# A PHRAGMÉN-LINDELÖF THEOREM WITH APPLICATIONS TO $\mathcal{M}(u, v)$ FUNCTIONS

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A well-known theorem of Paley and Wiener asserts that if $f$ is an entire function, its restriction to the real line belongs to the Hilbert space $\mathscr{F}^*L^2(-\tau, \tau)$ (where $\mathscr{F}$ is the Fourier-Plancherel operator) if and only if $f$ is square integrable on the real axis and satisfies $|f(z)| \leq Ke^{\tau|\text{Im } z|}$ for some positive $K$. The "if" part of this result may be viewed as a Phragmén-Lindelöf type theorem. The pair $(e^{i\tau x}, e^{i\tau x})$ of inner functions can be associated with the above mentioned Hilbert space in a natural way. By replacing this pair by a more general pair $(u, v)$ of inner functions it is possible to define a space $\mathscr{M}(u, v)$ of analytic functions similar to the Paley-Wiener space. For a certain class of inner functions (those of "type $\mathfrak{C}$") it is shown that membership in $\mathscr{M}(u, v)$ is implied by an inequality analogous to the exponential inequality above.

A second application of our results is to star-invariant subspaces of the Hardy space $H^2$. It is well known that if $u$ is an inner function on the circle and $f$ is in $H^2$, then in order for $f$ to be in $(uH^2)^\perp$ it is necessary for $f$ to have a meromorphic pseudocontinuation to $|z| > 1$ satisfying

$$|f(z)|^2 \leq K \frac{1 - |u(z)|^2}{1 - |z|^2}, \ |z| > 1.$$

If $u$ is inner of type $\mathfrak{C}$, it is proved that this necessary condition is also sufficient.

Let $\Gamma = \{e^{i\theta}: 0 < \theta < 2\pi\}$ be the unit circle and

$$R = \{x : -\infty < x < \infty\}$$

the real line considered as point sets in the complex plane $C$. Let $D$ and $D_-$ be the interior and exterior of the unit circle and let $\Omega$ and $\Omega_-$ be the open upper and open lower half-planes in $C$. A function $\Phi$ is *outer* on $D$ or $\Omega$ if $\Phi$ is holomorphic on $D$ or $\Omega$ and of the form

$$\Phi(z) = \exp \int_\Gamma \frac{e^{i\xi} + z}{e^{i\xi} - z} k_1(e^{i\xi}) \, \sigma(d\xi), \ z \in D,$$

or

$$\Phi(z) = \exp \frac{1}{\pi i} \int_R \frac{1 + tz}{t - z} k_2(t) dt, \ z \in \Omega,$$

where $k_1$, $k_2$ are real with $k_1 \in L^1(\Gamma)$, $k_2 \in L^1(R)$, and $\sigma$ is normalized Lebesgue measure on $\Gamma$. A function $F$ on $D$ or $\Omega$ is in $\mathfrak{R}^+$ if $F$ is holomorphic on $D$ or $\Omega$ and if there exists an outer function $\Phi$ that is not identically zero and such that $\Phi F$ is a bounded holomorphic function on $D$ or $\Omega$. If $F$ is in $\mathfrak{R}^+$ on $D$ or $\Omega$, then $f(e^{i\theta}) = \lim F(re^{i\theta})$ exists for almost all $e^{i\theta} \in \Gamma$, or

$$f(x) = \lim_{y \downarrow 0} F(x + iy)$$

exists for almost all $x$ in $R$. Such $f$ form the class $\mathcal{N}^+$ of functions on $\Gamma$ and $R$ respectively. We shall systematically use capital letters $F$, $G$, $\cdots$ for functions in $\mathfrak{R}^+$ and lower case letters $f$, $g$, $\cdots$ for the corresponding functions in $\mathcal{N}^+$.

Every outer function is in $\mathfrak{R}^+$. A function $U$ in $\mathfrak{R}^+$ is *inner* if $|u| = 1$ a.e.. Every function $F$ in $\mathfrak{R}^+$ has a factorization of the form $F = UG$, where $U$ is inner and $G$ is outer.

Suppose $U$ and $V$ are inner functions, say, on $\Omega$. $\mathcal{M}(u, v, R)$ is the set of functions $f$ on $R$ such that $uf$ and $vf^*$ are in $\mathcal{N}^+$ on $R$. ($f^*$ is the complex conjugate of $f$). $\mathcal{M}(u, v, \Gamma)$ is similarly defined. As shown in [5] one can associate with each $f$ in $\mathcal{M}(u, v, R)$ a unique function $F$ separately meromorphic in $\Omega$ and $\Omega_-$ such that $UF \in \mathfrak{R}^+$, $V\widetilde{F} \in \mathfrak{R}^+$, and

$$(1) \qquad f(x) = \lim_{y \downarrow 0} F(x + iy) = \lim_{y \downarrow 0} F(x - iy)$$

for almost all $x$ in $R$, where $\widetilde{F}(z) = F^*(z^*)$, $z \in \Omega$. If $F$ is meromorphic in $\Omega$, then an extension of $F$ to a meromorphic function on $\Omega \cup \Omega_-$ satisfying (1) is said to be a *meromorphic pseudocontinuation* (relative to $R$) of $F$. Similarly, to each $f$ in $\mathcal{M}(u, v, \Gamma)$ one associates a unique $F$ meromorphic in $D \cup D_-$ such that $UF \in \mathfrak{R}^+$, $V\widetilde{F} \in \mathfrak{R}^+$, and

$$(2) \qquad f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta})$$

for almost all $e^{i\theta} \in \Gamma$ where $\widetilde{F}(z) = F^*(z^{*-1})$, $z \in D$. Meromorphic pseudocontinuation is defined relative to $\Gamma$ in a manner analogous to the $R$ definition.

Considerations about $\mathcal{M}(u, v, R)$ may be motivated by examining the special case when $U(z) = V(z) = e^{iz\tau}$, $\tau \geqq 0$. Then

$$\mathcal{M}(u, v, R) \cap L^2(R)$$

is the class of functions that are the restrictions to $R$ of entire functions of exponential type $\leqq \tau$ such that $\int_R |F(x)|^2 dx < \infty$. Such entire $F$ can be characterized by this integral condition together

with the inequality

$$|F(z)|^2 < K|y|^{-1}|\sinh(2\tau y)|$$

for all $z \in \Omega \cup \Omega_-$, where $K > 0$. The object of this paper is to extend this type of function-theoretic characterization to more general $\mathscr{M}(u, v)$ classes. The above mentioned application to star-invariant subspaces arises from the fact that $M(1, v) \cap L^2(R) = H^2(\Omega) \ominus vH^2(\Omega)$, where $H^2(\Omega)$ is the Hardy space of the upper half-plane. In §3 and 4 applications are given to factorization problems for nonnegative operator-valued functions and to generalized Paley-Wiener representations.

**1. A Phragmén-Lindelöf Theorem.** In this section we shall derive a Phragmén-Lindelöf type theorem for certain functions holomorphic on $D$, and then transcribe the result to obtain a like theorem for functions on $\Omega$. A rather different Phragmén-Lindelöf type theorem is discussed by Helson in [2, p. 33].

Recall that a Blaschke product $B$ on $D$ has a representation

$$(3) \qquad B(z) = \prod_{j \geq 1} B_j(z), \quad B_j(z) = \frac{z_j^*}{|z_j|} \frac{z_j - z}{1 - z_j^* z}, \quad z \in D,$$

where $\sum_{j \geq 1}(1 - |z_j|) < \infty$. We take $z_j^*/|z_j| = 1$ if $z_j = 0$. The *support* supp $B$ of $B$ is the intersection of $\Gamma$ with the closure of $\{z_j\}_{j \geq 1}$. A singular inner function $S$ has a representation

$$(4) \qquad S(z) = \exp\left(-\int_\Gamma \frac{e^{i\xi} + z}{e^{i\xi} - z}\mu(d\xi)\right), \quad z \in D,$$

where $\mu$ is a positive singular measure on $\Gamma$. The *support* supp $S$ is the closed support of the measure $\mu$.

Any inner function $U$ on $D$ can be factored in the form $U = cBS$, where $c \in C$, $|c| = 1$, $B$ is a Blaschke product and $S$ is a singular inner function. The *support* supp $U$ of $U$ is supp $B \cup$ supp $S$.

A closed set $N$ on $\Gamma$ is a *Carleson set* if $N$ has zero Lebesgue measure and if the complement of $N$ in $\Gamma$ is a union of open arcs $I_j$ of lengths $\varepsilon_j$ such that $\sum_{j \geq 1} \varepsilon_j \log \varepsilon_j > -\infty$.

THEOREM 1.1. (Carleson [1]). *A closed subset $N$ of $\Gamma$ is a Carleson set on $\Gamma$ if and only if there exists an outer function $G$ on $D$ that satisfies a Lipschitz condition and such that*

$$g(e^{i\theta}) \overset{\text{def}}{=} \lim_{r \uparrow 1} G(re^{i\theta})$$

*vanishes on $N$.*

DEFINITION 1.2. An inner function $U$ on $D$ is of *type* $\mathfrak{E}$ if
( i )  supp $U$ is a Carleson set, and
( ii )  $\sum_{j \geq 1} [\text{dist}(z_j, \text{supp } U)] < \infty$,
where $\{z_j\}_{j \leq 1}$ are the zeros of $U$ in $D$ repeated according to multiplicity.

LEMMA 1.3. *Let $B$ be the Blaschke product given by* (3) *and suppose $B$ is of type* $\mathfrak{E}$. *If $G$ is a Lipschitz outer function on $D$ such that $g(e^{i\theta}) = \lim_{r \uparrow 1} G(re^{i\theta})$ vanishes on supp $B$, then*

$$(5) \qquad \sum_{j \geq 1} (1 - |z_j|^2) \int |(1 - z_j^* e^{i\theta})^{-1} g(e^{i\theta})|^2 \, \sigma(d\theta) < \infty .$$

*Proof.* Since $G$ is Lipschitz there exists $K > 0$ such that

$$|g(e^{i\theta})| \leq K |e^{i\theta} - \lambda|$$

for all $e^{i\theta}$ in $\Gamma$ and $\lambda$ in supp $B$. Thus for $\lambda$ in supp $B$,

$$(1 - |z_j|^2) \int |(1 - z_j^* e^{i\theta})^{-1} g(e^{i\theta})|^2 \, \sigma(d\theta)$$

$$\leq (1 - |z_j|^2) K^2 \int |(1 - z_j^* e^{i\theta})^{-1} (e^{i\theta} - \lambda)|^2 \, \sigma(d\theta) .$$

Applying Parseval's equality to the Fourier series for the function $(1 - z_j^* e^{i\theta})^{-1} (e^{i\theta} - \lambda)$ shows that this last expression is equal to

$$K^2 (|z_j - \lambda|^2 + (1 - |z_j|^2)) .$$

Since $\sum_{j \geq 1}(1 - |z_j|^2) < \infty$ and we are free to let $\lambda$ vary over supp $B$ this inequality implies (5).

The following theorem is our Phragmén-Lindelöf result for functions on $D$.

THEOREM 1.4. *Let $U$ be an inner function of type* $\mathfrak{E}$ *on $D$. Suppose $F$ is holomorphic in $D$ and there exists $M > 0$ such that*

$$(6) \qquad |F(z)|^2 \leq M(1 - |z|^2)^{-1} (1 - |U(z)|^2), \quad z \in D .$$

*Then $F \in \mathfrak{R}^+$.*

*Proof.* $U$ has the factorization $U = cBS$, where $|c| = 1$, $B$ is a Blaschke product of type $\mathfrak{E}$ and $S$ is a singular inner function of type $\mathfrak{E}$. We have

$$(7) \quad (1 - |z|^2)^{-1} (1 - |U(z)|^2)$$
$$= (1 - |z|^2)^{-1} (1 - |B(z)|^2) + |B(z)|^2 (1 - |z|^2)^{-1} (1 - |S(z)|^2)$$
$$\leq (1 - |z|^2)^{-1} (1 - |B(z)|^2) + (1 - |z|^2)^{-1} (1 - |S(z)|^2), \quad z \in D .$$

If $B$ is given by (3), then

$$1 - |B(z)|^2 = 1 - |B_1(z)|^2 + \sum_{n \geq 2} \left| \prod_{j=1}^{n-1} B_j(z) \right|^2 (1 - |B_n(z)|^2)$$

$$\leq \sum_{j \geq 1} (1 - |B_j(z)|^2) .$$

Thus

$$(8) \qquad (1 - |z|^2)^{-1} (1 - |B(z)|^2) \leq \sum_{j \geq 1} (1 - |z_j|^2) |1 - z_j^* z|^{-2} .$$

If $S$ is given by (4), then

$$|S(z)|^2 = \exp\left\{ -2 \int_\Gamma (1 - |z|^2) |e^{i\xi} - z|^{-2} \mu(d\xi) \right\}, \ z \in D .$$

Applying the elementary inequality $(1 - e^{-ah})/h) \leq a$ if $a, h \geq 0$, with $h = 1 - |z|^2$ and $a = 2 \int_\Gamma |e^{i\xi} - z|^{-2} \mu(d\xi)$ yields

$$(9) \qquad (1 - |z|^2)^{-1} (1 - |S(z)|^2) \leq 2 \int_\Gamma |e^{i\xi} - z|^{-2} \mu(d\xi), \ z \in D .$$

Suppose now that (6) holds and let $G$ be a Lipschitz outer function such that $g(e^{i\theta}) = \lim_{r \uparrow 1} G(re^{i\theta})$ vanishes on supp $U$. We have from (6) — (9) that

$$|G(z)F(z)|^2 \leq M \sum_{j \geq 1} (1 - |z_j|^2) |1 - z_j^* z|^{-2} |G(z)|^2$$

$$+ 2M \int_\Gamma |e^{i\xi} - z|^{-2} |G(z)|^2 \mu(d\xi), \ z \in D .$$

But for some $K > 0$

$$|G(z)|^2 \leq K^2 |e^{i\xi} - z|^2 \ \text{if} \ e^{i\xi} \in \text{supp } U ,$$

and $\mu$ is supported on supp $S \subsetneq$ supp $U$. Thus for all $z \in D$

$$|G(z)F(z)|^2 \leq M \sum_{j \geq 1} (1 - |z_j|^2) |1 - z_j^* z|^{-2} |G(z)|^2 + 2M K^2 \mu(\Gamma) .$$

It now follows from Lemma 1.3 that

$$\sup_{0 \leq r < 1} \int_\Gamma |G(re^{i\theta})F(re^{i\theta})|^2 \sigma(d\theta) \qquad < \infty \qquad ,$$

so $GF \in H^2$. It is easy to multiply $G$ by an outer function $G_1$ and obtain $G_1 GF$ bounded, and so $F$ is in $\mathfrak{N}^+$.

We shall next recast Theorem 1.4 for functions holomorphic on $\Omega$. Any inner function $U$ on $\Omega$ has a factorization $U = cBSV^a$, where $c \in C$, $|c| = 1$, $B$ is a Blaschke product on $\Omega$, $S$ is a singular function on $\Omega$, and $V^a(z) = e^{iaz}$, where $0 \leq a \in R$. Then supp $B$ is defined to be the set of limit points on $R \cup \{\infty\}$ of the zeros of $B$,

and supp $S$ is defined to be the support of the singular measure in the representation for $S$ analogous to (4), (Hoffman [3] p. 132–133). We define supp $V^a$ to be empty if $a = 0$, and $\{\infty\}$ if $a > 0$. The *support* supp $U$ of $U$ is supp $B \bigcup$ supp $S \bigcup$ supp $V^a$.

A closed subset $N$ of the extended real line $R \cup \{\infty\}$ is a *Carleson set* if $N \cap R$ has Lebesgue measure zero, $\infty \in N$, and the complement of $N$ in $R \cup \{\infty\}$ is a union of open intervals

$$I_j = (a_j, b_j), \ -\infty \leqq a_j < b_j \leqq \infty, \ j = 1, 2, \cdots$$

such that $\sum_{j>1} \delta_j \log \delta_j > -\infty$, where

$$\delta_j = \frac{b_j - a_j}{(1 + b_j^2)^{1/2} (1 + a_j^2)^{1/2}}, \quad j = 1, 2, \cdots.$$

We understand in the above that $\infty / \infty = 1$.

Now let $\alpha : \bar{D} \to \bar{\Omega} \cup \{\infty\}$ be the mapping defined by

$$\alpha(z) = i(1 + z)(1 - z)^{-1}$$

if $z \neq 1$ and $\alpha(1) = \infty$, and let $\beta$ be the inverse of $\alpha$. Then if $z_1, z_2 \in \bar{\Omega}$,

$$| \beta(z_1) - \beta(z_2) |^2 = 4 \frac{| z_1 - z_2 |^2}{| z_1 + i |^2 | z_2 + i |^2}.$$

Moreover $\beta$ maps $(-\infty, \infty]$ onto $\Gamma$ and $N$ is a Carleson set on $R \cup \{\infty\}$ if and only if $\beta(N) \cup \{1\}$ is a Carleson set on $\Gamma$. If $U$ is inner on $\Omega$ then $U \circ \alpha$ is inner on $D$ and supp $(U \circ \alpha) = \beta$ (Supp $U$). Furthermore if $\{z_j\}_{j>1}$ is the sequence of zeros of $U$, then $\{\beta(z_j)\}_{j \geqq 1}$ is the sequence of zeros of $U \circ \alpha$.

DEFINITION 1.5. Let $U$ be an inner function on $\Omega$. $U$ is of *type* $\mathfrak{E}$ if supp $U \cup \{\infty\}$ is a Carleson set on $R \cup \{\infty\}$ and

$$\sum_{j \geqq 1} \left( \inf_{\lambda \in \text{supp } U} \frac{| z_j - \lambda |^2}{(1 + \lambda^2)(1 + | z_j |^2)} \right) < \infty,$$

where $\{z_j\}_{j \geqq 1}$ is the sequence of zeros of $U$ in $\Omega$ repeated according to multiplicity.

The following lemma follows from the above discussion.

LEMMA 1.6. *Let $U$ be inner on $\Omega$. Then $U$ is of type $\mathfrak{E}$ if and only if $U \circ \alpha$ is of type $\mathfrak{C}$ on $D$.*

We can now recast Theorem 1.4 for the half-plane.

THEOREM 1.7. *Let $F$ be holomorphic in $\Omega$ and suppose that $U$ is inner of type $\mathfrak{E}$ in $\Omega$. Suppose that there exists $K > 0$ such that*

(10)    $|F(z)|^2 \leqq K(\operatorname{Im} z)^{-1}(1 + |z|^2)(1 - |U(z)|^2)$ *for* $z \in \Omega$.

*Then* $F \in \mathfrak{N}^+$ *on* $\Omega$.

   *Proof.* Set $G = F \circ \alpha$, so $G$ is meromorphic on $D$ and

$$|G(z)|^2 \leqq K[\operatorname{Im} \alpha(z)]^{-1}(1 + |\alpha(z)|^2)(1 - |U(\alpha(z))|^2), \quad z \in D.$$

We can replace $1 + |\alpha(z)|^2$ by $|i + \alpha(z)|^2$ and the inequality still holds but for a different constant $K$. Now

$$\operatorname{Im} \alpha(z) = (1 - |z|^2)|1 - z|^{-2}$$

and

$$|i + \alpha(z)|^2 = 4|1 - z|^{-2},$$

so

$$|G(z)|^2 \leqq K'(1 - |z|^2)^{-1}(1 - |U(\alpha(z)|^2), \quad z \in D.$$

But by Lemma 1.6 $U \circ \alpha$ is of type $\mathfrak{E}$, and thus Theorem 1.4 implies that $G \in \mathfrak{N}^+$ on $D$. We then deduce that $F = G \circ \beta$ is in $\mathfrak{N}^+$ on $\Omega$.

   2. **The classes $\mathscr{M}(u, v, \Gamma)$ and $\mathscr{M}(u, v, R)$.** Suppose $U$ is inner in $D$. Then $U$ has a meromorphic pseudocontinuation to a function $U$ on $D \cup D_-$ that is given by

(11)    $$U(z) = \begin{cases} U(z), & z \in D \\ 1/U^*(z^{*-1}), & z \in D_- . \end{cases}$$

If supp $U \neq \Gamma$, then $U$ on $D$ has a single valued meromorphic continuation to $D_-$ that coincides with $U$ as given by (11). If $F$ is meromorphic on $D_-$ then $\widetilde{F}(z) = F^*(z^{*-1})$ defines $\widetilde{F}$ to be meromorphic on $D$. Of course $\widetilde{F}$ need not be a pseudocontinuation of $F$.

   Analogous definitions are made for $\Omega$. Suppose $U$ is inner on $\Omega$. Then $U$ has a meromorphic pseudocontinuation on $\Omega \cup \Omega_-$ given by

(12)    $$U(z) = \begin{cases} U(z) & z \in \Omega \\ 1/U^*(z^*) & z \in \Omega_- . \end{cases}$$

If $F$ is meromorphic on $\Omega$, then $\widetilde{F}(z) = F^*(z^*)$ defines $\widetilde{F}$ to be meromorphic on $\Omega_-$.

   We say that $F$ is $\mathfrak{N}_0^+$ on $D$ if $F \in \mathfrak{N}^+$ on $D$ and $F(0) = 0$. $\mathscr{N}_0^+$ is defined to be the set of all $f$ such that $f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta})$ a.e., where $F \in \mathfrak{N}_0^+$ on $D$.

   Suppose $U, V$ are inner functions on $D$. $\mathscr{M}_0(u, v, \Gamma)$ is the set

of all functions $f$ on $\Gamma$ such that $uf \in \mathscr{N}^+$ and $vf^* \in \mathscr{N}_0^+$. $\mathscr{M}_0(u, v, \Gamma)$ can be characterized as follows: $f \in \mathscr{M}_0(u, v, \Gamma)$ if and only if there exists a function $F$ separately meromorphic in $D$ and $D_-$ and such that

(13)            $$f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta}) \quad \text{a.e.,}$$

with

(14)            $$UF \in \mathfrak{R}^+ \text{ on } D \text{ and } V\widetilde{F} \in \mathfrak{R}_0^+ \text{ on } D .$$

In case $U$ and $V$ are of type $\mathfrak{E}$ we can deduce (14) from an inequality involving $F$, $U$ and $V$.

THEOREM 2.1. *Suppose $U$ and $V$ are of type $\mathfrak{E}$, and $F$ is meromorphic in $D$ and has a meromorphic pseudocontinuation to a function $F$ on $D \cup D_-$. Further suppose there exists $K > 0$ such that*

(15)        $$|F(z)|^2 \leqq K(1 - |z|^2)^{-1} (|U(z)|^{-2} - |V(z)|^2), \quad |z| \neq 1.$$

*Then $f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) \in \mathscr{M}_0(u, v, \Gamma)$.*

*Proof.* If $F$ satisfies (15) on $D$ then

$$|U(z)F(z)|^2 \leqq K(1 - |z|^2)^{-1} (1 - |U(z)V(z)|^2) ,$$

so $UF \in \mathfrak{R}^+$ by Theorem 1.4.

If $F$ satisfies (15) on $D_-$, then for all $z \in D$,

$$|V(z)\widetilde{F}(z)|^2 \leqq K|z|^2 (1 - |z|^2)^{-1} (1 - |U(z)V(z)|^2)$$

so $V\widetilde{F} \in \mathfrak{R}^+$ by 1.4. But we also deduce that $V(0)\widetilde{F}(0) = 0$, so $V\widetilde{F} \in \mathfrak{R}_0^+$. It therefore follows from the characterization of $\mathscr{M}_0(u, v, \Gamma)$ given in (13) and (14) that $f \in \mathscr{M}_0(u, v, \Gamma)$.

In case $f \in L^2(\Gamma)$, i.e., in case $\int |f|^2 d\sigma < \infty$, we have a stronger result.

THEOREM 2.2. *Assume that $U$, $V$ are inner of type $\mathfrak{E}$ on $D$ and $f \in L^2(\Gamma)$. Then $f \in \mathscr{M}_0(u, v, \Gamma)$ if and only if there exists a function $F$ satisfying the hypotheses of Theorem 2.1 such*

$$f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) \quad \text{a.e..}$$

*Proof.* It follows from Theorem 2.1 that if $F$ satisfies (15) then $f \in \mathscr{M}_0(u, v, \Gamma)$. Conversely, suppose $f \in \mathscr{M}_0(u, v, \Gamma) \cap L^2(\Gamma)$. Then $uf \in \mathscr{N}^+ \cap L^2(\Gamma) = H^2$ and $vf^* \in \mathscr{N}_0^+ \cap L^2(\Gamma) \subsetneqq H^2$ with $\int vf^* d\sigma = 0$.

Thus $uf$ and $v\chi^*f^*$ are in $(uvH^2)^\perp \cap H^2$, where $\chi(e^{i\theta}) = e^{i\theta}$.

Now any $g \in (uvH^2)^\perp \cap H^2$ is the boundary value function of

$$G(z) = \int (1 - ze^{-i\xi})^{-1} (1 - u^*(e^{i\xi})v^*(e^{i\xi})U(z)V(z))g(e^{i\xi})\sigma(d\xi), \quad z \in D \,.$$

But then it follows from the Schwarz inequality that

$$(16) \qquad |G(z)|^2 \leqq K(1 - |z|^2)^{-1} (1 - |U(z)V(z)|^2), \quad z \in D \,,$$

where $K = \int |g|^2 \, d\sigma$.

By applying (16) to $g = uf$ and $g = v\chi^*f^*$ we obtain

$$(17) \qquad |U(z)F(z)|^2 \leqq K(1 - |z|^2)^{-1} (1 - |U(z)V(z)|^2), \quad z \in D \,,$$

and

$$(18) \qquad |V(z)\widetilde{F}(z)|^2 \leqq K|z|^2 (1 - |z|^2)^{-1} (1 - |U(z)V(z)|^2), \quad z \in D \,,$$

where $K = \int_\Gamma |f|^2 \, d\sigma$.

It is easily seen that (17) and (18) together is equivalent to (15).

COROLLARY 2.3. *Assume that $V$ is inner of type $\mathfrak{C}$ on $D$ and $f \in H^2$ on $\Gamma$. Then $f \in (vH^2)^\perp$ if and only if there exists a meromorphic function $F$ on $D \cup D_-$ such that*

$$(19) \qquad f(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta}) = \lim_{r\downarrow 1} F(re^{i\theta}) \quad \text{a.e.,}$$

*for which there exists $K > 0$ with*

$$|F(z)|^2 \leqq K(1 - |z|^2)^{-1} (1 - |V(z)|^2), \quad z \in D \cup D_- \,.$$

*Proof.* Note that $(vH^2)^\perp \cap H^2 = \mathcal{M}_0(1, v, \Gamma)$, and use 2.2.

COROLLARY 2.4. *Assume that $U, V$ are inner of type $\mathfrak{C}$ on $D$ and $f \in L^2(\Gamma)$. Then $f \in \mathcal{M}(u, v, \Gamma)$ if and only if there exists a function $F$ meromorphic in $D$ with pseudocontinuation $F$ such that (19) holds and there exists $K > 0$ such that*

$$|F(z)|^2 \leqq K(1 - |z|^2)^{-1} (|U(z)|^{-2} - |zV(z)|^2), \quad z \in D \,.$$

*Proof.* Note that $\mathcal{M}(u, v, \Gamma) = \mathcal{M}_0(u, \chi v, \Gamma)$.

The same kind of problem can be considered on $\Omega$ with minor modifications in the proofs.

THEOREM 2.5. *Suppose $F$ is meromorphic on $\Omega$ and has a mero-*

*morphic pseudocontinuation to a function $F$ on $\Omega \cup \Omega_-$. Assume that $U$ and $V$ are inner functions of type $\mathbb{C}$ on $\Omega$. Further suppose that there exists $K > 0$ such that*

$$| F(z) |^2 \leqq K(\operatorname{Im} z)^{-1} (1 + | z |^2) (| U(z) |^{-2} - | V(z) |^2), \ z \in \Omega \cap \Omega_- \ .$$

Then $f(x) = \lim_{y \downarrow 0} F(x + iy) \in \mathscr{M}(u, v, R)$.

**THEOREM 2.6.** *Assume that $U$, $V$ are inner of type $\mathbb{C}$ on $\Omega$ and $f \in L^2(R)$. Then $f \in \mathscr{M}(u, v, R)$ if and only if there exists a function satisfying the hypotheses of Theorem 2.5 such that*

$$f(x) = \lim_{y \downarrow 0} F(x + iy) \quad \text{a.e..}$$

**3. Factorization of nonnegative functions.** In this section we shall reformulate an operator factorization theorem of the type set down in [5] in terms of inequalities of the type discussed in § 1 and 2. Throughout $\mathscr{C}$ is a complex separable Hilbert space and $B(\mathscr{C})$ the space of bounded operators on $\mathscr{C}$. We shall restrict ourselves to considerations involving $\Omega$ rather than $D$ in order to simplify the exposition. Following [5] we say that a holomorphic function $F$ on $\Omega$ taking values in $B(\mathscr{C})$ is in $\mathfrak{N}_{B(\mathscr{C})}^+$ if there exists a nonzero complex-valued outer function $\Phi$ such that $\Phi F$ is a bounded holomorphic function on $\Omega$ that takes values in $B(\mathscr{C})$. Any $F$ in $\mathfrak{N}_{B(\mathscr{C})}^+$ has strong boundary values a.e., that is, the limit $\lim_{y \downarrow 0} F(x + iy) = f(x)$ exists a.e. in the strong operator topology.

We say that a holomorphic function $G$ in $\mathfrak{N}_{B(\mathscr{C})}^+$ has a *meromorphic pseudocontinuation* $G$ if $G$ is meromorphic in $\Omega_-$ and the strong limits $\lim_{y \uparrow 0} G(x - iy)$ and $\lim_{y \uparrow 0} G(x + iy)$ exist and are a.e. equal. For such $G$ we define $\widetilde{G}$ by $\widetilde{G}(z) = G^*(z^*)$, $z \in \Omega \cup \Omega_-$.

**THEOREM 3.1.** *Let $U$ be a complex-valued inner function on $\Omega$ and $F$ a meromorphic function on $\Omega$ taking values in $B(\mathscr{C})$ such that $UF \in \mathfrak{N}_{B(\mathscr{C})}^+$. Then $F(x + iy)$ has strong boundary values $f(x)$ a.e. as $y \downarrow 0$. Assume that $\langle f(x)c, c \rangle \geqq 0$ a.e. for each $c$ in $\mathscr{C}$.*

*Then $F$ has a factorization $F(z) = \widetilde{G}(z)G(z)$, $z \in \Omega$, where $G$ is in $\mathfrak{N}_{B(\mathscr{C})}^+$ and has a meromorphic pseudocontinuation $G$ such that $U\widetilde{G} \in \mathfrak{N}_{B(\mathscr{C})}^+$. If there is real interval $I$ such that $f(.)$ is a.e. bounded on $I$ and $U$ is analytically continuable across $I$, then $G$ is analytically continuable across $I$.*

*Proof.* This theorem is a summary of results proved in [5].

**THEOREM 3.2.** *Theorem 3.1 may be modified as follows:*

(i)  *The hypothesis "$UF \in \mathfrak{N}^+_{B(\mathscr{C})}$" may be replaced by the stronger hypothesis "there exists $K > 0$ such that*

(20) $$\| F(z) \|^2 \leq K (\mathrm{Im}\, z)^{-1} (1 + |z|^2) (|U(z)|^{-2} - |U(z)|^2)$$

*for all $z$ in $\Omega$".*

(ii)  *If in addition one assumes that $\int_{-\infty}^{\infty} \langle f(x)c, c \rangle\, dx < \infty$ for all $c$ in $\mathscr{C}$, then $G$ can be chosen to in addition satisfy*

(21) $$|\langle G(z)c, c \rangle|^2 \leq K_c (\mathrm{Im}\, z)^{-1} (1 + |z|^2) (1 - |U(z)|^2), \quad c \in \mathscr{C}$$

*for some $K_c > 0$ ($K_c$ depends on $c$) and all $z \in \Omega \cup \Omega_-$.*

*Proof.*  The proof of 1.4 shows that (20) implies that $UF \in \mathfrak{N}^+_{B(\mathscr{C})}$.

Assume the hypotheses of (ii). Now $f = g^*g$, where $g(x)$ are the strong boundary values of $G(x + iy)$ as $y \downarrow 0$ and $y \uparrow 0$. We have $|\langle g(\cdot)c, c \rangle|^2 \leq \| g(\cdot)c \|^2 \| c \|^2 = \langle f(\cdot)c, c \rangle \| c \|^2$ for all $c$ in $\mathscr{C}$, so $\langle g(\cdot)c, c \rangle \in L^2(R)$ for all $c$ in $\mathscr{C}$. (21) now follows from Theorem 2.6 and the fact that $\langle g(\cdot)c, c \rangle \in \mathscr{M}(1, u, R)$.

As an example suppose $F(\cdot)$ is an entire function taking values in $B(\mathscr{C})$ such that $\langle F(x)c, c \rangle \geq 0$ whenever $c \in \mathscr{C}$ and $x \in R$, and there exists $\tau \geq 0$ and $K > 0$ with

$$\| F(z) \|^2 \leq Ky^{-1} (1 + |z|^2) \sinh 2\tau y, \quad z = x + iy \in \Omega \,.$$

Then $F$ is factorable, $F(z) = \tilde{G}(z)G(z)$, where $G(\cdot)$ is an entire function taking values in $B(\mathscr{C})$. This follows from Theorems 3.1 and 3.2 (i) with $U(z) = e^{i\tau z}$. $G(\cdot)$ is entire by the last statement in Theorem 3.1. It also is deducible from Theorem 3.6 of [5].

If in addition to above $F(\cdot)$ satisfies $\int_{-\infty}^{\infty} \langle F(x)c, c \rangle\, dx < \infty$, then by (21) $G$ satisfies

$$|\langle G(z)c, c \rangle|^2 \leq K_c y^{-1} (1 + |z|^2) (1 - e^{-\tau y}) \,,$$

for all $z = x + iy$ with $y \neq 0$ and $c \in \mathscr{C}$. $K_c$ is a constant depending on $c$.

4.  **A Fourier type transform and the Paley-Wiener representation.**  As before let $U$ and $V$ be inner functions in $\Omega$ and denote the space $\mathscr{M}(u, v, R) \cap L^2(R)$ by $\mathscr{M}^2(u, v, R)$. This space is easily seen to be a Hilbert subspace of $L^2(R)$. As noted in the introduction $\mathscr{M}^2(e^{ix\tau}, e^{ix\tau}, R)$ is the restriction to the real axis of a classical Paley-Wiener space of entire functions. That

$$\mathscr{M}^2(e^{ix\tau}, e^{ix\tau}, R) = \mathscr{F}^* L^2(-\tau, \tau) \,,$$

(where $\mathscr{F}$ is the Fourier-Plancherel operator on $L^2(R)$), is the content of a well known theorem of Paley and Wiener.

In [4] one of the present authors generalized this theorem to give an integral representation for any of the spaces $\mathscr{M}^2(u, v, R)$. In this section we combine this result with Theorem 2.6. First we shall set down some basic facts from [4]. For simplicity we assume that $U$ and $V$ have no zeros and are normalized so that $U(i)$ and $V(i)$ are positive. $U$ then has a factorization $U(z) = S(z)e^{i\alpha z}$ where $S$ is a singular inner function in $\Omega$ and $\alpha \geqq 0$. Using the usual representation for singular inner functions we can combine the two factors in the following convenient form:

$$(22) \qquad U(z) = \exp\left( i \int_{R^*} \frac{1 + tz}{t - z} \, \mu(dt) \right)$$

where $\mu$ is a finite positive measure on the extended real numbers $R^* = R \cup \{\infty\}$ whose restriction to $R$ is singular and with $\mu(\{\infty\}) = \alpha$. In the integrand, and elsewhere below, we always take $(z \infty)/\infty = z$ for any complex $z$. $V$ has a similar representation with corresponding measure $\gamma$.

Let $\tau$ be the total variation of $\mu$ and suppose that $a$ is an $R^*$-valued measurable function defined on $[0, \tau]$ such that $m(a^{-1}(E)) = \mu(E)$ for every subinterval $E$ of $R^*$. For example, we could take $a(t) = \inf\{x \in R^*: \mu((-\infty, x]) \geqq t\}$. Extend the definition of $a$ to $[0, \infty)$ by setting $a(t) = \infty$ if $t > \tau$. For each $t \geqq 0$ let

$$U_t(z) = \exp\left( i \int_0^t \frac{1 + za(x)}{a(x) - z} \, dx \right).$$

It is clear from (22) and a change of variables that $U_\tau = U$. Moreover, $U_t$ is an inner function for each $t$ and $U_s$ divides $U_t$ if $0 \leqq s < t$.

In a like manner one can associate $\sigma, b: [0, \sigma] \to R^*$ and $V_t$ (analogous to $\tau$, $a$ and $U_t$) with the inner function $V$. Note that $V_\sigma = V$. $U_t$ and $V_t$ have pseudo-continuations to $\Omega_-$ given by (12). For any $z$ in $\Omega \cup \Omega_-$ let

$$H_z^+(t) = V_t(z) \frac{b(t) - i}{b(t) - z}$$

and

$$H_z^-(t) = U_t(z)^{-1} \frac{a(t) + i}{a(t) - z}, \qquad t \geqq 0 .$$

Now let $H^2(\Omega)$ and $H^2(\Omega_-)$ denote the usual Hardy spaces of functions analytic in $\Omega$ and $\Omega_-$ respectively, which can also be con-

sidered as orthogonal complements of each other in $L^2(R)$. It was shown in [4] that the mappings $W_1$ and $W_2$ given by

$$(W_1 g)(z) = (2\pi)^{-1/2} \int_0^\infty H_z^+(t) g(t)\, dt, \ \text{Im } z > 0$$

and

$$(W_2 g)(z) = (2\pi)^{-1/2} \int_0^\infty H_z^-(t) g(t)\, dt, \ \text{Im } z < 0 \,,$$

are isometries from $L^2(0, \infty)$ onto $H^2(\Omega)$ and $H^2(\Omega_-)$ respectively.

Let $E: L^2(-\infty, 0) \to L^2(0, \infty)$ be the operator $(Eg)(t) = g(-t)$. The $W_2 E \oplus W_1$ can be considered as a unitary operator from

$$L^2(-\infty, 0) \oplus L^2(0, \infty) = L^2(R)$$

onto $H^2(\Omega_-) \oplus H^2(\Omega) = L^2(R)$. This operator takes $L^2(-s, t)$ onto $\mathcal{M}^2(u_s, v_t, R)$ for all $s, t \geqq 0$. If $\mu$ and $\gamma$ are supported on the singleton $\{\infty\}$ or, equivalently, if $a(t) = b(t) = \infty$ a.e., then $W_2 E \oplus W_1$ is the adjoint of the Fourier-Plancherel operator. Combining this with Theorem 2.6 yields the following result.

THEOREM 4.1. *Let $U$ and $V$ be inner functions of type $\mathfrak{C}$. Let $F$ be analytic in $\Omega \cup \Omega_-$ and suppose that the two sided boundary function $f(x) = \lim_{|y| \to 0} F(x + iy)$ exists a.e. and lies in $L^2(R)$. Let $s, t \geqq 0$. Then the following are equivalent.*

( i )

$$| F(z) |^2 \leqq K\, (\text{Im } z)^{-1}\, (1 + | z |^2)\, (| U_s(z) |^{-2} - | V_t(z) |^2),\ z \in \Omega \cup \Omega_- \,.$$

(ii)    *There exist a.e. unique functions $g_1$ in $L^2(0, t)$ and $g_2$ in $L^2(0, s)$ such that*

$$F(z) = (2\pi)^{-1/2} \int_0^t H_z^+(z) g_1(x)\, dx$$

$$+ (2\pi)^{-1/2} \int_0^s H_z^-(x) g_2(x)\, dx, \ \text{Im } z \neq 0 \,.$$

*Moreover, $\| f \|_2^2 = \| g_1 \|_2^2 + \| g_2 \|_2^2$.*

*Added in proof.* We refer the reader to the papers.

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for more detailed information on meromorphic continuation and $(uH^2)^\perp$.

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# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

# Pacific Journal of Mathematics

## Vol. 43, No. 1      March, 1972