times W)$ and let $\mathscr{U}_W$ be the restriction of $\mathscr{U}$ to W. As in the proof of Theorem 2.3, an application of Lemma 2.1 yields a semimetric ρ on W which satisfies (i) the uniformity defined by ρ on W is weaker than $\mathscr{U}_W$, and (ii) for any $r, s \in {}^*W$, $r \equiv_{^*A} s$ if and only if $^*\rho(r, s)$ is finite.

Select w_0 in W and let f be the function defined on X by

$$f(x) = \begin{cases} w_0 & \text{if} \quad x \notin W \\ x & \text{if} \quad x \in W . \end{cases}$$

Then f is constant on $\equiv_A$ equivalence classes so that f is uniformly continuous as a map from $(X, \mathscr{U})$ to $(W, \mathscr{U}_W)$. It follows that the semimetric ρ' defined on X by

$$\rho'(x, y) = \rho(f(x), f(y))$$

defines a weaker uniformity on X than $\mathscr{U}$. By assumption, this means that $^*\rho'(p, q) = {}^*\rho(p, q)$ is finite. Therefore $p \equiv_{^*A} q$ by (ii) above, completing the proof.

COROLLARY 4.5. *If $(X, \mathscr{U})$ is chain connected and $\mathscr{S}$ is the set* *of all semimetrics which define weaker uniformities on X than $\mathscr{U}$, then the $\mathscr{U}$-galaxies form the same partition of $^*X$ as do the galaxies determined by $\mathscr{S}$.*

REMARK. As was noted above, if A is in $\mathscr{U}$, then each $\equiv_A$ equivalence class is open and closed in the $\mathscr{U}$-topology on X. Therefore if X is connected in the $\mathscr{U}$-topology, then $(X, \mathscr{U})$ must be chain connected. Applying the same reasoning to the uniform space $(^*X, \widetilde{\mathscr{U}})$ shows that any subset of $^*X$ which is connected in the $\widetilde{\mathscr{U}}$-topology must be entirely contained in one $\mathscr{U}$-galaxy.

THEOREM 4.6. *If $(X, \mathscr{U})$ is chain connected, then the following conditions are equivalent*:

(i) *There is a semimetric ρ which defines a uniformity weaker than $\mathscr{U}$ and which satisfies: p and q are in the same $\mathscr{U}$-galaxy in $^*X$ if and only if $^*\rho(p, q)$ is finite*:

(ii) *There is an element A_0 of $\mathscr{U}$ which satisfies*:
for each $A \in \mathscr{U}$ there is an $n \geqq 1$ such that $A_0 \subset A^n$.

Proof. (i) $\rightarrow$ (ii): Let ρ be as in (i) and define

$$A_0 = \{(x, y) \mid \rho(x, y) \leqq 1\}$$

as that A_0 is in $\mathscr{U}$. If A_0 does not satisfy (ii), then there is an element A of $\mathscr{U}$ such that for no $n \geqq 1$ does A^n contain A_0. That is, for each $n \geqq 1$ there exists a pair x_n, y_n of elements X which satisfy $\rho(x_n, y_n) \leqq 1$ and $(x_n, y_n) \notin A^n$. Let ω be an infinite member of $^*N$. Then $^*\rho(^*x_\omega, {}^*y_\omega) \leqq 1$, so that by (i) there is a $^*A$-chain $q_0, \cdots, q_n$ from $^*x_\omega$ to $^*y_\omega$. That is, $(^*x_\omega, {}^*y_\omega)$ is an element of $(^*A)^n = {}^*(A^n)$. But since ω is not standard, this means that $(x_k, y_k) \in A^n$ holds for infinitely many values of k in N. This contradicts the choice of the pairs (x_k, y_k) and proves that A_0 satisfies (ii).

(ii) $\rightarrow$ (i): Assume that A_0 satisfies (ii). Then for each A in $\mathscr{U}$, $^*A_0 \subset {}^*A^n$ (for some n depending on A.) Therefore $p \equiv_{^*A_0} q$ implies $p \equiv_{^*A} q$, for every $p, q \in {}^*X$ and every $A \in \mathscr{U}$. Thus the $\equiv_{^*A_0}$ equivalence classes and the $\mathscr{U}$-galaxies are exactly the same. The existence of the semimetric required in (i) now follows, using Lemma 2.1 and the fact that $(X, \mathscr{U})$ is chain connected.

REMARK. Suppose $(X, \mathscr{U})$ is chain connected and $\mathscr{U}$ is defined by a metric ρ_0. If $(X, \mathscr{U})$ satisfies the conditions in Theorem 4.6, then there exists a metric ρ_1 which defines $\mathscr{U}$ and also satisfies: p and q are in the same $\mathscr{U}$-galaxy if and only if $^*\rho_1(p, q)$ is finite. That is, $\mathscr{U}$ can be "remetrized" so that the $\mathscr{U}$-galaxies and the galaxies defined by the metric coincide. To construct ρ_1, simply choose ρ as in 4.6.i and define

$$\rho_1(x, y) = \max\{\rho(x, y), \min(\rho_0(x, y), 1)\} .$$

The following two examples were developed in collaboration with L. C. Moore, and are based on ideas due to him. In each case the uniformity $\mathscr{U}$ is defined by a metric on X. The first example shows that a $\mathscr{U}$-finite point need not be in the same $\mathscr{U}$-galaxy with any standard point, even when $(X, \mathscr{U})$ is complete. The second example shows that even when the original space $(X, \mathscr{U})$ is arcwise connected, the smallest nonstandard hull of $(X, \mathscr{U})$ constructed in [4] need not even be chain connected (or, what is the same, the uniform space obtained by restricting $\widetilde{\mathscr{U}}$ to $\mathrm{fin}_{\mathscr{U}}({}^*X)$ need not be chain connected.)

EXAMPLE 1. In this example X is the set of all pairs $x = (x_1, x_2)$ of positive integers, and $\mathscr{U}$ is the uniformity defined by the metric ρ, where

$$\rho(x, y) = \begin{cases} \left|\dfrac{x_2}{x_1} - \dfrac{y_2}{y_1}\right| + \left|\dfrac{x_1}{x_2} - \dfrac{y_1}{y_2}\right| & \text{if } x_1 = y_1 \\ \left|\dfrac{x_2}{x_1} - \dfrac{y_2}{y_1}\right| + \dfrac{x_1}{x_2} + \dfrac{y_1}{y_2} & \text{if } x_1 \neq y_1 . \end{cases}$$

(The metric ρ is obtained in the following way: for each x in X let $\tilde{x}$ be the sequence $\tilde{x} = (a_0, a_1, a_2, \cdots)$, where

$$a_0 = \frac{x_2}{x_1},\ a_{x_1} = \frac{x_1}{x_2}$$

and all other a_n are 0. The distance $\rho(x, y)$ is then just the l_1 norm of $\tilde{x} - \tilde{y}$ as an element of the linear space of all sequences which have finite support.)

For an element (p, q) of ${}^*X$ to be $\mathscr{U}$-finite, it is necessary (by Lemma 4.2) that ${}^*\rho((1, 1), (p, q))$ be finite. This implies that p/q and q/p are finite elements of ${}^*R$ (or, what is the same, that p/q is finite but not infinitesimal.) Suppose, conversely, that q/p and p/q are finite. It will be shown that the element (p, q) of ${}^*X$ is $\mathscr{U}$-finite. If either p or q is finite, then the other must be. That is, (p, q) is in X. Assume therefore that p and q are both in ${}^*N \sim N$. Given a standard real number $\delta > 0$, a number r in ${}^*N$ may be chosen which satisfies the inequalities

$$r\left[\frac{p}{q^2} + \frac{1}{p}\right] < \delta \leqq (r + 1)\left[\frac{p}{q^2} + \frac{1}{p}\right] . \tag{4.1}$$

For any $k \in N$, the ${}^*\rho$-distance between the elements $(p, q + kr)$ and

$(p, q + kr + r)$ of $^*X$ is equal to

$$\left|\frac{p}{q + kr} - \frac{p}{q + kr + r}\right| + \left|\frac{q + kr}{p} - \frac{q + kr + r}{p}\right|$$

which is bounded above by

$$\frac{rp}{q^2} + \frac{r}{p} < \delta\,.$$

Now choose the smallest s in $^*N$ which satisfies

$$\frac{p}{q + sr} < \frac{\delta}{4}\,.$$

The inequalities (4.1), together with the fact that p/q is finite but not infinitesimal, implies that r/p is finite but not infinitesimal. This shows that s is actually in N, and the sequence $(p, q), (p, p + r), \cdots, (p, q + sr)$ is a δ-chain in $^*X$ with a finite number of steps.

Since p/q and r/p are each finite but not infinitesimal, there are standard integers m, n such that m/n is within $\delta/4$ of

$$\frac{p}{q + sr}$$

and n/m is within $\delta/4$ of the reciprocal

$$\frac{q + sr}{p}\,.$$

It follows that the $^*\rho$-distance between $(p, q + sr)$ and (m, n) is less than δ. This shows that there is a δ-chain from (p, q) to a standard element of $^*X$, for each standard $\delta > 0$. Therefore (p, q) is $\mathscr{U}$-finite, as claimed.

Given a $\mathscr{U}$-pre-nearstandard element (p, q) of $^*X$, p/q must be finite but not infinitesimal, by Theorem 1.4 and the previous argument. If (p, q) is not standard, then p is infinite. Therefore every standard element of $^*X$ is a $^*\rho$-distance of at least p/q away from (p, q). But p/q is not infinitesimal, so this is a contradiction. Therefore $\mathrm{pns}_{\mathscr{U}}(^*X)$ is simply the set of all standard elements of $^*X$. This shows that $(X, \mathscr{U})$ is complete and that the $\mathscr{U}$-topology on X is discrete.

Also, there are elements of $\mathrm{fin}_{\mathscr{U}}(^*X)$ which are not standard (for example, (ω, ω) is one whenever ω is infinite.) Since the $\mathscr{U}$-topology is discrete, each standard element of $^*X$ comprises a $\mathscr{U}$-galaxy by itself. Thus there are $\mathscr{U}$-finite points which are not in the same $\mathscr{U}$-galaxy with any standard point. In fact it can be shown, by an argument similar to the one used to characterize $\mathrm{fin}_{\mathscr{U}}(^*X)$, that the set A of non-standard, $\mathscr{U}$-finite elements of $^*X$ comprises a single $\mathscr{U}$-galaxy.

Note that if ω and ω' are distinct elements of ${}^*N$, then the ${}^*\rho$-distance between (ω, ω) and (ω', ω') is 2. Thus the image under π of $\text{fin}_{\mathscr{U}}({}^*X)$ in X_0 has at least as many elements as ${}^*N$. Since the enlargement ${}^*\mathscr{M}$ can be chosen to make the cardinality of ${}^*N$ arbitrarily large, this shows that the various nonstandard hulls of $(X, \mathscr{U})$ constructed in [4] depend on ${}^*\mathscr{M}$ as well as on $(X, \mathscr{U})$.

EXAMPLE 2. In this example X consists of a countable set of points $\{a_n \mid n \geqq 0\}$, together with certain arcs joining a_0 to the other distinguished points. For each $n \geqq 1$ the arcs joining a_0 to a_n form n subspaces $X(n, 1), \cdots, X(n, n)$, each two of which have only the elements a_0 and a_n in common. Moreover, if $1 \leqq j \leqq m, 1 \leqq k \leqq n$ and $n \neq m$, then $X(m, j)$ and $X(n, k)$ have only the element a_0 in common.

The metric ρ which defines $\mathscr{U}$ is given first on the subspaces

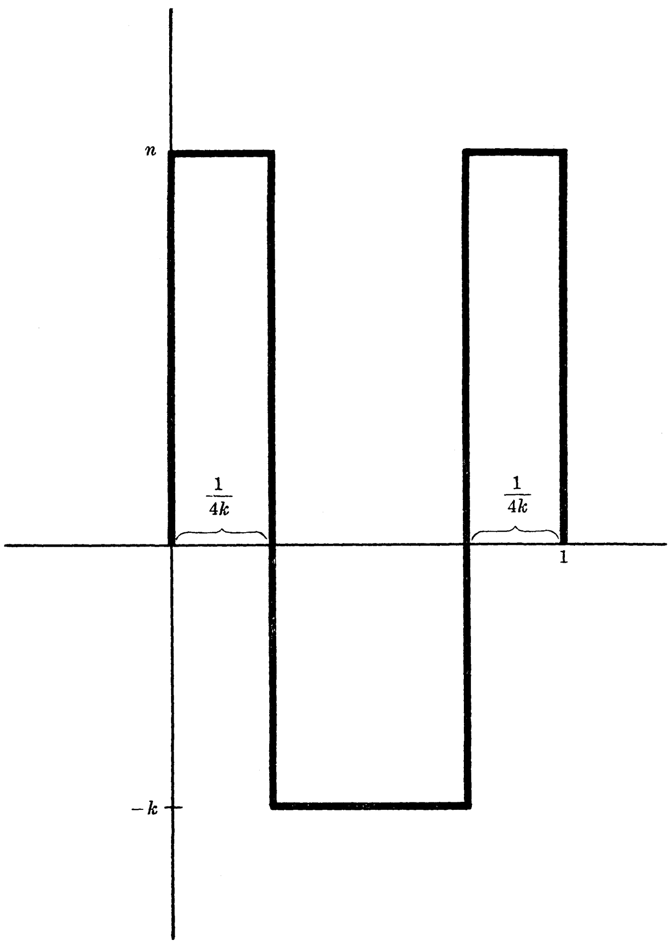

Figure 1.

$X(n, k)$ and then extended to all of X. For a given $1 \leqq k \leqq n$, ρ is defined on $X(n, k)$ in such a way as to make the subspace $X(n, k)$ isometric to the subspace of the Euclidean plane pictured in Figure 1. (This subspace consists of the seven line segments obtained by joining adjacent pairs of points in the sequence: $(0, 0)$, $(0, n)$, $(1/4k, n)$, $(1/4k, -k)$, $(1 - 1/4k, -k)$, $(1 - 1/4k, n)$, $(1, n)$, $(1, 0)$.) In each case the isometry is assumed to take a_0 to $(0, 0)$ and to take a_n to $(1, 0)$. Therefore there is a function f from X into R^2 whose restriction to a given subspace $X(n, k)$ yields the assumed isometry.

The metric ρ is defined on the rest of $X \times X$ as follows. Let $x, y \in X$ and suppose $\rho(x, y)$ is not yet defined. That is, $x \in X(m, j)$ and $y \in X(n, k)$, where the pairs (m, j) and (n, k) are distinct. If $n \neq m$, then $\rho(x, y)$ is defined to be $\rho(x, a_0) + \rho(a_0, y)$. If $n = m$, then $\rho(x, y)$ is defined to be

$$\min \{\rho(x, a_0) + \rho(a_0, y), \rho(x, a_n) + \rho(a_n, y)\} .$$

It will be shown first that for every $x, y \in X$ and $n \geqq 0$

$$\rho(x, y) \leqq \rho(x, a_n) + \rho(a_n, y) . \tag{4.2}$$

If $n = 0$ or if x and y are both elements of the union $X(n, 1) \cup \cdots \cup X(n, n)$, then (4.2) is obvious. Thus assume $x \in X(m, j)$ where $m \neq n$. In that case

$$\rho(x, a_n) = \rho(x, a_0) + \rho(a_0, a_n) . \tag{4.3}$$

If $y \in X(n, k)$ for some k, then

$$\rho(a_0, y) \leqq \rho(a_0, a_n) + \rho(a_n, y) .$$

This inequality, together with (4.3) and (4.2) when $n = 0$, proves (4.2) in the present case. By the symmetry of ρ, it remains only to consider the case when $y \in X(m, j)$ for some $m \neq n$. In that case

$$\rho(a_n, y) = \rho(a_0, a_n) + \rho(a_0, y) .$$

This, together with (4.3), shows that $\rho(x, a_n) + \rho(a_n, y)$ is bounded below by $\rho(x, a_0) + \rho(a_0, y)$. An application of (4.2) when $n = 0$ completes the proof.

To prove the triangle inequality in general, let $x, y, z \in X$ and assume $z \in X(n, k)$. If neither x nor y is in $X(n, k)$, then

$$\rho(x, z) + \rho(z, y) = \rho(x, b) + \rho(b, z) + \rho(z, c) + \rho(c, y) ,$$

where b and c are each either a_0 or a_n. Since b, c, z are all in $X(n, k)$, $\rho(b, c) \leqq \rho(b, z) + \rho(z, c)$. This, together with two uses of (4.2), proves the triangle inequality

$$(4.4) \qquad \rho(x, y) \leqq \rho(x, z) + \rho(z, y)$$

in this case. By the symmetry of ρ it remains only to consider the case when $x \in X(n, k)$ but $y \notin X(n, k)$. Then

$$\rho(x, z) + \rho(z, y) = \rho(x, z) + \rho(z, b) + \rho(b, y)$$

for $b = a_0$ or a_n. The triangle inequality applied to x, z, b (which are all elements of $X(n, k)$) together with one use of (4.2) yields (4.4) in this case, and completes the proof. Thus ρ is a metric on X.

In passing to consideration of $^*X$, note that there are subsets $^*X(\omega, \omega')$ of $^*X$ which correspond to the subsets $X(n, k)$ of X. In particular, for each p in $^*X$ there is at least one pair (ω, ω') which satisfies $1 \leqq \omega' \leqq \omega$ and $p \in {}^*X(\omega, \omega')$. Moreover, if p and q are both elements of $^*X(\omega, \omega')$, then $^*\rho(p, q) = {}^*d(^*f(p), {}^*f(q))$, where $^*d$ is the extension of the Euclidean metric to $^*R^2$.

The analysis of $\text{fin}_{\mathscr{U}}(^*X)$ depends on the following fact.

LEMMA. *If p is $\mathscr{U}$-finite and $p \in {}^*X(\omega, \omega')$, where $\omega' \in {}^*N$, $\omega \in {}^*N \sim N$ and $\omega' \leqq \omega$, then the standard part of the first coordinate of $^*f(p)$ is either* 0 *or* 1.

Proof. Let p, ω and ω' be as stated. Since p is $\mathscr{U}$-finite, $^*\rho(^*a_0, p)$ must be finite, by Theorem 4.2. Therefore $^*f(p)$ is a finite distance from $(0, 0)$ in $^*R^2$, so that the second coordinate of $^*f(p)$ must be finite. If ω' is infinite, this implies that the first coordinate of $^*f(p)$ must be one of the numbers: $0, 1/4\omega', 1 - 1/\omega'$, or 1. These numbers have standard part 0 or 1.

Thus it may be assumed that ω' is finite. Let A be the set of all q in $^*X(\omega, \omega')$ such that $^*f(q)$ has an infinite second coordinate or has a first coordinate different from 0 or 1. Then if $q \in A$ but $r \in {}^*X \sim A$, it follows that $^*\rho(q, r) > 1/8\omega'$. In addition, A has no standard element (since the only standard element of $^*X(\omega, \omega')$ is $^*a_0$.) Thus there is no $1/8\omega'$-chain from any element of A to any standard element. This shows that no element of A is $\mathscr{U}$-finite. Thus, in this case, $^*f(p)$ actually has first coordinate equal to 0 or 1.

Now consider the point $^*a_\omega$, where ω is any infinite element of $^*N$. For each standard k in N there is a $1/k$-chain from $^*a_\omega$ to $^*a_0$ in $^*X(\omega, k)$ (since the three segments in $^*f(^*X(\omega, k))$ which lie below the horizontal axis in $^*R^2$ have finite length when k is finite.) Therefore $^*a_\omega$ is $\mathscr{U}$-finite. However, there cannot be any sequence $q_0, \cdots, q_n$ of $\mathscr{U}$-finite points which satisfy: $q_0 = {}^*a_\omega$, $q_n = {}^*a_0$ and $^*\rho(q_i, q_{i+1}) < 1/2$ for all $i = 0, \cdots, n-1$. Otherwise, by the Lemma, there must exist $i, 0 \leqq i \leqq n-1$, such that the first coordinates of $^*f(q_i)$ and

${}^*f(q_{i+1})$ have standard parts 1 and 0 respectively. But this would imply ${}^*\rho(q_i, q_{i+1}) > 1/2$, which is a contradiction.

Thus it has been shown that the uniform space resulting from restricting $\widetilde{\mathscr{U}}$ to $\mathrm{fin}_{\mathscr{U}}({}^*X)$ is not chain connected. The example is completed by noting that since X is essentially a union of polygonal paths from a_0, the space $(X, \mathscr{U})$ is arcwise connected.

REMARK. The last example shows that restriction of $\widetilde{\mathscr{U}}$ to a $\mathscr{U}$-galaxy need not yield even a chain connected uniform space. In some cases, however, the $\mathscr{U}$-galaxies are exactly the connected components of ${}^*X$ under the $\widetilde{\mathscr{U}}$-topology. For example, let $\mathscr{U}$ be a uniformity defined by a metric ρ on X which satisfies the following convexity assumption: for each $x, y \in X$ and $\delta > 0$ there exists $z \in X$ which satisfies

$$\left|\rho(x, z) - \frac{1}{2}\rho(x, y)\right| < \delta$$

$$\left|\rho(y, z) - \frac{1}{2}\rho(x, y)\right| < \delta .$$

(This is equivalent to saying that the completion of (X, ρ) is metrically convex, and it is true, for example, when X is a normed linear space.)

Passing to ${}^*\mathscr{M}$, and letting δ be infinitesimal, it follows that for each $p, q \in {}^*X$ there exists $r \in {}^*X$ which satisfies

$$\mathrm{st}({}^*\rho(p, r)) = \mathrm{st}({}^*\rho(q, r)) = \frac{1}{2}\mathrm{st}({}^*\rho(p, q)) .$$

Used repeatedly, this shows that whenever ${}^*\rho(p, q)$ is finite, p and q must be in the same $\mathscr{U}$-galaxy. Moreover, the restriction of $\widetilde{\mathscr{U}}$ to any $\mathscr{U}$-galaxy yields a chain connected space. On such a galaxy Y the restriction of $\widetilde{\mathscr{U}}$ is defined by the semimetric $\tilde{\rho}$ defined by $\tilde{\rho}(p, q) = \mathrm{st}({}^*(p, q))$, as discussed in §1. If ${}^*\mathscr{M}$ is $\aleph_1$-saturated, then $(Y, \tilde{\rho})$ is a complete semimetric space, by the Remark following Theorem 1.4 (and the fact that $\mathscr{U}$-galaxies are closed in the $\mathscr{U}$-topology.) In fact, it has been shown above that $(Y, \tilde{\rho})$ is convex. As is well known, these facts imply that Y is arcwise connected in the $\tilde{\rho}$-topology. It follows, using the Remark following Corollary 4.5, that the $\mathscr{U}$-galaxies are identical to the connected components of ${}^*X$ in the $\widetilde{\mathscr{U}}$-topology.

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Alexander (Smbat) Abian, *The use of mitotic ordinals in cardinal arithmetic* 1
Helen Elizabeth. Adams, *Filtrations and valuations on rings* 7
Benno Artmann, *Geometric aspects of primary lattices* 15
Marilyn Breen, *Determining a polytope by Radon partitions* 27
David S. Browder, *Derived algebras in L_1 of a compact group* 39
Aiden A. Bruen, *Unimbeddable nets of small deficiency* 51
Michael Howard Clapp and Raymond Frank Dickman, *Unicoherent compactifications* 55
Heron S. Collins and Robert A. Fontenot, *Approximate identities and the strict topology* 63
R. J. Gazik, *Convergence in spaces of subsets* 81
Joan Geramita, *Automorphisms on cylindrical semigroups* 93
Kenneth R. Goodearl, *Distributing tensor product over direct product* 107
Julien O. Hennefeld, *The non-conjugacy of certain algebras of operators* 111
C. Ward Henson, *The nonstandard hulls of a uniform space* 115
M. Jeanette Huebener, *Complementation in the lattice of regular topologies* 139
Dennis Lee Johnson, *The diophantine problem $Y^2 - X^3 = A$ in a polynomial ring* 151
Albert Joseph Karam, *Strong Lie ideals* 157
Soon-Kyu Kim, *On low dimensional minimal sets* 171
Thomas Latimer Kriete, III and Marvin Rosenblum, *A Phragmén-Lindelöf theorem with applications to $\mathcal{M}(u, v)$ functions* 175
William A. Lampe, *Notes on related structures of a universal algebra* 189
Theodore Windle Palmer, *The reducing ideal is a radical* 207
Kulumani M. Rangaswamy and N. Vanaja, *Quasi projectives in abelian and module categories* 221
Ghulam M. Shah, *On the univalence of some analytic functions* 239
Joseph Earl Valentine and Stanley G. Wayment, *Criteria for Banach spaces* 251
Jerry Eugene Vaughan, *Linearly stratifiable spaces* 253
Zbigniew Zielezny, *On spaces of distributions strongly regular with respect to partial differential operators* 267