THE DIOPHANTINE PROBLEM $y^2 - x^3 = A$ IN A POLYNOMIAL RING

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Let $C[z]$ be the ring of polynomials in $z$ with complex coefficients; we consider the equation $Y^2 - X^3 = A$, with $A \in C[z]$ given, and seek solutions of this with $X, Y \in C[z]$ i.e. we treat the equation as a "polynomial diophantine" problem. We show that when $A$ is of degree 5 or 6 and has no multiple roots, then there are exactly 240 solutions $(X, Y)$ to the problem with $\deg X \leq 2$ and $\deg Y \leq 3$.

It is possible that, $A$ being of degree 6, solutions $(X, Y)$ exist with $\deg X > 2$ or $\deg Y > 3$. We "normalize" the problem so as to remove these from our consideration, and give the following definitions: if $A$ is any polynomial of degree $d$, we shall permit its formal degree to be any integer divisible by 6 and greater or equal to $d$. Given $A$ of formal degree $6k$, we require the solutions $X, Y$ of the equation to be of formal degrees $2k, 3k$ resp., i.e. $\deg X \leq 2k, \deg Y \leq 3k$. This problem will be called the problem of order $k$. The restriction on the degrees of $X, Y$ causes no loss in generality, for if $k$ is chosen large enough, it will exceed $1/2 \deg X$ and $1/3 \deg Y$. Furthermore, the classification by $k$ has a natural geometric interpretation. We confine our attention to the problem of order 1. The order restriction enables us to projectivize the equation to an equation of degree $6k$, with $\deg A = 6k, \deg X = 2k, \deg Y = 3k$.

Suppose then that $A$ has formal degree 6, and $(X, Y)$ is a solution of proper formal degree, $\deg X \leq 2, \deg Y \leq 3$. The projective curve $K: w^3 - 3Xw + 2Y = 0$ has the $z$-discriminant $Y^2 - X^3 = A$, so the function $z: K \to S^2$ (proj. line) has its branches among the roots of $A$, for finite $z$. At $z = \infty$ we introduce $\tilde{z} = 1/z, \tilde{w} = w/z = \tilde{z}w$ and get

$$\tilde{z}^2w^3 - 3\tilde{z}^3X\left(\frac{1}{\tilde{z}}\right)w + 2\tilde{z}^3Y\left(\frac{1}{\tilde{z}}\right) = 0 :$$

If $X = a_0z^2 + \cdots, Y = b_0z^3 + \cdots$, then

$$F = \tilde{w}^3 - 3(a_0 + a_1\tilde{z} + a_2\tilde{z}^2)\tilde{w} + 2(b_0 + b_1\tilde{z} + \cdots) = 0$$

and

$$\frac{\partial F}{\partial \tilde{w}}3\tilde{w}^2 - 3(a_0 + \cdots).$$

Now at $\tilde{z} = 0$ (i.e. $z = \infty$) $z$ has a branch point if and only if $\partial F/\partial \tilde{w} = 0$;
i.e. we must have
\[ \bar{w}^3 - 3a_0 \bar{w} + 2b_0 = 0 \]
and
\[ 3\bar{w}^3 - 3a_0 = 0 \]
which is true if and only if \( \Delta = -a_0^2 + b_0^3 = 0 \) i.e. if and only if \( \deg A < 6 \). Hence if \( \deg A < 6 \), we put a \textit{“formal root”} of \( A \) at \( \infty \) with multiplicity \( 6 - \deg A \).

We now assume the roots of \( A \) to be \textit{distinct}. This entails \( \deg A = 5 \) or \( 6 \), with no multiple (finite) roots. The roots will be called \( z_1, \ldots, z_s \). Note that if either \( X \) or \( Y \) were zero at \( z_i \), the other would also be, since \( A \) is zero there (for the case \( z_i = \infty \) just imagine the projective form of \( Y^2 - X^3 = A \); the statement then reads that \( \deg A < 6 \) and if \( \deg Y < 3 \) then \( \deg X < 2 \) and conversely). Hence \( A \) would have at least a \textit{double} zero at \( z_i \), (or at \( \infty : \deg A \leq 4 \)) contrary to hypothesis. Hence \( X, Y \neq 0 \) at \( z_i \), and \( \deg X = 2 \) or \( \deg Y = 3 \). Away from a branch point we may write locally:

\[
\begin{align*}
w_0 &= \omega^{\frac{3}{2}} (\sqrt{-Y + \sqrt{A}} + \sqrt{-Y - \sqrt{A}}) \\
w_1 &= \omega^{\frac{3}{2}} (\sqrt{-Y + \sqrt{A}} + \omega^2 \sqrt{-Y - \sqrt{A}}) \\
w_2 &= \omega^2 \sqrt{-Y + \sqrt{A}} + \omega^\frac{3}{2} \sqrt{-Y - \sqrt{A}}
\end{align*}
\]

for proper choice of the roots; as we go around \( z_i \), \( \sqrt{A} \) changes to \( -\sqrt{A} \), and we get a root permutation \( w_0 \leftrightarrow w_1, w_1 \leftrightarrow w_2 \). Thus the branching number \( b_i \) at \( z_i \) is 1, and the total branching is 6, so the genus is \( g = b/2 - r + 1 = 1 \), i.e. \( K \) is a torus.

We should also prove that \( K \) is irreducible; but if \( K \) were reducible, factoring as \((w - \alpha)(w^3 + \alpha w + \beta)\) (where \( \alpha, \beta \) are polynomials in \( z \) by Gauss’s lemma) i.e., we have \( 3X = \alpha^2 - \beta \) and \( 2Y = -\alpha \beta \), and \( A = Y^2 - X^3 = 4\beta^3 + 15\alpha^2 \beta^2 + 12\alpha^4 \beta - 4\alpha^6 = -(\alpha^2 - 4\beta)(2\alpha^2 + \beta)^2 \).

It is easy to see that \( \deg \alpha \leq 1 \), \( \deg \beta \leq 2 \), and hence \( \deg (\alpha^2 - 4\beta) \leq 2 \). Since \( \deg A \geq 5 \) we see that \( \deg (2\alpha^2 + \beta) \geq 1 \), whence \( A \) has double roots, contrary to hypothesis.

Thus, any solution \( X, Y \) gives us an elliptic curve \( K \) represented as a 3-sheeted branched covering of \( S^2 \) with branch points at \( z_i \), where \( z: K \rightarrow S^2 \) is an elliptic function of degree 3. Furthermore, \( w \) is also a function on \( K \), and its poles are among those of \( z \), and of order \( \leq \) the order of the \( z \)-poles: for expanding \( w \) at \( z = \infty \) we get

\[
w_i = \omega^{\frac{3}{2}} \left(-b_0 z^3 + \cdots + \sqrt{(b_0 - a_0^2)z^6} + \cdots + \omega^3 \sqrt{\text{etc.}} \right)
\]
i.e.

\[ w_1 = \left( \omega^3 \sqrt[3]{-b_0 + \sqrt{A}} + \omega^2 \sqrt[3]{-b_0 - \sqrt{A}} \right) + \text{lower powers of } z \]

i.e. the order of \( w \) is \( \leq \) order of \( z \) at all places \( z = \infty \). (Clearly \( w \) has no other poles). Note also that the sum \( \Sigma w_i \) of the three values of \( w \) over any \( z \) is zero.

Now suppose conversely that we are given a branched covering of \( S^2 \) with 6 simple branch points at the roots of \( A \); we then have an elliptic curve \( K \) and a meromorphic function \( z: K \to S^2 \) with 3 poles (one of which is double if a branch point is at \( \infty \)) at places \( k_1, k_2, k_3 \). Now the set of meromorphic functions \( w \) on \( K \) whose poles are among the \( k_i \) form a vector space \( V \) of dimension 3. Given any such \( w \), the sum \( w_0 + w_1 + w_2 \) of its 3 values over any \( z \) gives us a function which is:

1. finite for finite \( z \)
2. of order \( \leq \) the order of \( z \) at \( z = \infty \)
3. symmetric in the sheets, so rational in \( z \).

Hence \( \Sigma w \), must be linear in \( z \): \( \Sigma w_i = a_w z + b_w \), where \( a_w \) and \( b_w \) are constants depending on \( w \). Note that \( a_w \) and \( b_w \) are clearly complex-linear in \( w \), i.e. \( a, b: V \to C \) are linear maps. Furthermore, since both \( w = 1 \) and \( w = z \) are in \( V \) we have \( a \) and \( b \) are linearly independent: for

\[
\begin{align*}
a(1) &= 0 & a(z) &= 3 \\
b(1) &= 3 & b(z) &= 0
\end{align*}
\]

and so \( a_w = 0, b_w = 0 \) defines a one dimensional subspace of \( V \) i.e. \( a \neq 0 \), defined up to a constant multiple, of degree \( \leq 3 \), with its poles among those of \( z \), and with \( \Sigma w_i = 0 \). Hence \( w \) satisfies some equation

\[ w^3 - 3Pw + 2Q = 0, \text{ with } P & \text{ & } Q \text{ rational in } z; \]

but

\[ -3P = w_1 w_2 + w_2 w_3 + w_3 w_1 \text{ is finite for } z \text{ finite; } \]

hence \( P \) is a polynomial; also its degree is \( \leq 2 \) since the order of \( w \), is \( \leq \) that of \( z \) at \( \infty \). Likewise \( Q \) is a polynomial of degree \( \leq 3 \) in \( z \). Finally \( w \) is not rational in \( z \) since if it were, it would actually be linear, \( w = az + b \), and then

\[ \Sigma w_i = 3w = 3az + 3b = 0, \text{ i.e. } w \equiv 0 . \]

Hence \( w^3 - 3Pw + 2Q = 0 \) is irreducible, and thus defines the curve \( K \). Because of this, we must have the branch points as roots of the
discriminant $Q^2 - P^3 (\neq 0)$; i.e. $A \mid Q^2 - P^3$; $\deg Q^2 - P^3 \leq 6$, and is $<6$ if and only if as we have seen previously, $\infty$ is a branch point of $K$; in the latter case we also have $\deg A = 5$, and so in every case we have $\deg (Q^2 - P^3) = \deg A$, i.e. $A = k(Q^2 - P^3)$ for some constant $k \neq 0$. If now we replace $w$ by $w/\alpha (\alpha \in C)$, we replace $P$ by $P/\alpha^3$ and $Q$ by $Q/\alpha^3$ and $Q^2 - P^3$ by $(Q^2 - P^3)/\alpha^6$; Hence we may choose a scale factor $\alpha$, determined up to a 6th root of unity, and a rescaled $w$ such that $Q^2 - P^3 = A$, i.e. $(P, Q)$ is a solution. Thus we have shown that any 3 sheeted covering of $S^2$ with simple branches at $A = 0$ gives us exactly 6 solutions to the problem (These 6 solutions are distinct since two could be equal if and only if if $P$ or $Q = 0$, which is impossible). Furthermore, if we have two different such branched coverings $K_1, K_2$, then the corresponding solutions $(P_1, Q_1), (P_2, Q_2)$ must be distinct, since the data $(P, Q)$ actually define $K$.

Thus the only remaining problem is to enumerate the different coverings possible.

We choose a base point $g \in S^2$, distinct from the roots $z_i,$ and loops $p_t, (t = 1, \ldots, 6)$ encircling the roots $z_i$ acting as free generators of the fundamental group $\pi_1(S^2 - \bigcup z_i)$, subject only to the relation $p_1 \cdots p_6 = \text{identity}$. Choosing a numbering $1, 2, 3$ of the sheets over $q$, each $p_t$ determines a permutation $\pi_t$ (in $S_3$) of the sheets, and these completely determine the surface. Since the branches are all simple, these permutations must be transpositions: $(12), (23)$ or $(31)$. Also not all the $\pi_t$ can be equal, for then two sheets over $q$ would remain unconnected from the third. If we choose $\pi_1, \ldots, \pi_6$ arbitrarily then $\pi_6$ is determined by $\pi_1 \pi_2 \cdots \pi_6 = e$. Note however that $\pi_1, \ldots, \pi_5$ may not be chosen all equal, since $\pi_6$ would also be same by virtue of the relation. Hence we may choose $\pi_1, \ldots, \pi_5$ in $3^5 = 3$ $- 3$ ways, obtaining all possible coverings of the required nature. Two such choices $\pi_t, \pi'_t$ give the same covering if and only if they differ by a renumbering of the sheets over $q$, i.e. if and only if $\pi'_t = g \pi_t g^{-1}$ for some $g \in S_3$. Since at least two different transpositions occur among the $\pi_t$, conjugation by the elements of $S_3$ produces exactly 6 different equivalent choices of $\pi_t$; hence the total number of different surfaces is $(3^5 - 3)/6 = (3^4 - 1)/2 = 40$. Remembering that to each such surface there are 6 solutions, we have:

**Theorem.** If $A$ is a polynomial of degree 5 or 6 without multiple roots, then there are exactly 240 distinct solutions of the equation $Y^2 - X^3 = A$ in polynomials $X, Y$ for which $\deg X \leq 2$, $\deg Y \leq 3$.

It should be pointed out that, in principle at least, the determination of the solutions $(X, Y)$ for a given $A$ could be solved by classical elimination theory. For example, if $X = a_2 x^2 + a_1 x + a_0$ and
\( Y = b_0 z^2 + b_1 z + b_3 \) is a solution to \( Y^2 - X^3 = A = \alpha_0 z^6 + \ldots + \alpha_9 \), then treating the \( a_i \) and \( b_j \) as unknowns, formal manipulation and the equating of coefficients gives us 7 polynomial equations in 7 unknowns which presumably (assuming independence) gives a finite set of solutions for the unknowns \( a_i, b_j \). This also shows us that the \( a_i \) and \( b_j \) are algebraic over the field of the \( \alpha_k \). In practice, however, this elimination would probably not be computationally feasible.

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