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R is 2-torsion free semiprime with $2R = R$. A Lie ideal, U, of R-strong if $aua \in U$ for all $a \in R, u \in U$. One shows that U contains a nonzero two-sided ideal of R. If R has an involution, $*$, (with skew-symmetric elements K) a Lie ideal, U, of K is K-strong if $kuk \in U$ for all $k \in K$, $u \in U$. It is shown that if R is simple with characteristic not 2 and either the center, Z, is zero or the dimension of R over the center is greater than 4, then $U = K$. If R is a topological annihilator ring with continuous involution and if U is closed K-strong Lie ideal, $U = C \cap K$ where C is a closed two-sided ideal of R. A Lie ideal, U, of K is HK-strong if $u^3 \in U$ for all $u \in U$. A result similar to the above result for K-strong Lie ideals can be shown. Let R be a simple ring with involution such that $Z = (0)$ or the dimension of R over Z is greater than 4. Let ϕ be a nonzero additive map from R into a ring A such that the subring of A generated by $\{\phi(x): x \in R\}$ is a noncommutative, 2-torsion free prime ring. Suppose $\phi(xy - y^*x^*) = \phi(x)\phi(y) - \phi(y^*)\phi(x^*)$ for all $x, y \in R$. As an application of the above theory, ϕ is shown to be an associative isomorphism.

1. Introduction. R will denote a semiprime ring such that $2R = R$ and if $2r = 0$, then $r = 0$. We call the latter property 2-torsion free. Z will denote the center of R. If R has an involution, $*$, defined on it, S and K will be the set of symmetric and skew-symmetric elements respectively. The Lie and Jordan products are $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for any $x, y \in R$. If $X, Y \subseteqq R$, $[X, Y]$ will denate the additive subgroup generated by the set $\{[x, y]: x \in X \text{ and } y \in Y\}$. An additive subgroup, U, of R is a Lie ideal of R if $[U, R] \subseteqq U$. If R has an involution, we can similarly define a Lie ideal of K.

This paper is concerned with the study of different classes of Lie ideals of both R and K. A Lie ideal, U, of R is said to be R-strong if $aua \in U$ for all $a \in R$, $u \in U$. If U is a Lie ideal of K, U is K-(HK-) strong if $kuk \in U$ ($u^3 \in U$) for all $k \in K, u \in U$.

In the classical theory of the Lie structure of an associative ring, the main theorem [6; Th. 1.3] states: if R is simple and U is a Lie ideal of R, either $U \subseteqq Z$ or $[R, R] \subseteqq U$. We attempt to develop some criteria for differentiating between Lie ideals of R containing $[R, R]$ and R itself. Similar criteria are developed for Lie ideals of K. We will have occasion to use the following results of Herstein [6; pp 1, 5, 10, and 28]:

(i) R has no one-sided ideals which are nil of bounded index;

(ii) If $a \in R$ is such that $[a, [a, x]] = 0$ for all $x \in R$, then $a \in Z$;

(iii) Let R be simple with involution and characteristic not 2. If $Z = (0)$ or the dimension of R over Z is greater than 4, then $R = \bar{S} = \bar{K}$ where $\bar{S}$ and $\bar{K}$ are the subrings of R generated by S and K respectively.

If $X \subseteqq R$, $\mathscr{R}(X) = \{a \in R: Xa = (0)\}$ and $\mathscr{L}(X) = \{a \in R: aX = (0)\}$. The next two lemmas are analogs of a results of Baxter [3; p. 2].

LEMMA 1.1. *If U is a Lie ideal of R such that $u^2 = 0$ for all $u \in U$, then $U = (0)$.*

Proof. Let $u \in U, a \in R$. As $[u, a] \in U$, $[u, a]^2 = 0$. Therefore, $uauau = u[u, a]^2 = 0$ and uR is nil of bounded index. By the previously mentioned results, $uR = (0)$. But R is semiprime, so $\mathscr{L}(R) = (0)$. Thus $u = 0$.

LEMMA 1.2. *Let R have an involution, $*$. If U is a Lie ideal of K such that $u^2 = 0$ for all $u \in U$, then $U = (0)$.*

Proof. Let $u, v \in U$, then $0 = (u + v)^2 - u^2 - v^2 = uv + vu$. As $[u, v] \in U$, $2uv \in U$. Since $2R = R$, $[uv, K] \subseteqq U$. Thus, for each $k \in K$, $u \circ [uv, k] = 0$, and so, even more $v\{u \circ [uv, k]\} = 0$. Since u and v anti-commute, expansion of this expression yields $uvkuv = 0$. Now $suvs \in K$ for any $s \in S$. So $uv(suvs)uv = 0$. Therefore, given $a \in R, a = s + k$ where $s \in S$ and $k \in K$, then $(uv)a(uv)a(uv) = 0$. We conclude that uvR is nil of bounded index. This guarantees $uv = 0$ *for all* $u, v \in U$. Now, $-uku = u[u, k] = 0$. Repeating the previous arguments for $s \in S$ and $k \in K$, we conclude that $u = 0$.

2. R-strong Lie ideals. In this section U will denote an R-strong Lie ideal. If $a, b \in R$ and $u, v \in U$, one can easily show that the following are in U: $aub + bua$, $abu + uba$, and uau. We associate with U the set $B_U = \{b \in R: a \circ b \in U \text{ for all } a \in R\}$. This set is a Lie ideal of R and $u^2 \in B_U$ for all $u \in U$. The latter can be seen by observing that if we set $b = u$ above, we obtain $au^2 + u^2a \in U$. Thus, via Lemma 1.1, $U \neq (0)$ implies $B_U \neq (0)$.

LEMMA 2.1.

(i) *B_U is an R-strong Lie ideal*

(ii) *$u^2xu^2 \in B_U \cap U$ for all $u \in U$, $x \in R$.*

Proof.

(i) We know that B_U is a Lie ideal of R. For arbitrary $x, y \in R$ and $b \in B_U$, $[x\circ b, y]$ and $[x, b]\circ y$ are in U. Thus, by adding and subtracting these terms, we have that $xby - ybx$ and $bxy - yxb$ are in U. Now,

$$x(yby) + (yby)x = \{(xy)by - yb(xy)\} + \{yb(yx) - (yx)by\} + \{y(bx + xb)y\} .$$

Since each term on the right is in U, $x(yby) + (yby)x \in U$ and B_U is R-strong.

(ii) As $u^2 \in B_U$, $u^2xu^2 \in B_U$. Moreover, $u^2xu^2 = u(uxu)u \in U$. Therefore, $u^2xu^2 \in B_U \cap U$.

THEOREM 2.2. *$C = B_U \cap U$ is a nonzero two-sided ideal.*

Proof. Note that C is an R-strong Lie ideal. Also $C \neq (0)$ since if this were so, for each $u \in U$, u^2R would be a nil right ideal of bounded index. Let $b \in C$ and $x, y \in R$; $xb + bx \in U$. Also

$$(xb + bx)y + y(xb + bx) = \{x(by - yb) - (by - yb)x\} + \{(yx)b + b(yx)\} + \{b(xy) + (yx)b\} .$$

As each term on the right is in U, $(x\circ b)\circ y \in U$. Thus, $x\circ b \in C$. Now $2xb = x\circ b + [x, b] \in C$. Since $2R = R$, $Rb \subseteqq C$. Similarly, $bR \subseteqq C$. Thus C is a nonzero two-sided ideal of R.

We note that C is the same as the set $L_U = \{u \in U: ua \in U$ for all $a \in R\}$ which was used by Zuev [10] in his study of the Lie structure of R.

COROLLARY 2.3. *If R is simple and $U \neq (0)$, $U = R$.*

This corollary allows us to study the R-strong structure of the ring as it relates to minimal idempotents of R. If e is a minimal idempotent, eUe is an eRe-strong Lie ideal. Since eRe is a division ring either $eUe = (0)$ or $eUe = eRe$. We use this fact to prove the next theorem.

THEOREM 2.4. *Let H be the homogeneous component of the socle which contains e. Then either $H \subseteqq U$ or $H \subseteqq \mathscr{L}(U) \cap \mathscr{R}(U)$.*

Proof. Recall that H is a simple ring. The theorem then follows by considering $H \cap U$.

COROLLARY 2.5. *If R is completely reducible, U is the direct sum of the homogeneous components of the socle which it contains.*

This result is similar to that of Kaplansky [7].

Assume that R has the additional properties that $3R = R$ and R is 3-torsion free. Let W be any Lie ideal of R such that $u^3 \in W$ for all $u \in W$. Let $u, v \in W$. We have $\alpha = 2(v^2u + vuv + uv^2) = (u+v)^3 + (u - v)^3 - 2u^3 \in W$, $\beta = [v, [v, u]] \in W$ and $\gamma = [v^2, u] \in W$. From these we have: $3(v^2u + uv^2) = \alpha + \beta \in W$, $6vuv = \alpha - 2\beta \in W$, $6v^2u = \alpha + 3\gamma \in W$, and $6uv^2 = \alpha - 3\gamma \in W$. We now have enough to show a result similar to Theorem 2.2.

THEOREM 2.6. *Let W be a Lie ideal of R such that $u^3 \in W$ for all $u \in W$. Then either W contains a nonzero two-sided ideal or $u^2 \in Z$ for all $u \in W$.*

Proof. Let $a, b \in R$ and $u \in W$. Since $2a[a, u] = [a, [a, u]] + [a^2, u] \in W$ and $2R = R$, $a[a, u] \in W$. Linearization of this expression yields $a[b, u] + b[a, u] \in W$. Upon multiplication by 6 and replacement of b by v^2, we obtain $6\{a[v^2, u] + v^2[a, u]\} \in W$. As $6v^2[a, u] \in W$, $6a[v^2, u] \in W$ and this implies $a[v^2, u] \in W$. It immediately follows that $R[v^2, u]R \subseteq W$ of $R[v^2, u]R \neq (0)$, we are finished.

Assume $R[v^2, u]R = (0)$ for all $u, v \in W$, then $[v^2, u]R$ is a nilpotent ideal, hence $[v^2, u] = 0$ for all $u, v \in W$. As $[v^2, a] = [v, va + av] \in W$, $[v^2, [v^2, a]] = 0$. Thus, by remarks in §1, $v^2 \in Z$.

The obvious corollary holds in the case where R is simple.

3. *K*-strong Lie ideals. Let R have an involution, $*$, and let U be a K-strong Lie ideal. For $u, v \in U$ and $k, l \in K$, the following are in U: $kul + luk$, $klu + ulk$, and uku. We associate with U the set $B(U) = \{b \in R\colon ba - a^*b^* \in U \text{ for all } a \in R\}$. This is the analog for Lie ideals of the set which Baxter [3] uses in his study of the Jordan structure of S. When there is no confusion, we write $B(U) = B$.

LEMMA 3.1.
(i) *B is a right ideal*
(ii) *$KB \subseteq B$*
(iii) *$u^2 \in B$ for all $u \in U$*

Proof. The proofs of (i) and (ii) are straightforward. We prove (iii). As $u \in U$, $u^2a - a^*(u^2)^* = u^2a - a^*u^2$. Then

$$u^2a - a^*u^2 = \{[u, ua + a^*u]\} + \{u(a - a^*)u\} .$$

The first $\{\ \}$ is in U since $ua + a^*u \in K$. The second $\{\ \}$ is in U since $(a - a^*) \in K$ and U is K-strong.

Now from Lemma 1.2, we know that if $U \neq (0)$, $B \neq (0)$.

For $u \in U$, $k \in K$, $a \in R$ and $b, c \in B$, direct computation leads to the following facts: $ac^*b \in B$, $c^*b \in B$, $bkb^* \in B \cap U$, and $uku \in B \cap U$.

THEOREM 3.2. *Let R be a simple ring with characteristic not 2. If $Z = (0)$ or the dimension of R over Z is greater than 4, then $U = K$.*

The proof of this essentially the same as the proof of Theorem 7 [3; p. 7]. As a corollary, we include a slight extension of a theorem of Baxter [1; p. 74].

COROLLARY 3.3. *Let R be as in the theorem. $S \circ K$, the additive subgroup of R generated by the set $\{s \circ k\colon s \in S$ and $k \in K\}$ is a K-strong Lie ideal and hence $S \circ K = K$.*

The following results on $\mathscr{L}(B)$ and $\mathscr{L}(U)$ will be particularly useful in the next section.

THEOREM 3.4. *$\mathscr{L}(B)$ is a self-adjoint two-sided ideal.*

Proof. The proof is similar to the proof of Theorem 2 [4; p. 563].

Knowing that $\mathscr{L}(B)$ is a two-sided ideal, we can easily show that $\mathscr{L}(B) \cap B = (0)$ and $\mathscr{L}(B) \cap U = (0)$.

THEOREM 3.5. *$\mathscr{L}(U \cap B) = \mathscr{L}(U)$.*

Proof. It suffices to show $\mathscr{L}(U \cap B) \subseteqq \mathscr{L}(U)$. Let $b \in U \cap B$, $k \in K$, and $x \in \mathscr{L}(U \cap B)$. As $bk - kb \in U \cap B$, $xkb = -x(bk - kb) = 0$. Thus, $\mathscr{L}(U \cap B)K \subseteqq \mathscr{L}(U \cap B)$.

Let $u \in U$, then $u^3 \in U \cap B$ so $xu^3 = 0$. Since $u^2k + ku^2 \in U \cap B$, $xu^2ku = x(u^2k + ku^2)u = 0$. Let $a \in R$; $ua^* + au \in K$, therefore $0 = xu^2(ua^* + au)u = xu^2au^2$. If we replace a by ax, we have $(xu^2a)^2 = 0$. That is, xu^2R is a nil ideal of bounded index and so $xu^2 = 0$ for any

$u \in U$. Upon linearization we obtain

(3.5.1) $$xuv = -xvu \quad \text{for} \quad u, v \in U .$$

Since $xuvu = -xvu^2 = 0$ and $vkv \in U$, we have

(3.5.2) $$xu(vkv)u = 0 .$$

Let $w \in U$ and $s \in S$; $xuv(ws + sw)vu = 0$. Replacement of x by xw, expansion of the expression, and repeated use of (3.5.1) yields, $0 = -xwvuswvu$. By repeated use of (3.5.1) and finally (3.5.2), we have $xwvukwvu = 0$. Given $a \in R$, since $a = s + k$ for some $s \in S$ and $k \in K$, we can write $xwvuawvu = 0$. Replace a by ax to obtain

$$xwvu(ax)wvu = 0 .$$

Then $xwvuR$ is a nilpotent ideal so $xwvu = 0$. As $uk - ku \in U$.

(3.5.3) $$0 = xwv(uk - ku) = -xwvku .$$

Let $s \in S$; $xwv(ws + sw)v = 0$. Moreover, since $xwvwsv = 0$, we have $xwvswv = 0$. From (3.5.3), $xwvkwv = 0$. As before, this implies

(3.5.4) $$xwv = 0 .$$

Immediately, $0 = xw(vk - kv) = -xwkv$. In particular $xwkw = 0$. Since $sws \in K$, $xw(sws)w = 0$. Also, $0 = xw(swk - kws)w = xwswkw$. Again, letting $a = s + k$ for $a \in R$, we have $xwawaw = 0$. Via the same techniques, $xw = 0$ or $x \in \mathscr{L}(U)$. Hence, $\mathscr{L}(U \cap B) \subseteqq \mathscr{L}(U)$.

4. Topological annihilator rings. In this section R will denote a semiprime topological annihilator ring with continuous involution such that $2R = R$ and if $\{2x_\alpha\}$ is a net convergent to $0 \in R$, then $\{x_\alpha\}$ is also a net convergent to 0. U will be a closed K-strong Lie ideal.

The definition of an annihilator ring says that $\mathscr{L}(R) = \mathscr{R}(R) = (0)$ and if $A(L)$ is a closed right (left) ideal not equal to R, then $\mathscr{L}(A) \neq (0)$ $\mathscr{R}(L) \neq (0)$. So if $B = B(U)$, $H = \mathscr{L}(B) \oplus B$ is dense in R. It is easy to show that if U is closed, B is closed. If $X \subseteqq R$, $Cl(X)$ will denote to topolopical closure of X.

The following results have proofs which are similar to those given by Baxter in [3; p. 4].

THEOREM 4.1.
(i) *B is a two-sided ideal*
(ii) $\{\mathscr{L}(B)\}^* = \mathscr{L}(B^*)$

(iii) $B = B^*$
(iv) $U \subseteqq B$.

For any $x, y \in R$, we adopt the following notation: $(x, y)_L = xy - y^*x^*$ and $(x, y)_J = xy + y^*x^*$. Using the results of the last theorem, we prove

THEOREM 4.2. *$U = C \cap K$ where C is a closed two-sided ideal.*

Proof. Let V be the additive subgroup of S generated by the set $\{(u, a)_J : u \in U \text{ and } a \in R\}$. If we show $(U + V)$ to be a right ideal, since it is self-adjoint, it must be a two-sided ideal.

Since $U \subseteqq B$, $(u, a)_L = ua + a^*u \in U$ for all $a \in R$. Let $c \in R$, then

$$auc + c^*ua^* = ((a, u)_L, c)_L + (u, (-a^*c))_L \in V$$

and

$$auc - c^*ua^* = ((a, u)_L, c)_J + (u, (-a^*c))_J \in V .$$

Since $2R = R$, for any $2d \in R$, $u(2d) = (u, d)_L + (u, d)_J \in U + V$. Thus, $UR \subseteqq U + V$. Also,

$$(u, a)_J(2d) = (u, ad)_L + \{a^*u(-d) + (-d)^*ua\} + (u, ad)_J + \{d^*ua - a^*ud\} \in U + V$$

and $VR \subseteqq U + V$. Thus $(U + V)R \subseteqq U + V$, or the desired conclusion that $(U + V)$ is a two-sided ideal.

Let $C = Cl(U + V)$. $U \subseteqq C \cap K$. Let $x \in C \cap K$. There exists a net $\{u_\alpha + v_\alpha\}$ such that $u_\alpha + v_\alpha \rightarrow x$ where $u_\alpha \in U$ and $v_\alpha \in V$. As $x \in K$, $(u_\alpha + v_\alpha)^* = -u_\alpha + v_\alpha \rightarrow x^* = -x$. Thus $u_\alpha - v_\alpha \rightarrow x$. By subtracing these expressions we obtain $2u_\alpha \rightarrow 2x$. Therefore $u_\alpha \rightarrow x$. Since $u_\alpha \in U$ and U is closed, $x \in U$. Hence, $C \cap K = U$.

5. **HK-strong Lie ideals.** In this section U is an HK-strong Lie ideal. R will have those properties as described in §1. We further assume that $3R = R$ and R is 3-torsion free. HK-strong Lie ideals were defined by Herstein [5]. Baxter [2; p. 393] showed that if R is simple with either $Z = (0)$ or the dimension of R over Z greater than 16 with $U \nsubseteqq Z$, then $U = K$. This can be refined by using entirely different techniques.

As before, we associate with U the set $B(U)$. B is a right ideal and $KB \subseteqq B$. However, we are no longer guaranteed that $u^2 \in B$ for all $u \in U$. Hence the possibility that $B = (0)$ does arise.

LEMMA 5.1. Let $u, v, w \in U$ and $k \in K$.

(i) $6vuv \in U$
(ii) $6(uvw + wvu) \in U$
(iii) $uv(wk - kw) + (wk - kw)vu \in U$
(iv) $u^2v - vu^2 \in B$.

Proof. (i) and (ii) follow in a manner similar to the remarks preceding Theorem 2.6. (iii) holds because $2R = R$ and $3R = R$. Finally (iv) can be verified in the same manner as [6; p. 33].

If $B = (0)$, $u^2v - vu^2 = 0$ for all $u, v \in U$. Let $s \in S$. Since $[u^2, s] = [u, us + su] \in U$, $[u^2, [u^2, s]] = 0$. Also, if $k \in K$, $[u^2, [u, k]] = 0$, therefore $[u^2, [u^2, k]] = [u^2, u \circ [u, k]] = 0$. We know that this implies

$$[u^2, [u^2, a]] = 0$$

for all $a \in R$. Thus, from the first section, $u^2 \in Z$.

We now refine Baxter's theorem.

THEOREM 5.2. *Let R be simple and of characteristic not 2 or 3. If $Z = (0)$ or the dimension of R over Z is greater than 4, then either $U = K$ or $U^2 \in Z$ for all $u \in U$.*

Proof. If $B \neq (0)$, by the remarks preceding Lemmas 1.1 and 5.1 we have the alternative result.

We relate the notations of K- and HK-strong Lie ideals by calling attention to the fact that if U is HK-strong, $B \cap U$ is K-strong. Clearly $B \cap U$ is a Lie ideal. If $k \in K$ and $u \in B \cap U$, then $[k, [k, u]] = k^2u + uk^2 - 2kuk$. Now, $k^2u + uk^2 \in B \cap U$ by the definition of B. Therefore, $kuk \in B \cap U$ since $2R = R$.

Herstein [6; p. 28] has shown that K^2 is a Lie ideal of R. It is not difficult to show that if U is an HK-strong Lie ideal such that $B \cap U = (0)$, then any $x \in B \cap S$ commutes with every element in K^2. We need this fact to prove

THEOREM 5.3. *Let R be a topological anninilator ring with properties as described in the previous section. Assume also that $3R = R$ and if $\{3x_\alpha\}$ is a net convergent to $0 \in R$, $\{x_\alpha\}$ is a net converging to 0. If U is a closed HK-strong Lie ideal, then either $u^2 \in Z$ for all $u \in U$, U contains the intersection of K with a closed two-sided ideal, or $u^2v - vu^2 \in \mathscr{L}(K)$ for all $u, v \in U$.*

Proof. If $B = (0)$, $u^2 \in Z$. Assume $B \neq (0)$ and $B \cap U \neq (0)$.

Since $B \cap U$ is K-strong, Theorem 4.2 guarantees the existence of C, a closed two-sided ideal, such that $C \cap K = B \cap U \subseteqq U$.

Let $B \cap U = (0)$. As K^2 is a Lie ideal of R, $t = u^2v - vu^2 \in K^2 \cap (B \cap S)$. Also, by the remarks preceding the theorem, $[t, [t, a]] = 0$ for all $a \in R$. Therefore, $t \in Z$. Let $k \in K$; $tk + kt = tk - k^*t^* \in B \cap U = (0)$. Therefore, $tk = 0$ or $t = u^2v - vu^2 \in \mathscr{L}(K)$.

7. **Application.** We now parallel some of the results obtained by Small [9] and Riedlinger [8] concerning an additive mapping whose multiplicative property is defined relative to an involution. Let R be a simple ring with involution, $*$, and characteristic not 2 such that $Z = (0)$ or the dimension of R over Z is greater than 4. Notice that under these conditions R cannot be commutative. Let ϕ be a nozero additive mapping from R into an associative ring A. Assume $R' = \overline{\phi(R)}$, the subring of A generated by $\{\phi(r): r \in R\}$, is a noncommutative prime ring such that $2R' = R'$ and R' is 2-torsion free. Let ϕ enjoy the further property that $\phi(xy - y^*x^*) = \phi(x)\phi(y) - \phi(y^*)\phi(x^*)$ for all $x, y \in R$. We would like to show that ϕ is an associative isomorphism. We will have occasion to use the following theorem by Baxter [1; p. 73] which was slightly modified by Herstein [6; p. 29]: If R is such that $2R = R$ and $\bar{K} = R$, then $S = K \circ K$, the additive subgroup of R generated by the set $\{k \circ l: k, l \in K\}$.

The next lemma is the key to much of what follows.

LEMMA 6.1. $\operatorname{Ker} \phi \cap K = (0)$.

Proof. We show $\operatorname{Ker} \phi \cap K$ to be a K-strong Lie ideal. Let $l \in \operatorname{Ker} \phi \cap K$ and $k \in K$. Since $\phi([k, l]) = [\phi(k), \phi(l)] = 0$, $\operatorname{Ker} \phi \cap K$ is a Lie ideal of K. Thus $[k, [k, l]] \in \operatorname{Ker} \phi \cap K$ or $\phi([k, [k, l]]) = (0)$. We may expand this and obtain

$$\phi([k, [k, l]]) = \phi(k^2l - 2klk + lk^2) = \phi(k^2l + lk^2) - 2\phi(klk) = 0 .$$

Now, $\phi(k^2l + lk^2) = \phi(k^2)\phi(l) + \phi(l)\phi(k^2) = 0$. Therefore $\phi(klk) = 0$ or $\operatorname{Ker} \phi \cap K$ is a K-strong Lie ideal.

By Theorem 3.2 either $\operatorname{Ker} \phi \cap K = (0)$ or $\operatorname{Ker} \phi \cap K = K$. Assume the latter. For $s, t \in S$ and $k, l \in K$, $[\phi(k), \phi(l)] = 0$ and $[\phi(k), \phi(s)] = 0$. As $[s, t] \in K$, $0 = \phi([s, t]) = [\phi(s), \phi(t)]$. Because any $x \in R$ can be written as $x = s + k$, we have $[\phi(x), \phi(y)] = 0$ for all $x, y \in R$. Therefore, R' is commutative, a contradiction. Thus $\operatorname{Ker} \phi \cap K = (0)$.

Let $x, y \in R$, then

$$\phi((xy - y^*x^*)x^* - x(xy - y^*x^*)^*) = \{\phi(x)\phi(y) - \phi(y^*)\phi(x^*)\}\phi(x^*) - \phi(x)\{\phi(y^*)\phi(x^*) - \phi(x)\phi(y)\} .$$

If $y = s$, we can write,

$$\phi((xy - y^*x^*)x^* - x(y^*x^* - xy)) = \phi(x^2s - sx^{*2}) = \phi(x^2)\phi(s) - \phi(s)\phi(x^{*2})$$

and

$$\{\phi(x)\phi(y) - \phi(y^*)\phi(x^*)\}\phi(x^*) - \phi(x)\{\phi(y^*)\phi(x^*) - \phi(x)\phi(y)\} = (\phi(x))^2\phi(s) - \phi(s)(\phi(x^*))^2 .$$

This can be rewritten as

$$\text{(6.1.1)} \qquad \{\phi(x^2) - (\phi(x))^2\}\phi(s) = \phi(s)\{\phi(x^{*2}) - (\phi(x^*))^2\}$$

for all $x \in R$ and $s \in S$.

LEMMA 6.2. *For any* $s \in S$ *and*

$$k \in K, \{\phi(s^2) - (\phi(s))^2\} \quad \text{and} \quad \{\phi(k^2) - (\phi(k))^2\}$$

are in Z', *the center of* R'.

Proof. Set u equal to either $\{\phi(s^2) - (\phi(s))^2\}$ or $\{\phi(k^2) - (\phi(k))^2\}$. From (6.1.1), $\phi(s)u = u\phi(s)$. Consider $2\phi(t_1t_2 \cdots t_n)$ where $t_1 \in S$. We write

$$\begin{aligned} 2\phi(t_1t_2 \cdots t_n) &= \phi(t_1t_2 \cdots t_n + t_n \cdots t_2t_1) \\ &\quad + \phi(t_1t_2 \cdots t_n - t_n \cdots t_2t_1) \\ &= \phi(t_1t_2 \cdots t_n + t_n \cdots t_2t_1) \\ &\quad + \{\phi(t_1)\phi(t_2 \cdots t_n) - \phi(t_n \cdots t_2)\phi(t_1)\} . \end{aligned}$$

By induction, u commutes with $\phi(t_2 \cdots t_n)$ and $\phi(t_n \cdots t_2)$. Since $t_1t_2 \cdots t_n + t_n \cdots t_2t_1 \in S$, u commutes with $\phi(t_1t_2 \cdots t_n + t_n \cdots t_2t_1)$. Thus, $[u, \phi(t_1t_2 \cdots t_n)] = 0$. That is, u commutes with $\phi(\bar{S})$. But under our hypothesis, $\bar{S} = R$. Hence, u commutes with $\phi(R)$ and, indeed, with $\overline{\phi(R)} = R'$. Thus $u \in Z'$.

COROLLARY 6.3.

$$\text{(6.3.1)} \qquad \{\phi(x^2) - (\phi(x))^2\} \in Z' \quad \textit{for all} \quad x \in R .$$

Proof. If $x = s + k$, since $\phi(sk + ks) - \{\phi(s)\phi(k) + \phi(k)\phi(s)\} = 0$, $\{\phi(x^2) - (\phi(x))^2\} = \{\phi(s^2) - (\phi(s))^2\} + \{\phi(k^2) - (\phi(k))^2\} \in Z'$.

Let $x, y \in R$. If we linearize (6.3.1), we obtain

$$\phi(xy + yx) - \{\phi(x)\phi(y) + \phi(y)\phi(x)\} \in Z' .$$

In particular, for $s, t \in S$, $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} \in Z'$. Also, $\phi(st - ts) - \{\phi(s)\phi(t) - \phi(t)\phi(s)\} = 0$. Addition of these terms leads us to $\phi(st) - \phi(s)\phi(t) \in Z'$. Similarly, we can show that $\phi(kl) - \phi(k)\phi(l) \in Z'$ for $k, l \in K$.

For notational convenience, let $\phi(xy) - \phi(x)\phi(y) = x^y$ for any $x, y \in R$. Thus the above says that $s^t, k^l \in Z'$. The definition of ϕ tells us that $s^k = -k^s$. Also, we have $k^l = l^k$. Since these terms are in Z', $\phi(s)k^l - l^k\phi(s) = 0$. Upon expansion and rearrangement of terms, we obtain

$$\{\phi(skl - lks)\} - \{\phi(s)\phi(k)\phi(l) - \phi(l)\phi(k)\phi(s)\} = 0 . \tag{6.4.1}$$

We can write $\phi(sk - ks) = \phi(sk)\phi(l) - \phi(l)\phi(ks)$. Replacement of this in (6.4.1) and rearrangement of terms yields

$$s^k\phi(l) - \phi(l)k^s = 0$$

or

$$s^k\phi(l) = \phi(l)k^s = -\phi(l)s^k . \tag{6.4.2}$$

Let $m \in K$, by the above, there exists $z' \in Z'$ such that $\phi(ml+lm) = \phi(m)\phi(l) + \phi(l)\phi(m) + z'$. As a result of (6.4.2) and this relation we have that $s^k\phi(ml + lm) = \phi(ml + lm)s^k$ or s^k commutes with $\phi(K \circ K)$. The preliminary remarks guarantee for us that $K \circ K = S$. So, using an argument exactly like that in Lemma 6.2, we can show

$$s^k \in Z' . \tag{6.4.3}$$

LEMMA 6.4. *$x^y \in Z'$ for all $x, y \in R$.*

The proof follows directly from (6.4.3) and the remarks immediately after Corollary 6.3.

COROLLARY 6.5. *If $Z' = (0)$, ϕ is an associative isomorphism.*

Proof. As $Z' = (0)$, $\phi(xy) - \phi(x)\phi(y) = 0$. Thus ϕ is an associative homomorphism and $\overline{\phi(R)} = \phi(R)$. Moreover, since R is simple, ϕ is an associative isomorphism.

Let $z'(\neq 0) \in Z'$. Since $\mathscr{A}(z') = \{r' \in R' : r'z' = 0\}$ is a two-sided ideal in a prime ring, $\mathscr{A}(z') = (0)$.

LEMMA 6.6. *$k^s = s^k = 0$ for all $s \in S, k \in K$.*

Proof. From (6.4.2) $s^k\phi(l) = -\phi(l)s^k$ for $l \in K$. By Lemma 6.4, $s^k \in$

Z', therefore $s^k\phi(l) = 0$. Suppose $s^k \neq 0$. By the remarks preceding the lemma, we have $\phi(l) = 0$, that is, $K \subseteqq \operatorname{Ker}\phi$. Therefore, $\operatorname{Ker}\phi \cap K = K$, a contradiction. We conclude that $0 = s^k = -k^s$.

COROLLARY 6.7. *$\phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for $x, y \in R$.*

We have shown that when $Z' = (0)$, then ϕ is an associative isomorphism. Therefore, the following theorem is proved except when $Z' \neq (0)$.

THEOREM 6.8. *ϕ is an associative isomorphism.*

Proof. From Lemma 6.6, $(s^2)^k - \phi(s)s^k = 0$. Expansion and rearrangement of terms leads to $(s^2)^k - \phi(s)s^k = (s)^{sk} - s^s\phi(k) = 0$. From Lemma 6.4, $(s)^{sk} \in Z'$ so $s^s\phi(k) \in Z'$. Let $l \in K$. There exist z_1' and z_2' in Z' such that $s^s\phi(k) = z_1'$ and $s^s\phi(l) = z_2'$. As $s^s \in Z'$, we can write $0 = [z_1', z_2'] = (s^s)^2[\phi(k), \phi(l)]$ for all $s \in S$ and $k, l \in K$.

If $(s^s)^2 \neq 0$ for some $s \in S$, then by the remarks preceding Lemma 6.6, $[\phi(k), \phi(l)] = 0$ for all $k, l \in K$. As $\phi([k, l]) = [\phi(k), \phi(l)] = 0$, we conclude that $[K, K] \subseteqq \operatorname{Ker}\phi \cap K = (0)$. This implies $\bar{K} = R$ is commutative, a contradiction. So $(s^s)^2 = 0$ for all $s \in S$. Since the center of a prime ring is an integral domain, $s^s = 0$. Upon linearization of this expression, we obtain $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} = 0$ for all $t, s \in S$.

For $k, l \in K$, $k^l \in Z'$. Thus there exists $z_3' \in Z'$ such that $k^l - z_3' = 0$. Since $k^2 \in S$, $(k^2)^l = 0$ and so $(k^2)^l - \phi(k)\{k^l - z_3'\} = 0$. Expansion and rearrangement of terms leads to $k^{kl} - k^k\phi(l) + z_3'\phi(k) = 0$. In view of Lemma 6.4, there is an element $z_4' \in Z'$ such that $k^{kl} = z_4'$. Therefore we can always find $z_3', z_4', \in Z'$ such that $k^k\phi(l) = z_3'\phi(k) + z_4'$ where k is an arbitrary fixed element in K and l is allowed to vary in K. Note that $k^k \in Z'$. For $m \in K$, there are z_5' and z_6' in Z' such that $k^k\phi(m) = z_5'\phi(k) + z_6'$. Thus $0 = (k^k)^2[\phi(l), \phi(m)] = [k^k\phi(l), k^k\phi(m)]$. Via the same argument as above, we can show $k^k = 0$. Linearization of this expression leads to $\phi(kl + lk) - \{\phi(k)\phi(l) + \phi(l)\phi(k)\} = 0$. Now, using this fact and the fact that both $\phi(sk) - \phi(s)\phi(k) = 0$ and $\phi(st + ts) - \{\phi(s)\phi(t) + \phi(t)\phi(s)\} = 0$, we have that

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$$

for all $x, y \in R$. From Corollary 6.7, we know

$$\phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x) .$$

Addition of these two expressions yields $\phi(xy) = \phi(x)\phi(y)$ or that ϕ is an associative homomorphism. Therefore, $\overline{\phi(R)} = \phi(R)$ and $\operatorname{Ker}\phi = (0)$

since R is simple. Hence ϕ is an associative isomorphism.

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