STRONG LIE IDEALS

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$R$ is 2-torsion free semiprime with $2R = R$. A Lie ideal, $U$, of $R$-strong if $aua \in U$ for all $a \in R$, $u \in U$. One shows that $U$ contains a nonzero two-sided ideal of $R$. If $R$ has an involution, $\ast$, (with skew-symmetric elements $K$) a Lie ideal, $U$, of $K$ is $K$-strong if $kuk \in U$ for all $k \in K$, $u \in U$. It is shown that if $R$ is simple with characteristic not 2 and either the center, $Z$, is zero or the dimension of $R$ over the center is greater than 4, then $U = K$. If $R$ is a topological annihilator ring with continuous involution and if $U$ is closed $K$-strong Lie ideal, $U = C \cap K$ where $C$ is a closed two-sided ideal of $R$. A Lie ideal, $U$, of $K$ is $HK$-strong if $\phi(x) \phi(u) \in U$ for all $x \in X, u \in U$. A result similar to the above result for $K$-strong Lie ideals can be shown. Let $R$ be a simple ring with involution such that $Z = \{0\}$ or the dimension of $R$ over $Z$ is greater than 4. Let $\phi$ be a nonzero additive map from $R$ into a ring $A$ such that the subring of $A$ generated by $\{\phi(x) : x \in R\}$ is a noncommutative, 2-torsion free prime ring. Suppose $\phi(x) \phi(y) - \phi(y^\ast) \phi(x^\ast) = \phi(x) \phi(y) - \phi(y) \phi(x^\ast)$ for all $x, y \in R$. As an application of the above theory, $\phi$ is shown to be an associative isomorphism.

1. Introduction. $R$ will denote a semiprime ring such that $2R = R$ and if $2r = 0$, then $r = 0$. We call the latter property 2-torsion free. $Z$ will denote the center of $R$. If $R$ has an involution, $\ast$, defined on it, $S$ and $K$ will be the set of symmetric and skew-symmetric elements respectively. The Lie and Jordan products are $[x, y] = xy - yx$ and $x \ast y = xy + yx$ for any $x, y \in R$. If $X, Y \subseteq R$, $[X, Y]$ will denote the additive subgroup generated by the set $\{[x, y] : x \in X$ and $y \in Y\}$. An additive subgroup, $U$, of $R$ is a Lie ideal of $R$ if $[U, R] \subseteq U$. If $R$ has an involution, we can similarly define a Lie ideal of $K$.

This paper is concerned with the study of different classes of Lie ideals of both $R$ and $K$. A Lie ideal, $U$, of $R$ is said to be $R$-strong if $aua \in U$ for all $a \in R$, $u \in U$. If $U$ is a Lie ideal of $K$, $U$ is $K$-(HK-) strong if $kuk \in U$ ($u^\ast \in U$) for all $k \in K$, $u \in U$.

In the classical theory of the Lie structure of an associative ring, the main theorem [6; Th. 1.3] states: if $R$ is simple and $U$ is a Lie ideal of $R$, either $U \subseteq Z$ or $[R, R] \subseteq U$. We attempt to develop some criteria for differentiating between Lie ideals of $R$ containing $[R, R]$ and $R$ itself. Similar criteria are developed for Lie ideals of $K$. We
will have occasion to use the following results of Herstein [6; pp 1, 5, 10, and 28]:

(i) \( R \) has no one-sided ideals which are nil of bounded index;
(ii) If \( a \in R \) is such that \([a, [a, x]] = 0 \) for all \( x \in R \), then \( a \in Z \);
(iii) Let \( R \) be simple with involution and characteristic not 2. If \( Z = (0) \) or the dimension of \( R \) over \( Z \) is greater than 4, then \( R = \bar{S} = \bar{K} \) where \( \bar{S} \) and \( \bar{K} \) are the subrings of \( R \) generated by \( S \) and \( K \) respectively.

If \( X \subseteq R \), \( \mathcal{R}(X) = \{a \in R; Xa = (0)\} \) and \( \mathcal{I}(X) = \{a \in R; aX = (0)\} \). The next two lemmas are analogs of a results of Baxter [3; p. 2].

**Lemma 1.1.** If \( U \) is a Lie ideal of \( R \) such that \( u^2 = 0 \) for all \( u \in U \), then \( U = (0) \).

**Proof.** Let \( u \in U \), \( a \in R \). As \([u, a] \in U \), \([u, a]^2 = 0 \). Therefore, \( uuaua = u[u, a]^2 = 0 \) and \( uR \) is nil of bounded index. By the previously mentioned results, \( uR = (0) \). But \( R \) is semiprime, so \( \mathcal{I}(R) = (0) \). Thus \( u = 0 \).

**Lemma 1.2.** Let \( K \) have an involution, * \( \). If \( U \) is a Lie ideal of \( K \) such that \( u^2 = 0 \) for all \( u \in U \), then \( U = (0) \).

**Proof.** Let \( u, v \in U \), then \( 0 = (u + v)^2 - u^2 - v^2 = uv + vu \). As \([u, v] \in U \), \( 2uv \in U \). Since \( 2R = R \), \([uv, K] \subseteq U \). Thus, for each \( k \in K \), \( u[uv, k] = 0 \), and so, even more \( v[uv, k] = 0 \). Since \( u \) and \( v \) anti-commute, expansion of this expression yields \( uvkw = 0 \). Now \( svws \in K \) for any \( s \in S \). So \( wfs \in K \) for any \( s \in S \). Therefore, given \( a \in R \), \( a = s + k \) where \( s \in S \) and \( k \in K \), then \( (uv)a(uw)(uv) = 0 \). We conclude that \( uvR \) is nil of bounded index. This guarantees \( uv = 0 \) for all \( u, v \in U \). Now, \(-wku = u[w, k] = 0 \). Repeating the previous arguments for \( s \in S \) and \( k \in K \), we conclude that \( u = 0 \).

2. \( R \)-strong Lie ideals. In this section \( U \) will denote an \( R \)-strong Lie ideal. If \( a, b \in R \) and \( u, v \in U \), one can easily show that the following are in \( U \): \( abu + uba, abu + uba \), and \( uab \). We associate with \( U \) the set \( B_u = \{b \in R; a \cdot b \in U \text{ for all } a \in R\} \). This set is a Lie ideal of \( R \) and \( w \in B_u \) for all \( u \in U \). The latter can be seen by observing that if we set \( b = u \) above, we obtain \( aw + w^2a \in U \). Thus, via Lemma 1.1, \( U \neq (0) \) implies \( B_u \neq (0) \).

**Lemma 2.1.**

(i) \( B_u \) is an \( R \)-strong Lie ideal
(ii) \( u^2xu^2 \in B_U \cap U \) for all \( u \in U, x \in R \).

**Proof.**

(i) We know that \( B_U \) is a Lie ideal of \( R \). For arbitrary \( x, y \in R \) and \( b \in B_U, \) \([x \circ b, y]\) and \([x, b] \circ y\) are in \( U \). Thus, by adding and subtracting these terms, we have that \( xby - ybx \) and \( bxy - yxb \) are in \( U \). Now,

\[
x(yby) + (yby)x = \{(xy)by - yb(xy)\} \\
\quad + \{yb(yx) - (yx)by\} + \{y(bx + xb)y\}.
\]

Since each term on the right is in \( U \), \( x(yby) + (yby)x \in U \) and \( B_U \) is \( R \)-strong.

(ii) As \( u^2 \in B_U \), \( u^2xu^2 \in B_U \). Moreover, \( u^2xu^2 = u(uxu)u \in U \). Therefore, \( u^2xu^2 \in B_U \cap U \).

**Theorem 2.2.** \( C = B_U \cap U \) is a nonzero two-sided ideal.

**Proof.** Note that \( C \) is an \( R \)-strong Lie ideal. Also \( C \neq (0) \) since if this were so, for each \( u \in U \), \( u^2R \) would be a nil right ideal of bounded index. Let \( b \in C \) and \( x, y \in R; xb + bx \in U \). Also

\[
(xb + bx)y + y(xb + bx) = \{x(by - yb) - (by - yb)x\} \\
\quad + \{(yx)b + b(yx)\} \\
\quad + \{b(xy) + (yx)b\}.
\]

As each term on the right is in \( U \), \( (x \circ b) \circ y \in U \). Thus, \( x \circ b \in C \). Now \( 2xb = x \circ b + [x, b] \in C \). Since \( 2R = R, Rb \subseteq C \). Similarly, \( bR \subseteq C \). Thus \( C \) is a nonzero two-sided ideal of \( R \).

We note that \( C \) is the same as the set \( L_U = \{u \in U; uR \subseteq U \} \) which was used by Zuev [10] in his study of the Lie structure of \( R \).

**Corollary 2.3.** If \( R \) is simple and \( U \neq (0), U = R \).

This corollary allows us to study the \( R \)-strong structure of the ring as it relates to minimal idempotents of \( R \). If \( e \) is a minimal idempotent, \( eUe \) is an \( eRe \)-strong Lie ideal. Since \( eRe \) is a division ring either \( eUe = (0) \) or \( eUe = eRe \). We use this fact to prove the next theorem.

**Theorem 2.4.** Let \( H \) be the homogeneous component of the socle which contains \( e \). Then either \( H \subseteq U \) or \( H \subseteq L(U) \cap \mathcal{R}(U) \).
Proof. Recall that $H$ is a simple ring. The theorem then follows by considering $H \cap U$.

**Corollary 2.5.** If $R$ is completely reducible, $U$ is the direct sum of the homogeneous components of the socle which it contains.

This result is similar to that of Kaplansky [7].

Assume that $R$ has the additional properties that $3R = R$ and $R$ is 3-torsion free. Let $W$ be any Lie ideal of $R$ such that $u^2 \in W$ for all $u \in W$. Let $u, v \in W$. We have $\alpha = 2(u^3 + u^2 v) = (u + v)^3 - 2u^3 \in W$, $\beta = [v, [v, u]] \in W$ and $\gamma = [v, u] \in W$. From these we have: $3(u^3 + u^2 v) = \alpha + \beta \in W$, $6uv = \alpha - 2\beta \in W$, $6\alpha u = \alpha + 3\gamma \in W$, and $6uv^3 = \alpha - 3\gamma \in W$. We now have enough to show a result similar to Theorem 2.2.

**Theorem 2.6.** Let $W$ be a Lie ideal of $R$ such that $u^2 \in W$ for all $u \in W$. Then either $W$ contains a nonzero two-sided ideal or $u^2 \in Z$ for all $u \in W$.

Proof. Let $a, b \in R$ and $u \in W$. Since $2a[a, u] = [a, [a, u]] + [a^2, u] \in W$ and $2R = R$, $a[a, u] \in W$. Linearization of this expression yields $a[b, u] + b[a, u] \in W$. Upon multiplication by 6 and replacement of $b$ by $v^2$, we obtain $6[a[v^2, u] + v^2[a, u]] \in W$. As $6v^2[a, u] \in W$, $6a[v^2, u] \in W$ and this implies $a[v^2, u] \in W$. It immediately follows that $R[v^2, u]R \subseteq W$ of $R[v^2, u]R \neq (0)$, we are finished.

Assume $R[v^2, u]R = (0)$ for all $u, v \in W$, then $[v^2, u]R$ is a nilpotent ideal, hence $[v^2, u] = 0$ for all $u, v \in W$. As $[v^2, a] = [v, va + av] \in W$, $[v^2, [v^2, a]] = 0$. Thus, by remarks in §1, $v^2 \in Z$.

The obvious corollary holds in the case where $R$ is simple.

3. $K$-strong Lie ideals. Let $R$ have an involution, $*$, and let $U$ be a $K$-strong Lie ideal. For $u, v \in U$ and $k, l \in K$, the following are in $U$: $kul + lku$, $klu + ulk$, and $uku$. We associate with $U$ the set $B(U) = \{b \in R: ba - a^*b^* \in U$ for all $a \in R\}$. This is the analog for Lie ideals of the set which Baxter [3] uses in his study of the Jordan structure of $S$. When there is no confusion, we write $B(U) = B$.

**Lemma 3.1.**
(i) $B$ is a right ideal
(ii) $KB \subseteq B$
(iii) $u^2 \in B$ for all $u \in U$
Proof. The proofs of (i) and (ii) are straightforward. We prove (iii). As \( u \in U \), \( u^2a - a^*(u^2)^* = u^2a - a^*u^2 \). Then
\[
u^2a - a^*u^2 = \{[u, ua + a^*u]\} + \{u(a - a^*)u\} \cdot
\]
The first \{ \} is in \( U \) since \( ua + a^*u \in K \). The second \{ \} is in \( U \) since \((a - a^*) \in K \) and \( U \) is \( K \)-strong.

Now from Lemma 1.2, we know that if \( U \neq (0) \), \( B \neq (0) \).
For \( u \in U \), \( k \in K \), \( a \in R \) and \( b, c \in B \), direct computation leads to the following facts: \( ae^*b \in B \), \( c^*b \in B \), \( bkb^* \in B \cap U \), and \( uku \in B \cap U \).

**Theorem 3.2.** Let \( R \) be a simple ring with characteristic not 2. If \( Z = (0) \) or the dimension of \( R \) over \( Z \) is greater than 4, then \( U = K \).

The proof of this essentially the same as the proof of Theorem 7 [3; p. 7]. As a corollary, we include a slight extension of a theorem of Baxter [1; p. 74].

**Corollary 3.3.** Let \( R \) be as in the theorem. \( S^cK \), the additive subgroup of \( R \) generated by the set \{\( s^c = k; s \in S \) and \( k \in K \)\} is a \( K \)-strong Lie ideal and hence \( S^cK = K \).

The following results on \( \mathcal{L}(B) \) and \( \mathcal{L}(U) \) will be particularly useful in the next section.

**Theorem 3.4.** \( \mathcal{L}(B) \) is a self-adjoint two-sided ideal.

Proof. The proof is similar to the proof of Theorem 2 [4; p. 563].

Knowing that \( \mathcal{L}(B) \) is a two-sided ideal, we can easily show that \( \mathcal{L}(B) \cap B = (0) \) and \( \mathcal{L}(B) \cap U = (0) \).

**Theorem 3.5.** \( \mathcal{L}(U \cap B) = \mathcal{L}(U) \).

Proof. It suffices to show \( \mathcal{L}(U \cap B) \subseteq \mathcal{L}(U) \). Let \( b \in U \cap B \), \( k \in K \), and \( x \in \mathcal{L}(U \cap B) \). As \( bk - kb \in U \cap B \), \( xkb = -x(bk - kb) = 0 \). Thus, \( \mathcal{L}(U \cap B)K \subseteq \mathcal{L}(U \cap B) \).

Let \( u \in U \), then \( u^i \in U \cap B \) so \( xu^i = 0 \). Since \( u^i k + ku^i \in U \cap B \), \( xu^i ku = x(u^i k + ku^i)u = 0 \). Let \( a \in R; ua^* + au \in K \), therefore \( 0 = xu^i(ua^* + au)u = xu^iau^i \). If we replace \( a \) by \( ax \), we have \( (xu^ia)^i = 0 \). That is, \( xu^iR \) is a nil ideal of bounded index and so \( xu^i = 0 \) for any...
Upon linearization we obtain

\( (3.5.1) \quad xuv = -xvu \quad \text{for} \quad u, v \in U. \)

Since \( xuvu = -xvu^2 = 0 \) and \( vkv \in U \), we have

\( (3.5.2) \quad xu(vkv)u = 0. \)

Let \( w \in U \) and \( s \in S; xuv(\omega s + s\omega)v = 0. \) Replacement of \( x \) by \( xw, \) expansion of the expression, and repeated use of \( (3.5.1) \) yields,

\( 0 = xwuvuswvu. \) By repeated use of \( (3.5.1) \) and finally \( (3.5.2), \) we have \( xwvukwvu = 0. \) Given \( a \in R, \) since \( a = s + k \) for some \( s \in S \) and \( k \in K, \) we can write \( xuvuaawvu = 0. \) Replace \( a \) by \( ax \) to obtain

\( xwv(a\omega)xwvu = 0. \)

Then \( xwvuR \) is a nilpotent ideal so \( xwvu = 0. \) As \( uk - ku \in U. \)

\( (3.5.3) \quad 0 = xwv(uk - ku) = -xwvku. \)

Let \( s \in S; xwv(\omega s + s\omega)v = 0. \) Moreover, since \( xwvusv = 0, \) we have \( xwvsuv = 0. \) From \( (3.5.3), \) \( xwvkwv = 0. \) As before, this implies

\( (3.5.4) \quad xwv = 0. \)

Immediately, \( 0 = xw(vk - kv) = -xwkv. \) In particular \( xwkv = 0. \)

Since \( sws \in K, xw(sws)w = 0. \) Also, \( 0 = xw(swk - ksw)s = xswskw. \)

Again, letting \( a = s + k \) for \( a \in R, \) we have \( xwawaw = 0. \) Via the same techniques, \( xw = 0 \) or \( x \in \mathcal{L}(U). \) Hence, \( \mathcal{L}(U \cap B) \subseteq \mathcal{L}(U). \)

4. Topological annihilator rings. In this section \( R \) will denote a semiprime topological annihilator ring with continuous involution such that \( 2R = R \) and if \( \{2x_n\} \) is a net convergent to \( 0 \in R, \) then \( \{x_n\} \) is also a net convergent to \( 0. \) \( U \) will be a closed \( K \)-strong Lie ideal.

The definition of an annihilator ring says that \( \mathcal{L}(R) = \mathcal{R}(R) = (0) \) and if \( A(L) \) is a closed right (left) ideal not equal to \( R, \) then \( \mathcal{L}(A) \neq (0), \mathcal{R}(L) \neq (0). \) So if \( B = B(U), H = \mathcal{L}(B) \oplus B \) is dense in \( R. \) It is easy to show that if \( U \) is closed, \( B \) is closed. If \( X \subseteq R, Cl(X) \) will denote to topological closure of \( X. \)

The following results have proofs which are similar to those given by Baxter in [3; p. 4].

**Theorem 4.1.**

(i) \( B \) is a two-sided ideal

(ii) \( \{\mathcal{L}(B)\}^* = \mathcal{L}(B^*) \)
(iii) \( B = B^* \)
(iv) \( U \subseteq B \).

For any \( x, y \in R \), we adopt the following notation: \( (x, y)_L = xy - y^*x^* \) and \( (x, y)_r = xy + y^*x^* \). Using the results of the last theorem, we prove

**Theorem 4.2.** \( U = C \cap K \) where \( C \) is a closed two-sided ideal.

**Proof.** Let \( V \) be the additive subgroup of \( S \) generated by the set \( \{(u, a)_L; u \in U \) and \( a \in R\} \). If we show \((U + V)\) to be a right ideal, since it is self-adjoint, it must be a two-sided ideal.

Since \( U \subseteq B \), \((u, a)_L = ua + a^*u \in U \) for all \( a \in R \). Let \( c \in R \), then \( auc + c^*ua^* = ((a, u)_L, c)_L + (u, (a^*c))_L \in V \)

and \( auc - c^*ua^* = ((a, u)_L, c)_r + (u, (a^*c))_r \in V \).

Since \( 2R = R \), for any \( 2d \in R \), \( u(2d) = (u, d)_L + (u, d)_r \in U + V \). Thus, \( UR \subseteq U + V \). Also,

\[
(u, a)_L(2d) = (u, ad)_L + \{a^*u(-d) + (-d)^*ua\} + (u, ad)_r
\]

and \( VR \subseteq U + V \). Thus \((U + V)R \subseteq U + V \), or the desired conclusion that \((U + V)\) is a two-sided ideal.

Let \( C = Cl(U + V) \). \( U \subseteq C \cap K \). Let \( x \in C \cap K \). There exists a net \( \{u_a + v_a\} \) such that \( u_a + v_a \rightarrow x \) where \( u_a \in U \) and \( v_a \in V \). As \( x \in K \), \((u_a + v_a)^* = -u_a + v_a \rightarrow x^* = -x \). Thus \( u_a \rightarrow -x \). By subtracting these expressions we obtain \( 2u_a \rightarrow 2x \). Therefore \( u_a \rightarrow x \). Since \( u_a \in U \) and \( U \) is closed, \( x \in U \). Hence, \( C \cap K = U \).

5. **HK-strong Lie ideals.** In this section \( U \) is an \( HK \)-strong Lie ideal. \( R \) will have those properties as described in §1. We further assume that \( 3R = R \) and \( R \) is 3-torsion free. \( HK \)-strong Lie ideals were defined by Herstein [5]. Baxter [2; p. 393] showed that if \( R \) is simple with either \( Z = (0) \) or the dimension of \( R \) over \( Z \) greater than 16 with \( U \nsubseteq Z \), then \( U = K \). This can be refined by using entirely different techniques.

As before, we associate with \( U \) the set \( B(U) \). \( B \) is a right ideal and \( KB \subseteq B \). However, we are no longer guaranteed that \( u^2 \in B \) for all \( u \in U \). Hence the possibility that \( B = (0) \) does arise.

**Lemma 5.1.** Let \( u, v, w \in U \) and \( k \in K \).
(i) $\beta vuv \in U$
(ii) $6(uvw + wvu) \in U$
(iii) $uv(wk - kw) + (wk - kw)vu \in U$
(iv) $\omega v - v\omega \in B$.

Proof. (i) and (ii) follow in a manner similar to the remarks preceding Theorem 2.6. (iii) holds because $2R = R$ and $3R = R$. Finally (iv) can be verified in the same manner as [6; p. 33].

If $B = (0)$, $\omega^2 v - vv^2 = 0$ for all $u, v \in U$. Let $s \in S$. Since $[\omega^2, s] = [u, us + sv] \in U$, $[\omega^2, [\omega^2, s]] = 0$. Also, if $k \in K$, $[\omega^2, [u, k]] = 0$, therefore $[\omega^2, [u, k]] = 0$. We know that this implies

$$[\omega^2, [\omega^2, a]] = 0$$

for all $a \in R$. Thus, from the first section, $\omega^2 \in Z$.

We now refine Baxter's theorem.

**Theorem 5.2.** Let $R$ be simple and of characteristic not 2 or 3. If $Z = (0)$ or the dimension of $R$ over $Z$ is greater than 4, then either $U = K$ or $U^2 \in Z$ for all $u \in U$.

Proof. If $B \neq (0)$, by the remarks preceding Lemmas 1.1 and 5.1 we have the alternative result.

We relate the notations of $K$- and $HK$-strong Lie ideals by calling attention to the fact that if $U$ is $HK$-strong, $B \cap U$ is $K$-strong. Clearly $B \cap U$ is a Lie ideal. If $k \in K$ and $u \in B \cap U$, then $[k, [k, u]] = k^2 u + uk^2 - 2kuk$. Now, $k^2 u + uk^2 \in B \cap U$ by the definition of $B$. Therefore, $kuk \in B \cap U$ since $2R = R$.

Herstein [6; p. 28] has shown that $K^2$ is a Lie ideal of $R$. It is not difficult to show that if $U$ is an $HK$-strong Lie ideal such that $B \cap U = (0)$, then any $x \in B \cap S$ commutes with every element in $K^2$. We need this fact to prove

**Theorem 5.3.** Let $R$ be a topological annihilator ring with properties as described in the previous section. Assume also that $3R = R$ and if $\{\lambda_x\}$ is a net convergent to $0 \in R$, $\{x_x\}$ is a net converging to 0. If $U$ is a closed $HK$-strong Lie ideal, then either $\omega^2 \in Z$ for all $u \in U$, $U$ contains the intersection of $K$ with a closed two-sided ideal, or $\omega^2 v - vv^2 \in \mathcal{L}(K)$ for all $u, v \in U$.

Proof. If $B = (0)$, $\omega^2 \in Z$. Assume $B \neq (0)$ and $B \cap U \neq (0)$.
Since $B \cap U$ is $K$-strong, Theorem 4.2 guarantees the existence of $C$, a closed two-sided ideal, such that $C \cap K = B \cap U \subseteq U$.

Let $B \cap U = (0)$. As $K^2$ is a Lie ideal of $R$, $t = w^*v - vu^* \in K^2 \cap (B \cap S)$. Also, by the remarks preceding the theorem, $[t, [t, a]] = 0$ for all $a \in R$. Therefore, $t \in Z$. Let $k \in K$; $tk + kt = tk - k^*t^* \in B \cap U = (0)$. Therefore, $tk = 0$ or $t = u^*v - vu^* \in \mathcal{L}(K)$.

7. Application. We now parallel some of the results obtained by Small [9] and Riedlinger [8] concerning an additive mapping whose multiplicative property is defined relative to an involution. Let $R$ be a simple ring with involution, $*$, and characteristic not 2 such that $Z = (0)$ or the dimension of $R$ over $Z$ is greater than 4. Notice that under these conditions $R$ cannot be commutative. Let $\phi$ be a nonzero additive mapping from $R$ into an associative ring $A$. Assume $R' = \phi(R)$, the subring of $A$ generated by $\{\phi(r) : r \in R\}$, is a noncommutative prime ring such that $2R = R'$ and $R'$ is 2-torsion free. Let $\phi$ enjoy the further property that $\phi(xy - y^*x^*) = \phi(x)\phi(y) - \phi(y^*)\phi(x^*)$ for all $x, y \in R$. We would like to show that $\phi$ is an associative isomorphism. We will have occasion to use the following theorem by Baxter [1; p. 73] which was slightly modified by Herstein [6; p. 29]: If $R$ is such that $2R = R$ and $K = R$, then $S = K \circ K$, the additive subgroup of $R$ generated by the set $\{kl : k, l \in K\}$.

The next lemma is the key to much of what follows.

**Lemma 6.1.** $\ker \phi \cap K = (0)$.

**Proof.** We show $\ker \phi \cap K$ to be a $K$-strong Lie ideal. Let $l \in \ker \phi \cap K$ and $k \in K$. Since $\phi([k, l]) = [\phi(k), \phi(l)] = 0$, $\ker \phi \cap K$ is a Lie ideal of $K$. Thus $[k, [k, l]] \in \ker \phi \cap K$ or $\phi([k, [k, l]]) = (0)$. We may expand this and obtain

$$\phi([k, [k, l]]) = \phi(k^2l - 2klk + lk^2) = \phi(k^2l + lk^2) - 2\phi(klk) = 0.$$ 

Now, $\phi(k^2l + lk^2) = \phi(k^2)\phi(l) + \phi(l)\phi(k^2) = 0$. Therefore $\phi(klk) = 0$ or $\ker \phi \cap K$ is a $K$-strong Lie ideal.

By Theorem 3.2 either $\ker \phi \cap K = (0)$ or $\ker \phi \cap K = K$. Assume the latter. For $s, t \in S$ and $k, l \in K$, $[\phi(k), \phi(l)] = 0$ and $[\phi(k), \phi(s)] = 0$. As $[s, t] \in K$, $0 = \phi([s, t]) = [\phi(s), \phi(t)]$. Because any $x \in R$ can be written as $x = s + k$, we have $[\phi(x), \phi(y)] = 0$ for all $x, y \in R$. Therefore, $R'$ is commutative, a contradiction. Thus $\ker \phi \cap K = (0)$.

Let $x, y \in R$, then
\[ \phi((xy - y^*x^*)x^* - x(xy - y^*x^*)^*) = \{\phi(x)\phi(y) - \phi(y^*)\phi(x^*)\phi(x^*) \]
\[ - \phi(x)\{\phi(y^*)\phi(x^*) - \phi(x)\phi(y)\} \].

If \( y = s \), we can write,

\[ \phi((xy - y^*x^*)x^* - x(y^*x^* - xy)) = \phi(x^2s - sx^*) = \phi(x^2)\phi(s) - \phi(s)\phi(x^2) \]

and

\[ \{\phi(x)\phi(y) - \phi(y^*)\phi(x^*)\phi(x^*) - \phi(x)\{\phi(y^*)\phi(x^*) - \phi(x)\phi(y)\} \]
\[ = (\phi(x))^2\phi(s) - \phi(s)(\phi(x^2))^2 \).

This can be rewritten as

(6.1.1) \[ \{\phi(x^2) - (\phi(x))^2\}\phi(s) = \phi(s)[\phi(x^2) - (\phi(x))^2] \]

for all \( x \in R \) and \( s \in S \).

**Lemma 6.2.** For any \( s \in S \) and

\[ k \in K, \{\phi(s^2) - (\phi(s))^2\} \text{ and } \{\phi(k^2) - (\phi(k))^2\} \]

are in \( Z' \), the center of \( R' \).

**Proof.** Set \( u \) equal to either \( \{\phi(s^2) - (\phi(s))^2\} \) or \( \{\phi(k^2) - (\phi(k))^2\} \). From (6.1.1), \( \phi(s)u = u\phi(s) \). Consider \( 2\phi(t_1, t_2, \ldots, t_n) \) where \( t_1 \in S \). We write

\[ 2\phi(t_1, t_2, \ldots, t_n) = \phi(t_1, t_2, \ldots, t_n + t_n \ldots t_n) \]
\[ + \phi(t_1, t_2, \ldots, t_n - t_n \ldots t_n) \]
\[ = \phi(t_1, t_2, \ldots, t_n + t_n \ldots t_n) \]
\[ + \{\phi(t_1, t_2, \ldots, t_n - t_n \ldots t_n)\phi(t_1)\} \].

By induction, \( u \) commutes with \( \phi(t_1, t_2, \ldots, t_n) \) and \( \phi(t_1, t_2, \ldots, t_n) \). Since \( t_1, t_2, \ldots, t_n + t_n \ldots t_n, \in S \), \( u \) commutes with \( \phi(t_1, t_2, \ldots, t_n + t_n \ldots t_n) \). Thus, \( [u, \phi(t_1, t_2, \ldots, t_n)] = 0 \). That is, \( u \) commutes with \( \phi(S) \). But under our hypothesis, \( S = R \). Hence, \( u \) commutes with \( \phi(R) \) and, indeed, with \( \phi(R) = R' \). Thus \( u \in Z' \).

**Corollary 6.3.**

(6.3.1) \[ \{\phi(x^2) - (\phi(x))^2\} \in Z' \text{ for all } x \in R \].

**Proof.** If \( x = s + k \), since \( \phi(sk + ks) - \{\phi(s)\phi(k) + \phi(k)\phi(s)\} = 0 \),

\[ \{\phi(x^2) - (\phi(x))^2\} = \{\phi(s^2) - (\phi(s))^2\} + \{\phi(k^2) - (\phi(k))^2\} \in Z' \].

Let \( x, y \in R \). If we linearize (6.3.1), we obtain
In particular, for \( s, t \in S \), \( \phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} \in Z' \). Also, \( \phi(st - ts) - \{\phi(s)\phi(t) - \phi(t)\phi(s)\} = 0 \). Addition of these terms leads us to \( \phi(st) - \phi(s)\phi(t) \in Z' \). Similarly, we can show that \( \phi(kl) - \phi(k)\phi(l) \in Z' \) for \( k, l \in K \).

For notational convenience, let \( \phi(xy) - \phi(x)\phi(y) = x^y \) for any \( x, y \in R \). Thus the above says that \( s^t, k^l \in Z' \). The definition of \( \phi \) tells us that \( s^k = -k^s \). Also, we have \( k^l = l^k \). Since these terms are in \( Z' \), \( \phi(s)k^l - l^k\phi(s) = 0 \). Upon expansion and rearrangement of terms, we obtain

\[
(6.4.1) \quad \{\phi(skl - lks)\} - \{\phi(s)\phi(k)\phi(l) - \phi(l)\phi(k)\phi(s)\} = 0 .
\]

We can write \( \phi(sk - ks) = \phi(sk)\phi(l) - \phi(l)\phi(ks) \). Replacement of this in (6.4.1) and rearrangement of terms yields

\[
s^k\phi(l) - \phi(l)k^s = 0
\]

or

\[
(6.4.2) \quad s^k\phi(l) = \phi(l)k^s = -\phi(l)s^k .
\]

Let \( m \in K \), by the above, there exists \( z' \in Z' \) such that \( \phi(ml + lm) = \phi(m)\phi(l) + \phi(l)\phi(m) + z' \). As a result of (6.4.2) and this relation we have that \( s^k\phi(ml + lm) = \phi(ml + lm)s^k \) or \( s^k \) commutes with \( \phi(KoK) \). The preliminary remarks guarantee for us that \( KoK=S \). So, using an argument exactly like that in Lemma 6.2, we can show

\[
(6.4.3) \quad s^k \in Z' .
\]

**Lemma 6.4.** \( x^y \in Z' \) for all \( x, y \in R \).

The proof follows directly from (6.4.3) and the remarks immediately after Corollary 6.3.

**Corollary 6.5.** If \( Z' = (0) \), \( \phi \) is an associative isomorphism.

**Proof.** As \( Z' = (0) \), \( \phi(xy) - \phi(x)\phi(y) = 0 \). Thus \( \phi \) is an associative homomorphism and \( \overline{\phi(R)} = \phi(R) \). Moreover, since \( R \) is simple, \( \phi \) is an associative isomorphism.

Let \( z'(\neq 0) \in Z' \). Since \( \mathcal{A}(z') = \{r' \in R': r'z' = 0\} \) is a two-sided ideal in a prime ring, \( \mathcal{A}(z') = (0) \).

**Lemma 6.6.** \( s^k = s^b = 0 \) for all \( s \in S, k \in K \).

**Proof.** From (6.4.2) \( s^k\phi(l) = -\phi(l)s^k \) for \( l \in K \). By Lemma 6.4, \( s^k \in Z' \).
Suppose $s_k \neq 0$. By the remarks preceding the lemma, we have $\phi(l) = 0$, that is, $K \subseteq \ker \phi$. Therefore, $\ker \phi \cap K = K$, a contradiction. We conclude that $0 = s^k = -k^s$.

**Corollary 6.7.** $\phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for $x, y \in R$.

We have shown that when $Z' = (0)$, then $\phi$ is an associative isomorphism. Therefore, the following theorem is proved except when $Z' \neq (0)$.

**Theorem 6.8.** $\phi$ is an associative isomorphism.

**Proof.** From Lemma 6.6, $(s^k)^k - \phi(s)s^k = 0$. Expansion and rearrangement of terms leads to $(s^k)^k - \phi(s)s^k = (s^k)^k - s^k\phi(k) = 0$. From Lemma 6.4, $(s^k)^k \in Z'$ so $s^k\phi(k) \in Z'$. Let $l \in K$. There exist $z'_1$ and $z'_2$ in $Z'$ such that $s^k\phi(k) = z'_1$ and $s^k\phi(l) = z'_2$. As $s^k \in Z'$, we can write $0 = [z'_1, z'_2] = (s^k)[\phi(k), \phi(l)]$ for all $s \in S$ and $k, l \in K$.

If $(s^k)^2 \neq 0$ for some $s \in S$, then by the remarks preceding Lemma 6.6, $[\phi(k), \phi(l)] = 0$ for all $k, l \in K$. As $\phi([k, l]) = [\phi(k), \phi(l)] = 0$, we conclude that $[K, K] \subseteq \ker \phi \cap K = (0)$. This implies $K = R$ is commutative, a contradiction. So $(s^k)^2 = 0$ for all $s \in S$. Since the center of a prime ring is an integral domain, $s^2 = 0$. Upon linearization of this expression, we obtain $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} = 0$ for all $t, s \in S$.

For $k, l \in K, k' \in Z'$. Thus there exists $z'_2 \in Z'$ such that $k^l - z'_2 = 0$. Since $k^2 \in S, (k^2)^l = 0$ and so $(k^2)^l - \phi(k)[k^l - z'_2] = 0$. Expansion and rearrangement of terms leads to $k^{kl} - k^l\phi(l) + z'_2\phi(k) = 0$. In view of Lemma 6.4, there is an element $z'_2 \in Z'$ such that $k^{kl} = z'_2$. Therefore we can always find $z'_1, z'_2 \in Z'$ such that $k^l\phi(l) = z'_1\phi(k) + z'_2$ where $k$ is an arbitrary fixed element in $K$ and $l$ is allowed to vary in $K$. Note that $k^l \in Z'$. For $m \in K$, there are $z'_1$ and $z'_2$ in $Z'$ such that $k^m\phi(m) = z'_1\phi(k) + z'_2$. Thus $0 = (k^m)[\phi(l), \phi(m)] = [k^l\phi(l), k^m\phi(m)]$. Via the same argument as above, we can show $k^l = 0$. Linearization of this expression leads to $\phi(kl + lk) - \{\phi(k)\phi(l) + \phi(l)\phi(k)\} = 0$. Now, using this fact and the fact that both $\phi(sk) - \phi(s)\phi(k) = 0$ and $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} = 0$, we have that

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$$

for all $x, y \in R$. From Corollary 6.7, we know

$$\phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x).$$

Addition of these two expressions yields $\phi(xy) = \phi(x)\phi(y)$ or that $\phi$ is an associative homomorphism. Therefore, $\phi(R) = \phi(R)$ and $\ker \phi = (0)$.
since $R$ is simple. Hence $\phi$ is an associative isomorphism.

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