A PHRAGMÉN-LINDELÖF THEOREM WITH APPLICATIONS TO $M(u, v)$ FUNCTIONS

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A well-known theorem of Paley and Wiener asserts that if \( f \) is an entire function, its restriction to the real line belongs to the Hilbert space \( \mathcal{F}L^2(\mathbb{R}) \) (where \( \mathcal{F} \) is the Fourier-Plancherel operator) if and only if \( f \) is square integrable on the real axis and satisfies
\[
|f(z)| \leq K e^{\tau |\text{Im}z|}
\]
for some positive \( K \). The "if" part of this result may be viewed as a Phragmén-Lindelöf type theorem. The pair \( (e^{i\tau}, e^{i\tau}) \) of inner functions can be associated with the above mentioned Hilbert space in a natural way. By replacing this pair by a more general pair \( (u, v) \) of inner functions it is possible to define a space \( \mathcal{M}(u, v) \) of analytic functions similar to the Paley-Wiener space. For a certain class of inner functions (those of "type \( \mathcal{C} \)") it is shown that membership in \( \mathcal{M}(u, v) \) is implied by an inequality analogous to the exponential inequality above.

A second application of our results is to star-invariant subspaces of the Hardy space \( H^2 \). It is well known that if \( u \) is an inner function on the circle and \( f \) is in \( H^2 \), then in order for \( f \) to be in \( (uH^2)^\perp \) it is necessary for \( f \) to have a meromorphic pseudocontinuation to \( |z| > 1 \) satisfying
\[
|f(z)|^2 \leq K \frac{1 - |u(z)|^2}{1 - |z|^2}, \quad |z| > 1.
\]

If \( u \) is inner of type \( \mathcal{C} \), it is proved that this necessary condition is also sufficient.

Let \( \Gamma = \{e^{i\theta}: 0 < \theta < 2\pi\} \) be the unit circle and
\[
R = \{x: -\infty < x < \infty\}
\]
the real line considered as point sets in the complex plane \( C \). Let \( D \) and \( D_- \) be the interior and exterior of the unit circle and let \( \Omega \) and \( \Omega_- \) be the open upper and open lower half-planes in \( C \). A function \( \Phi \) is outer on \( D \) or \( \Omega \) if \( \Phi \) is holomorphic on \( D \) or \( \Omega \) and of the form
\[
\Phi(z) = \exp \int_D \frac{e^{it} + z}{e^{it} - z} k_t(e^{it}) \sigma(d\xi), \quad z \in D,
\]
or
\[
\Phi(z) = \exp \frac{1}{\pi i} \int_R \frac{1 + tz}{t - z} k_t(t) dt, \quad z \in \Omega,
\]

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where $k_1, k_2$ are real with $k_1 \in L^1(\Gamma)$, $k_2 \in L^1(R)$, and $\sigma$ is normalized Lebesgue measure on $\Gamma$. A function $F$ on $D$ or $\Omega$ is in $\mathcal{R}^+$ if $F$ is holomorphic on $D$ or $\Omega$ and if there exists an outer function $\Phi$ that is not identically zero and such that $\Phi F$ is a bounded holomorphic function on $D$ or $\Omega$. If $F$ is in $\mathcal{R}^+$ on $D$ or $\Omega$, then $f(e^{i\theta}) = \lim F(re^{i\theta})$ exists for almost all $e^{i\theta} \in \Gamma$, or

$$f(x) = \lim_{y \downarrow 0} F(x + iy)$$

exists for almost all $x$ in $R$. Such $f$ form the class $\mathcal{M}^+$ of functions on $\Gamma$ and $R$ respectively. We shall systematically use capital letters $F, G, \cdots$ for functions in $\mathcal{R}^+$ and lower case letters $f, g, \cdots$ for the corresponding functions in $\mathcal{M}^+$.

Every outer function is in $\mathcal{R}^+$. A function $U$ in $\mathcal{R}^+$ is inner if $|u| = 1$ a.e.. Every function $F$ in $\mathcal{R}^+$ has a factorization of the form $F = UG$, where $U$ is inner and $G$ is outer.

Suppose $U$ and $V$ are inner functions, say, on $\Omega$. $\mathcal{M}(u, v, R)$ is the set of functions $f$ on $R$ such that $uf$ and $vf^*$ are in $\mathcal{N}^+$ on $R$. ($f^*$ is the complex conjugate of $f$). $\mathcal{M}(u, v, \Gamma)$ is similarly defined. As shown in [5] one can associate with each $f$ in $\mathcal{M}(u, v, R)$ a unique function $F$ separately meromorphic in $\Omega$ and $\Omega_-$ such that $UF \in \mathcal{R}^+$, $V\bar{F} \in \mathcal{R}^+$, and

$$f(x) = \lim_{y \downarrow 0} F(x + iy) = \lim_{y \downarrow 0} F(x - iy)$$

for almost all $x$ in $R$, where $\bar{F}(z) = F^*(z^*)$, $z \in \Omega$. If $F$ is meromorphic in $\Omega$, then an extension of $F$ to a meromorphic function on $\Omega \cup \Omega_-$ satisfying (1) is said to be a meromorphic pseudoccontinuation (relative to $R$) of $F$. Similarly, to each $f$ in $\mathcal{M}(u, v, \Gamma)$ one associates a unique $F$ meromorphic in $D \cup D_-$ such that $UF \in \mathcal{R}^+$, $V\bar{F} \in \mathcal{R}^+$, and

$$f(e^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta})$$

for almost all $e^{i\theta} \in \Gamma$ where $\bar{F}(z) = F^*(z^{*-1})$, $z \in D$. Meromorphic pseudoccontinuation is defined relative to $\Gamma$ in a manner analogous to the $R$ definition.

Considerations about $\mathcal{M}(u, v, R)$ may be motivated by examining the special case when $U(z) = V(z) = e^{iz}$, $\tau \geq 0$. Then

$$\mathcal{M}(u, v, R) \cap L^\tau(R)$$

is the class of functions that are the restrictions to $R$ of entire functions of exponential type $\leq \tau$ such that $\int_R |F(x)|^2 \, dx < \infty$. Such entire $F$ can be characterized by this integral condition together
with the inequality
\[ |F(z)|^2 < K |y|^{-1} \sinh(2\tau y) \]
for all \( z \in \Omega \cup \Omega_\infty \), where \( K > 0 \). The object of this paper is to extend this type of function-theoretic characterization to more general \( \mathcal{A}(u, v) \) classes. The above mentioned application to star-invariant subspaces arises from the fact that \( \mathcal{M}(1, v) \cap L^1(R) = H^2(\Omega) \cap vH^2(\Omega) \), where \( H^2(\Omega) \) is the Hardy space of the upper half-plane. In § 3 and 4 applications are given to factorization problems for nonnegative operator-valued functions and to generalized Paley-Wiener representations.

1. A Phragmén-Lindelöf Theorem. In this section we shall derive a Phragmén-Lindelöf type theorem for certain functions holomorphic on \( D \), and then transcribe the result to obtain a like theorem for functions on \( \Omega \). A rather different Phragmén-Lindelöf type theorem is discussed by Helson in [2, p. 33].

Recall that a Blaschke product \( B \) on \( D \) has a representation
\[ B(z) = \prod_{j \in \mathbb{Z}} B_j(z), \quad B_j(z) = \frac{z_j^+ z_j - z}{z_j - z_j^+}, \quad z \in D, \]
where \( \sum_{j \in \mathbb{Z}} (1 - |z_j|) < \infty \). We take \( z_j^+/|z_j| = 1 \) if \( z_j = 0 \). The support \( \text{supp} B \) of \( B \) is the intersection of \( \Gamma \) with the closure of \( \{z_j\}_{j \in \mathbb{Z}} \). A singular inner function \( S \) has a representation
\[ S(z) = \exp \left( -\int \frac{e^{it} + z}{e^{it} - z} \mathrm{d}\mu(\xi) \right), \quad z \in D, \]
where \( \mu \) is a positive singular measure on \( \Gamma \). The support \( \text{supp} S \) is the closed support of the measure \( \mu \).

Any inner function \( U \) on \( D \) can be factored in the form \( U = eBS \), where \( e \in \mathbb{C}, |e| = 1, B \) is a Blaschke product and \( S \) is a singular inner function. The support \( \text{supp} U \) of \( U \) is \( \text{supp} B \cup \text{supp} S \).

A closed set \( N \) on \( \Gamma \) is a Carleson set if \( N \) has zero Lebesgue measure and if the complement of \( N \) in \( \Gamma \) is a union of open arcs \( I_j \) of lengths \( \varepsilon_j \) such that \( \sum_{j \in \mathbb{Z}} \varepsilon_j \log \varepsilon_j > -\infty \).

**Theorem 1.1.** (Carleson [1]). A closed subset \( N \) of \( \Gamma \) is a Carleson set on \( \Gamma \) if and only if there exists an outer function \( G \) on \( D \) that satisfies a Lipschitz condition and such that
\[ g(e^{i\theta}) \overset{\text{def}}{=} \lim_{r \to 1} G(re^{i\theta}) \]
vanishes on \( N \).
DEFINITION 1.2. An inner function $U$ on $D$ is of type $G_e$ if
(i) $\text{supp } U$ is a Carleson set, and
(ii) $\sum_{j \in \mathbb{Z}} [\text{dist } (z_j, \text{supp } U)] < \infty$,
where $\{z_j\}_{j \in \mathbb{Z}}$ are the zeros of $U$ in $D$ repeated according to multiplicity.

LEMMA 1.3. Let $B$ be the Blaschke product given by (3) and suppose $B$ is of type $G_e$. If $G$ is a Lipschitz outer function on $D$ such that $g(e^{i\theta}) = \lim_{r \uparrow 1} G(re^{i\theta})$ vanishes on $\text{supp } B$, then
\begin{equation}
\sum_{j \in \mathbb{Z}} (1 - |z_j|^2) \int |1 - z^*_j e^{i\theta}|^{-1} g(e^{i\theta}) |^2 \sigma(d\theta) < \infty.
\end{equation}

Proof. Since $G$ is Lipschitz there exists $K > 0$ such that $|g(e^{i\theta})| \leq K |e^{i\theta} - \lambda|$ for all $e^{i\theta}$ in $\Gamma$ and $\lambda$ in $\text{supp } B$. Thus for $\lambda$ in $\text{supp } B$,
\begin{equation}
(1 - |z_j|^2) \int |1 - z^*_j e^{i\theta}|^{-1} g(e^{i\theta}) |^2 \sigma(d\theta)
\leq (1 - |z_j|^2) K^2 \int |1 - z^*_j e^{i\theta}|^{-1} (e^{i\theta} - \lambda)^2 |^2 \sigma(d\theta) .
\end{equation}

Applying Parseval’s equality to the Fourier series for the function $(1 - z^*_j e^{i\theta})^{-1} (e^{i\theta} - \lambda)$ shows that this last expression is equal to $K^2 (|z_j - \lambda|^2 + (1 - |z_j|^2)).$

Since $\sum_{j \in \mathbb{Z}} (1 - |z_j|^2) < \infty$ and we are free to let $\lambda$ vary over $\text{supp } B$ this inequality implies (5).

The following theorem is our Phragmén-Lindelöf result for functions on $D$.

THEOREM 1.4. Let $U$ be an inner function of type $G_e$ on $D$. Suppose $F$ is holomorphic in $D$ and there exists $M > 0$ such that
\begin{equation}
|F(z)|^2 \leq M(1 - |z|^2)^{-1} (1 - |U(z)|^2), \quad z \in D .
\end{equation}

Then $F \in \mathcal{H}^+.$

Proof. $U$ has the factorization $U = cBS$, where $|c| = 1$, $B$ is a Blaschke product of type $G_e$ and $S$ is a singular inner function of type $G_e$. We have
\begin{equation}
(1 - |z|^2)^{-1} (1 - |U(z)|^2)
= (1 - |z|^2)^{-1} (1 - |B(z)|^2) + |B(z)|^2 (1 - |z|^2)^{-1} (1 - |S(z)|^2)
\leq (1 - |z|^2)^{-1} (1 - |B(z)|^2) + (1 - |z|^2)^{-1} (1 - |S(z)|^2), \quad z \in D .
\end{equation}
If $B$ is given by (3), then
\[
1 - |B(z)|^2 = 1 - |B_1(z)|^2 + \sum_{n \in \mathbb{Z}} \left| \prod_{j=1}^{n-1} B_j(z) \right|^2 (1 - |B_n(z)|^2) 
\leq \sum_{j \in \mathbb{Z}} (1 - |B_j(z)|^2).
\]

Thus
\[
(8) \quad (1 - |z|^2)^{-1} (1 - |B(z)|^2) \leq \sum_{j \in \mathbb{Z}} (1 - |z_j|^2) |1 - z_j^* z|^{-2}.
\]

If $S$ is given by (4), then
\[
|S(z)|^2 = \exp \left\{ -2 \int_{\mathbb{R}} (1 - |z|^2) \left| e^{it} - z \right|^{-2} \mu(d\xi) \right\}, \quad z \in \mathbb{D}.
\]

Applying the elementary inequality $(1 - e^{-ah})/h) \leq a$ if $a, h \geq 0$, with $h = 1 - |z|^2$ and $a = 2 \int_{\mathbb{R}} |e^{it} - z|^{-2} \mu(d\xi)$ yields
\[
(9) \quad (1 - |z|^2)^{-1} (1 - |S(z)|^2) \leq 2 \int_{\mathbb{R}} |e^{it} - z|^{-2} \mu(d\xi), \quad z \in \mathbb{D}.
\]

Suppose now that (6) holds and let $G$ be a Lipschitz outer function such that $g(e^{i\theta}) = \lim_{r \to 1} G(re^{i\theta})$ vanishes on supp $U$. We have from (6) — (9) that
\[
|G(z)F(z)|^2 \leq M \sum_{j \in \mathbb{Z}} (1 - |z_j|^2) |1 - z_j^* z|^{-2} |G(z)|^2 + 2M \int_{\mathbb{R}} |e^{it} - z|^{-2} |G(z)|^2 \mu(d\xi), \quad z \in \mathbb{D}.
\]

But for some $K > 0$
\[
|G(z)|^2 \leq K^2 |e^{it} - z|^2 \quad \text{if} \quad e^{it} \in \text{supp } U,
\]
and $\mu$ is supported on supp $S \subseteq \text{supp } U$. Thus for all $z \in \mathbb{D}$
\[
|G(z)F(z)|^2 \leq M \sum_{j \in \mathbb{Z}} (1 - |z_j|^2) |1 - z_j^* z|^{-2} |G(z)|^2 + 2MK^2 \mu(I').
\]

It now follows from Lemma 1.3 that
\[
\sup_{0 \leq r < 1} \int_{\mathbb{R}} |G(re^{i\theta})F(re^{i\theta})|^2 \sigma(d\theta) < \infty,
\]
so $GF \in H^\infty$. It is easy to multiply $G$ by an outer function $G_1$ and obtain $G_1GF$ bounded, and so $F$ is in $\mathcal{H}^\infty$.

We shall next recast Theorem 1.4 for functions holomorphic on $\Omega$. Any inner function $U$ on $\Omega$ has a factorization $U = cBV^a$, where $c \in \mathbb{C}$, $|c| = 1$, $B$ is a Blaschke product on $\Omega$, $S$ is a singular function on $\Omega$, and $V^a(z) = e^{iza}$, where $0 \leq a \in \mathbb{R}$. Then supp $B$ is defined to be the set of limit points on $R \cup \{\infty\}$ of the zeros of $B$,
and supp $S$ is defined to be the support of the singular measure in the representation for $S$ analogous to (4), (Hoffman [3] p.132-133). We define supp $V^a$ to be empty if $a = 0$, and $\{\infty\}$ if $a > 0$. The support supp $U$ of $U$ is supp $B \cup$ supp $S \cup$ supp $V^a$.

A closed subset $N$ of the extended real line $R \cup \{\infty\}$ is a Carleson set if $N \cap R$ has Lebesgue measure zero, $\infty \in N$, and the complement of $N$ in $R \cup \{\infty\}$ is a union of open intervals $I_j = (a_j, b_j)$, $-\infty \leq a_j < b_j \leq \infty$, $j = 1, 2, \ldots$ such that $\sum_{j>1} \delta_j \log \delta_j > -\infty$, where

$$\delta_j = \frac{b_j - a_j}{(1 + b_j)^{1/2} (1 + a_j)^{1/2}}, \quad j = 1, 2, \ldots$$

We understand in the above that $\infty/\infty = 1$.

Now let $\alpha: \mathbb{D} \to \overline{\mathbb{D}} \cup \{\infty\}$ be the mapping defined by

$$\alpha(z) = i(1 + z)(1 - z)^{-i}$$

if $z \neq 1$ and $\alpha(1) = \infty$, and let $\beta$ be the inverse of $\alpha$. Then if $z_i, z_\infty \in \overline{\mathbb{D}}$,

$$|\beta(z_i) - \beta(z_\infty)|^2 = 4 \frac{|z_i - z_\infty|^2}{|z_i + i|^2 |z_\infty + i|^2}.$$ 

Moreover $\beta$ maps $(-\infty, \infty]$ onto $\Gamma$ and $N$ is a Carleson set on $R \cup \{\infty\}$ if and only if $\beta(N) \cup \{1\}$ is a Carleson set on $\Gamma$. If $U$ is inner on $\Omega$ then $U \circ \alpha$ is inner on $D$ and supp $(U \circ \alpha) = \beta$ (Supp $U$). Furthermore if $\{z_j\}_{j>1}$ is the sequence of zeros of $U$, then $\{\beta(z_j)\}_{j>1}$ is the sequence of zeros of $U \circ \alpha$.

**Definition 1.5.** Let $U$ be an inner function on $\Omega$. $U$ is of type $C$ if supp $U \cup \{\infty\}$ is a Carleson set on $R \cup \{\infty\}$ and

$$\sum_{j \geq 1} \left( \inf_{\lambda \in \text{supp} U} \frac{|z_j - \lambda|^2}{(1 + \lambda^2) (1 + |z_j|^2)} \right) < \infty,$$

where $\{z_j\}_{j>1}$ is the sequence of zeros of $U$ in $\Omega$ repeated according to multiplicity.

The following lemma follows from the above discussion.

**Lemma 1.6.** Let $U$ be inner on $\Omega$. Then $U$ is of type $C$ if and only if $U \circ \alpha$ is of type $C$ on $D$.

We can now recast Theorem 1.4 for the half-plane.
Theorem 1.7. Let \( F \) be holomorphic in \( \Omega \) and suppose that \( U \) is inner of type \( \mathcal{C} \) in \( \Omega \). Suppose that there exists \( K > 0 \) such that

\[
|F(z)|^2 \leq K(\text{Im } z)^{-1} (1 + |z|^2) (1 - |U(z)|^2) \quad \text{for } z \in \Omega.
\]

Then \( F \in \mathcal{R}^+ \) on \( \Omega \).

Proof. Set \( G = F \circ \alpha \), so \( G \) is meromorphic on \( \mathbb{D} \) and

\[
|G(z)|^2 \leq K [\text{Im } \alpha(z)]^{-1} (1 + |\alpha(z)|^2) (1 - |U(\alpha(z))|^2), \quad z \in \mathbb{D}.
\]

We can replace \( 1 + |\alpha(z)|^2 \) by \( i + \alpha(z) \) and the inequality still holds but for a different constant \( K \). Now

\[
\text{Im } \alpha(z) = (1 - |z|^2) |1 - z|^2
\]

and

\[
|i + \alpha(z)|^2 = 4 |1 - z|^2,
\]

so

\[
|G(z)|^2 \leq K' (1 - |z|^2)^{-1} (1 - |U(\alpha(z))|^2), \quad z \in \mathbb{D}.
\]

But by Lemma 1.6 \( U \circ \alpha \) is of type \( \mathcal{C} \), and thus Theorem 1.4 implies that \( G \in \mathcal{R}^+ \) on \( \mathbb{D} \). We then deduce that \( F = G \circ \beta \) is in \( \mathcal{R}^+ \) on \( \Omega \).

2. The classes \( \mathcal{M}^+(u, v, \Gamma) \) and \( \mathcal{A}(u, v, R) \). Suppose \( U \) is inner in \( \mathbb{D} \). Then \( U \) has a meromorphic pseudocontinuation to a function \( U \) on \( \mathbb{D} \) that is given by

\[
U(z) = \begin{cases} 
U(z), & z \in \mathbb{D} \\
1/U^*(z^-), & z \in D_-
\end{cases}
\]

If \( \text{supp } U \neq \Gamma \), then \( U \) on \( \mathbb{D} \) has a single valued meromorphic continuation to \( D_- \) that coincides with \( U \) as given by (11). If \( F \) is meromorphic on \( D_- \) then \( \tilde{F}(z) = F^*(z^-) \) defines \( \tilde{F} \) to be meromorphic on \( \mathbb{D} \). Of course \( \tilde{F} \) need not be a pseudocontinuation of \( F \).

Analogous definitions are made for \( \Omega \). Suppose \( U \) is inner on \( \Omega \). Then \( U \) has a meromorphic pseudocontinuation on \( \Omega \) given by

\[
U(z) = \begin{cases} 
U(z), & z \in \Omega \\
1/U^*(z^*) & z \in \Omega_-
\end{cases}
\]

If \( F \) is meromorphic on \( \Omega \), then \( \tilde{F}(z) = F^*(z^*) \) defines \( \tilde{F} \) to be meromorphic on \( \Omega_- \).

We say that \( F \) is \( \mathcal{R}_+ \) on \( D \) if \( F \in \mathcal{R}^+ \) on \( D \) and \( F(0) = 0 \). \( \mathcal{R}_+ \) is defined to be the set of all \( f \) such that \( f(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta}) \) a.e., where \( F \in \mathcal{R}_e^+ \) on \( D \).

Suppose \( U, V \) are inner functions on \( D \). \( \mathcal{A}_0(u, v, \Gamma) \) is the set
of all functions $f$ on $\Gamma$ such that $uf \in \mathcal{M}_0^+$ and $vf^* \in \mathcal{N}_0^-$. $\mathcal{M}_0(u, v, \Gamma)$ can be characterized as follows: $f \in \mathcal{M}_0(u, v, \Gamma)$ if and only if there exists a function $F$ separately meromorphic in $D$ and $D_-$ and such that

$$f(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta}) = \lim_{r \to 1} F(re^{i\theta}) \text{ a.e.,}$$

with

$$UF \in \mathcal{R}^+ \text{ on } D \text{ and } V\tilde{F} \in \mathcal{R}^+ \text{ on } D.$$

In case $U$ and $V$ are of type $\mathcal{C}$ we can deduce (14) from an inequality involving $F$, $U$ and $V$.

**Theorem 2.1.** Suppose $U$ and $V$ are of type $\mathcal{C}$, and $F$ is meromorphic in $D$ and has a meromorphic pseudocontinuation to a function $F$ on $D \cup D_-$. Further suppose there exists $K > 0$ such that

$$|F(z)|^2 \leq K(1 - |z|^2)^{-1} (|U(z)|^2 - |V(z)|^2), \quad |z| \neq 1.$$

Then $f(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta}) \in \mathcal{M}_0(u, v, \Gamma)$.

**Proof.** If $F$ satisfies (15) on $D$ then

$$|U(z)F(z)|^2 \leq K(1 - |z|^2)(1 - |U(z)V(z)|^2),$$

so $UF \in \mathcal{R}^+$ by Theorem 1.4.

If $F$ satisfies (15) on $D_-$, then for all $z \in D$,

$$|V(z)\tilde{F}(z)|^2 \leq K|z|^2(1 - |z|^2)^{-1}(1 - |U(z)V(z)|^2)$$

so $V\tilde{F} \in \mathcal{R}^+$ by 1.4. But we also deduce that $V(0)\tilde{F}(0) = 0$, so $V\tilde{F} \in \mathcal{R}^+$.

It therefore follows from the characterization of $\mathcal{M}_0(u, v, \Gamma)$ given in (13) and (14) that $f \in \mathcal{M}_0(u, v, \Gamma)$.

In case $f \in L^2(\Gamma)$, i.e., in case $\int |f|^2d\sigma < \infty$, we have a stronger result.

**Theorem 2.2.** Assume that $U, V$ are inner of type $\mathcal{C}$ on $D$ and $f \in L^2(\Gamma)$. Then $f \in \mathcal{M}_0(u, v, \Gamma)$ if and only if there exists a function $F$ satisfying the hypotheses of Theorem 2.1 such

$$f(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta}) \text{ a.e.}$$

**Proof.** It follows from Theorem 2.1 that if $F$ satisfies (15) then $f \in \mathcal{M}_0(u, v, \Gamma)$. Conversely, suppose $f \in \mathcal{M}_0(u, v, \Gamma) \cap L^2(\Gamma)$. Then $uf \in \mathcal{N}^+ \cap L^2(\Gamma) = H^2$ and $vf^* \in \mathcal{N}_0^- \cap L^2(\Gamma) \subseteq H^2$ with $\int vf^*d\sigma = 0$.\[\int \]
Thus $uf$ and $v\chi^* f^*$ are in $(uvH^2)^\perp \cap H^2$, where $\chi(e^{i\theta}) = e^{i\theta}$.

Now any $g \in (uvH^2)^\perp \cap H^2$ is the boundary value function of

$$G(z) = \int (1 - ze^{-i\theta})^{-1} (1 - u^*(e^{i\theta})v^*(e^{i\theta})U(z)V(z))g(e^{i\theta})\sigma(d\xi), \ z \in D.$$ 

But then it follows from the Schwarz inequality that

$$|G(z)|^2 \leq K(1 - |z|^3)^{-1} (1 - |U(z)V(z)|^3), \ z \in D,$$

where $K = \int |g|^3 \sigma$.

By applying (16) to $g = uf$ and $g = v\chi^* f^*$ we obtain

$$|U(z)F(z)|^3 \leq K(1 - |z|^3)^{-1} (1 - |U(z)V(z)|^3), \ z \in D,$$

and

$$|V(z)F(z)|^3 \leq K |z|^3 (1 - |z|^3)^{-1} (1 - |U(z)V(z)|^3), \ z \in D,$$

where $K = \int |f|^3 \sigma$.

It is easily seen that (17) and (18) together is equivalent to (15).

**Corollary 2.3.** Assume that $V$ is inner of type $C$ on $D$ and $f \in H^2$ on $\Gamma$. Then $f \in (vH^2)^\perp$ if and only if there exists a meromorphic function $F$ on $D \cup D_- \ U$ such that

$$f(e^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) \ a.e.,$$

for which there exists $K > 0$ with

$$|F(z)|^3 \leq K (1 - |z|^3)^{-1} (1 - |V(z)|^3), \ z \in D \cup D_-.$$

**Proof.** Note that $(vH^2)^\perp \cap H^2 = \mathcal{M}_0(1, v, \Gamma)$, and use 2.2.

**Corollary 2.4.** Assume that $U, V$ are inner of type $C$ on $D$ and $f \in L^2(\Gamma)$. Then $f \in \mathcal{M}(u, v, \Gamma)$ if and only if there exists a function $F$ meromorphic in $D$ with pseudocontinuation $F'$ such that (19) holds and there exists $K > 0$ such that

$$|F(z)|^3 \leq K (1 - |z|^3)^{-1} (|U(z)|^{-2} - |zV(z)|^3), \ z \in D.$$

**Proof.** Note that $\mathcal{M}(u, v, \Gamma) = \mathcal{M}_0(u, \chi v, \Gamma)$.

The same kind of problem can be considered on $\Omega$ with minor modifications in the proofs.

**Theorem 2.5.** Suppose $F$ is meromorphic on $\Omega$ and has a mero-
morphic pseudocontinuation to a function $F$ on $\Omega \cup \Omega_-$. Assume that $U$ and $V$ are inner functions of type $\mathfrak{C}$ on $\Omega$. Further suppose that there exists $K > 0$ such that

$$|F(z)|^2 \leq K(\text{Im } z)^{-1}(1 + |z|^2)(|U(z)|^{-2} - |V(z)|^{-2}), \quad z \in \Omega \cap \Omega_-.$$  

Then $f(x) = \lim_{y \to 0} F(x + iy) \in \mathcal{M}(u, v, R)$.

**Theorem 2.6.** Assume that $U, V$ are inner of type $\mathfrak{C}$ on $\Omega$ and $f \in L^1(\mathbb{R})$. Then $f \in \mathcal{M}(u, v, R)$ if and only if there exists a function satisfying the hypotheses of Theorem 2.5 such that

$$f(x) = \lim_{y \to 0} F(x + iy) \text{ a.e.}$$

3. Factorization of nonnegative functions. In this section we shall reformulate an operator factorization theorem of the type set down in [5] in terms of inequalities of the type discussed in § 1 and 2. Throughout $\mathscr{H}$ is a complex separable Hilbert space and $B(\mathscr{H})$ the space of bounded operators on $\mathscr{H}$. We shall restrict ourselves to considerations involving $\Omega$ rather than $D$ in order to simplify the exposition. Following [5] we say that a holomorphic function $F$ on $\Omega$ taking values in $B(\mathscr{H})$ is in $\mathcal{S}_\mathfrak{C}$ if there exists a nonzero complex-valued outer function $\Phi$ such that $\Phi F$ is a bounded holomorphic function on $\Omega$ that takes values in $B(\mathscr{H})$. Any $F$ in $\mathcal{S}_\mathfrak{C}$ has strong boundary values a.e., that is, the limit $\lim_{y \to 0} F(x + iy) = f(x)$ exists a.e. in the strong operator topology.

We say that a holomorphic function $G$ in $\mathcal{S}_\mathfrak{C}$ has a meromorphic pseudocontinuation $G$ if $G$ is meromorphic in $\Omega_- \cup \Omega_+$ and the strong limits $\lim_{y \to 0} G(x - iy)$ and $\lim_{y \to 0} G(x + iy)$ exist and are a.e. equal. For such $G$ we define $\tilde{G}$ by $\tilde{G}(z) = G^*(z^*)$, $z \in \Omega \cup \Omega_-$.

**Theorem 3.1.** Let $U$ be a complex-valued inner function on $\Omega$ and $F$ a meromorphic function on $\Omega$ taking values in $B(\mathscr{H})$ such that $UF \in \mathcal{S}_\mathfrak{C}$. Then $F(x + iy)$ has strong boundary values $f(x)$ a.e. as $y \downarrow 0$. Assume that $\langle f(x)e, e \rangle \geq 0$ a.e. for each $e$ in $\mathscr{H}$.

Then $F$ has a factorization $F(x) = \tilde{G}(z)G(z)$, $z \in \Omega$, where $G$ is in $\mathcal{S}_\mathfrak{C}$ and has a meromorphic pseudocontinuation $G$ such that $UG \in \mathcal{S}_\mathfrak{C}$. If there is real interval $I$ such that $f(.)$ is a.e. bounded on $I$ and $U$ is analytically continuable across $I$, then $G$ is analytically continuable across $I$.

**Proof.** This theorem is a summary of results proved in [5].

**Theorem 3.2.** Theorem 3.1 may be modified as follows:
(i) The hypothesis "UF ∈ R^+(Ω)" may be replaced by the stronger hypothesis "there exists K > 0 such that

\[ ||F(z)||^2 ≤ K(\text{Im } z)^{-1} (1 + |z|^2) (||U(z)||^{-2} - |U(z)|^2) \]

for all z in Ω".

(ii) If in addition one assumes that \( \int_{-\infty}^{\infty} \langle f(x), c \rangle \, dx < \infty \) for all c in \( \mathbb{C} \), then G can be chosen to in addition satisfy

\[ ||G(z), c||^2 ≤ K_s(\text{Im } z)^{-1} (1 + |z|^2) (1 - |U(z)|^2), \quad c \in \mathbb{C} \]

for some \( K_s > 0 \) (\( K_s \) depends on \( c \)) and all \( z \in \Omega \cup \Omega_\).\)

Proof. The proof of 1.4 shows that (20) implies that \( UF ∈ R^+(Ω) \).

Assume the hypotheses of (ii). Now \( f = g \ast g \), where \( g(x) \) are the strong boundary values of \( G(x + iy) \) as \( y \downarrow 0 \) and \( y \uparrow 0 \). We have

\[ ||g(\cdot)c, c||^2 ≤ ||g(\cdot)c||^2 ||c||^2 = \langle f(\cdot)c, c \rangle ||c||^2 \]

for all \( c \) in \( \mathbb{C} \), so \( \langle g(\cdot)c, c \rangle \in L^2(R) \) for all \( c \) in \( \mathbb{C} \). (21) now follows from Theorem 2.6 and the fact that \( \langle g(\cdot)c, c \rangle \in M(1, u, R) \).

As an example suppose \( F(\cdot) \) is an entire function taking values in \( B(\mathbb{C}) \) such that \( \langle F(x), c \rangle \geq 0 \) whenever \( c \in \mathbb{C} \) and \( x \in R \), and there exists \( \tau \geq 0 \) and \( K > 0 \) with

\[ ||F(z)||^2 ≤ Ky^{-1} (1 + |z|^2) \sinh 2\tau y, \quad z = x + iy \in \Omega. \]

Then \( F \) is factorable, \( F(z) = \tilde{G}(z)G(z) \), where \( G(\cdot) \) is an entire function taking values in \( B(\mathbb{C}) \). This follows from Theorems 3.1 and 3.2 (i) with \( U(z) = e^{iz\tau} \). \( G(\cdot) \) is entire by the last statement in Theorem 3.1. It also is deducible from Theorem 3.6 of [5].

If in addition to above \( F(\cdot) \) satisfies \( \int_{-\infty}^{\infty} \langle F(x), c \rangle \, dx < \infty \), then \( G(\cdot) \) satisfies

\[ ||G(z)c, c||^2 ≤ K_s y^{-1} (1 + |y|^2) (1 - e^{-\tau y}) \]

for all \( z = x + iy \) with \( y \neq 0 \) and \( c \in \mathbb{C} \). \( K_s \) is a constant depending on \( c \).

4. A Fourier type transform and the Paley-Wiener representation. As before let \( U \) and \( V \) be inner functions in \( \Omega \) and denote the space \( M(u, v, R) \cap L^2(R) \) by \( M^2(u, v, R) \). This space is easily seen to be a Hilbert subspace of \( L^2(R) \). As noted in the introduction \( M^2(e^{iz\tau}, e^{iz\tau}, R) \) is the restriction to the real axis of a classical Paley-Wiener space of entire functions. That

\[ M^2(e^{iz\tau}, e^{iz\tau}, R) = \mathcal{F}L^2(-\tau, \tau), \]
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(where is the Fourier-Plancherel operator on ), is the content of a well known theorem of Paley and Wiener.

In [4] one of the present authors generalized this theorem to give an integral representation for any of the spaces .

In this section we combine this result with Theorem 2.6. First we shall set down some basic facts from [4]. For simplicity we assume that and have no zeros and are normalized so that and are positive. Then has a factorization where is a singular inner function in and . Using the usual representation for singular inner functions we can combine the two factors in the following convenient form:

\[
U(z) = \exp \left( i \int_{\mathbb{R}} \frac{1 + tz}{t - z} \mu(dt) \right)
\]

where is a finite positive measure on the extended real numbers whose restriction to is singular and with . In the integrand, and elsewhere below, we always take \((z\infty)/\infty = z\) for any complex \(z\). has a similar representation with corresponding measure \(\gamma\).

Let \(\tau\) be the total variation of \(\mu\) and suppose that \(a\) is an \(\mathbb{R}^*\)-valued measurable function defined on \([0, \tau]\) such that \(m(a^-(E)) = \mu(E)\) for every subinterval \(E\) of \(\mathbb{R}^*\). For example, we could take \(a(t) = \inf \{x \in \mathbb{R}^* : \mu((-\infty, x]) \geq t\}\). Extend the definition of \(a\) to \([0, \infty)\) by setting \(a(t) = \infty\) if \(t > \tau\). For each \(t \geq 0\) let

\[
U_t(z) = \exp \left( i \int_0^t \frac{1 + za(x)}{a(x) - z} \, dx \right).
\]

It is clear from (22) and a change of variables that \(U_t = U\). Moreover, \(U_t\) is an inner function for each \(t\) and \(U_s\) divides \(U_t\) if \(0 \leq s < t\).

In a like manner one can associate \(\sigma, b: [0, \sigma] \to \mathbb{R}^*\) and \(V_t\) (analogous to \(\tau, a\) and \(U_t\)) with the inner function \(V\). Note that \(V_s = V\). \(U_t\) and \(V_t\) have pseudo-continuations to \(\mathbb{Q}^-\) given by (12). For any \(z\) in \(\mathbb{Q} \cup \mathbb{Q}^-\) let

\[
H^+\!(t) = V_t(z) \frac{b(t) - i}{b(t) - z}
\]

and

\[
H^-\!(t) = U_t(z)^{-1} \frac{a(t) + i}{a(t) - z}, \quad t \geq 0.
\]

Now let \(H^+(\mathbb{Q})\) and \(H^-(\mathbb{Q}^-)\) denote the usual Hardy spaces of functions analytic in \(\mathbb{Q}\) and \(\mathbb{Q}^-\) respectively, which can also be con-
sidered as orthogonal complements of each other in $L^2(\mathbb{R})$. It was shown in [4] that the mappings $W_1$ and $W_2$ given by

$$(W_1g)(z) = (2\pi)^{-1/2} \int_0^\infty H_z^+(t)g(t) \, dt, \quad \text{Im } z > 0$$

and

$$(W_2g)(z) = (2\pi)^{-1/2} \int_0^\infty H_z^-(t)g(t) \, dt, \quad \text{Im } z < 0,$$

are isometries from $L^2(0, \infty)$ onto $H^2(\Omega)$ and $H^2(\Omega_-)$ respectively.

Let $E: L^2(-\infty, 0) \to L^2(0, \infty)$ be the operator $(Eg)(t) = g(-t)$. The $W_2E \oplus W_1$ can be considered as a unitary operator from $L^2(-\infty, 0) \oplus L^2(0, \infty) = L^2(\mathbb{R})$ onto $H^2(\Omega_-) \oplus H^2(\Omega) = L^2(\mathbb{R})$. This operator takes $L^2(-s, t)$ onto $\mathcal{H}^2(u, v, R)$ for all $s, t \geq 0$. If $\mu$ and $\gamma$ are supported on the singleton $\{\infty\}$ or, equivalently, if $a(t) = b(t) = \infty$ a.e., then $W_2E \oplus W_1$ is the adjoint of the Fourier-Plancherel operator. Combining this with Theorem 2.6 yields the following result.

**Theorem 4.1.** Let $U$ and $V$ be inner functions of type $\mathbb{C}$. Let $F$ be analytic in $\Omega \cup \Omega_-$ and suppose that the two sided boundary function $f(x) = \lim_{y \to 0} F(x + iy)$ exists a.e. and lies in $L^2(\mathbb{R})$. Let $s, t \geq 0$. Then the following are equivalent.

(i) \[ |F(z)|^2 \leq K (\text{Im } z)^{-i} (1 + |z|^2) (|U_s(z)|^{-2} - |V_t(z)|^{-2}), \quad z \in \Omega \cup \Omega_- . \]

(ii) There exist a.e. unique functions $g_1$ in $L^2(0, t)$ and $g_2$ in $L^2(0, s)$ such that

$F(z) = (2\pi)^{-1/2} \int_0^t H_z^+(x)g_1(x) \, dx$

$+ (2\pi)^{-1/2} \int_s^\infty H_z^-(x)g_2(x) \, dx, \quad \text{Im } z \neq 0 . \]

Moreover, \[ ||f||_2^2 = ||g_1||_2^2 + ||g_2||_2^2. \]

**Added in proof.** We refer the reader to the papers.


and,

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