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ON THE UNIVALENCE OF SOME ANALYTIC FUNCTIONS

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Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

be analytic and satisfy

$$\operatorname{Re}\left(f(z)/[\lambda f(z)+(1-\lambda)\,g(z)]\right)>0$$

or

$$|\,f(z)/[\lambda f(z) + (1-\lambda)\,g(z)] - 1\,|\,<\,1$$
 for $|\,z\,|\,<\,1$, $0 \le \lambda < 1$.

We propose to determine the values of R such that f(z) is univalent and starlike for $\mid z\mid < R$ under the assumption

(i) $\operatorname{Re}(g(z)/z) > 0$, or (ii) $\operatorname{Re}(zg'(z)/g(z)) > \alpha$, $0 \le \alpha < 1$.

We also consider the case when n=1 and $\mathrm{Re}\,(g(z)/z)>1/2$ and show that under condition (a) f(z) is univalent and starlike for $|z|<(1-\lambda)/(3+\lambda)$.

2. LEMMA 1. If $p(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ is analytic and satisfies $\text{Re}(p(z)) > \alpha$, $0 \le \alpha < 1$, for |z| < 1, then

(1)
$$p(z) = [1 + (2\alpha - 1)z^n u(z)]/[1 + z^n u(z)], \quad \textit{for } |z| < 1,$$
 where $u(z)$ is analytic and $|u(z)| \le 1$ for $|z| < 1.$

Proof. Let

(2)
$$F(z) = [p(z) - \alpha]/(1 - \alpha) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

F(z) is analytic and Re (F(z)) > 0 for |z| < 1 and hence

$$(3) h(z) = [1 - F(z)]/[1 + F(z)] = d_n z^n + d_{n+1} z^{n+1} + \cdots,$$

is analytic and |h(z)| < 1 for |z| < 1. Thus, by Schwarz's lemma

$$h(z) = z^n u(z) ,$$

where u(z) is analytic and $|u(z)| \le 1$ for |z| < 1. Now equations (2), (3) and (4) prove (1).

LEMMA 2. Under the hypothesis of Lemma 1 we have for |z| < 1

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$$|zp'(z)/p(z)| \leq 2nz^n(1-\alpha)/\{(1-|z|^n)[1+(1-2\alpha)|z|^n]\}.$$

Proof. Proceeding as in the proof of Lemma 1, we have in view of (3) and a result of Goluzin [1] that for |z| < 1

$$|h'(z)| \leq n |z|^{n-1} (1 - |h(z)|^2) / (1 - |z|^{2n}).$$

Using (3), the inequality (5) takes the form

$$|F'(z)| \le 2n |z|^{n-1} \operatorname{Re} (F(z))/(1 - |z|^{2n})$$
.

Hence, in view of (2),

$$|p'(z)| \le 2n |z|^{n-1} [\operatorname{Re}(p(z)) - \alpha]/(1 - |z|^{2n})$$

or,

$$|zp'(z)/p(z)| \leq 2n |z|^n (1-\alpha/(|p(z)|)/(1-|z|^{2n}).$$

Equation (4) gives

$$|h(z)| \leq |z|^n \qquad \text{for } |z| < 1,$$

and hence, by virtue of (3),

(9)
$$|F(z)| \le (1 + |z|^n)/(1 - |z|^n)$$
 for $|z| < 1$.

From (2) and (9),

$$| p(z) | = | \alpha + (1 - \alpha)F(z) |$$

 $\leq \alpha + (1 - \alpha) | F(z) |$
 $\leq [1 + (1 - 2\alpha) | z|^n]/(1 - | z|^n).$

The inequality (7), because of the last inequality, reduces to

$$|zp'(z)/p(z)| \le 2n |z|^n (1-\alpha)/\{(1-|z|^n)[1+(1-2\alpha)|z|^n]\}$$
 for $|z| < 1$ and this completes the proof.

We remark that in the case $\alpha = 0$, the above lemma reduces to a result of MacGregor [2; Lemma 1] and the inequality (6) with $\alpha = 0$, n = 1, gives another result of MacGregor [2, Lemma 2].

LEMMA 3. Under the hypothesis of Lemma 1 we have for |z| < 1 Re $(p(z)) \ge [1 + (2\alpha - 1) |z|^n]/(1 + |z|^n)$.

Proof. We have from equation (3), $F(z) = \frac{[1 - h(z)]}{[1 + h(z)]}$ and also from (8), $|h(z)| \le |z|^n$ for |z| < 1. Hence the image of |z| < r (0 < r < 1) under F(z) lies in the interior of the circle with the line segment joining the points $\frac{(1 - r^n)}{(1 + r^n)}$ and $\frac{(1 + r^n)}{(1 - r^n)}$ as a diameter. Consequently $\operatorname{Re}(F(z)) \ge \frac{(1 - |z|^n)}{(1 + |z|^n)}$ for

|z| < 1. The result now follows from the last inequality involving F(z) and equation (2).

LEMMA 4. ([6]). If $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$ is analytic and Re (h(z)) > 0 for |z| < 1, then

$$[1 - \lambda \mid h(z) \mid]^{-1} \leq (1 - \mid z \mid^{n}) / [(1 - \mid z \mid^{n}) - \lambda(1 + \mid z \mid^{n})]$$

for $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$, where $0 \le \lambda < 1$.

3. THEOREM 1. Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$, and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ are analytic and $\operatorname{Re}(g(z)/z) > 0$ for |z| < 1. If $\operatorname{Re}(f(z)/[\lambda f(z) + (1-\lambda)g(z)]) > 0$, $0 \le \lambda < 1$, for |z| < 1, then f(z) is univalent and starlike for $|z| < R^{1/n}$, where $R = \{[(2n + \lambda - n\lambda)^2 + (1 - \lambda^2)]^{1/2} - (2n + \lambda - n\lambda)\}/(1 + \lambda)$.

Proof. Let

$$h(z) = f(z)/[\lambda f(z) + (1-\lambda)g(z)] = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots,$$

then h(z) is analytic and Re (h(z)) > 0 for |z| < 1. Now

(10)
$$f(z) [1 - \lambda h(z)] = (1 - \lambda) h(z) z p(z) ,$$

where $p(z) = g(z)/z = 1 + b_{n+1}z^n + b_{n+2}z^{n+1} + \cdots$. Multiplying the logarithmic derivative of both sides of equation (10) by z we have

(11)
$$zf'(z)/f(z) = 1 + zp'(z)/p(z) + zh'(z)/\{h(z)[1 - \lambda h(z)]\}$$
.

Equation (11) is valid for those z for which $1 - \lambda h(z) \neq 0$ and |z| < 1. Since $|h(z)| \leq (1 + |z|^n)/(1 - |z|^n)$, $1 - \lambda h(z) \neq 0$ in particular if $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$. Now from equation (11), we have

$$|zf'(z)/f(z) - 1| \le |zp'(z)/p(z)| + |zh'(z)/h(z)| |1 - \lambda h(z)|^{-1}$$

and by using Lemma 2 with $\alpha = 0$ and Lemma 4, this gives

$$|zf'(z)/f(z) - 1| \leq \frac{2n |z|^n}{1 - |z|^{2n}} + \frac{2n |z|^n}{(1 - |z|^{2n}) - \lambda(1 + |z|^n)^2},$$

$$= \frac{2n |z|^n [(1 - |z|^n) - \lambda(1 + |z|^n) + (1 - |z|^n)]}{(1 - |z|^{2n}) [(1 - |z|^n) - \lambda(1 + |z|^n)]}$$

provided that $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$.

The fact that |zf'(z)/f(z) - 1| < 1 implies that Re(zf'(z)/f(z)) > 0, it follows from the inequality (12) that Re(zf'(z)/f(z) > 0 if

$$|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$$

and if

(13)
$$G(|z|^n) \equiv (1+\lambda) |z|^{3n} + (4n+2n\lambda+\lambda-1) |z|^{2n} + (2n\lambda-4n-\lambda-1) |z|^n + (1-\lambda) > 0.$$

Let $|z|^n=t$ and consider the cubic polynomial G(t) for $0 \le t \le 1$. G(t) has at most two positive zeros. Since $G(0)=(1-\lambda)>0$, $G[(1-\lambda)/(1+\lambda)]=-4\lambda n(1-\lambda)/(1+\lambda)^2<0$ and $G(1)=4\lambda n>0$, it follows that $G(t_1)=0$ for some t_1 such that $0< t_1<(1-\lambda)/(1+\lambda)$ and G(t)>0 for $0 \le t < t_1$ and G(t)<0 for $t_1< t<(1-\lambda)/(1+\lambda)$. Hence Re (zf'(z)/f(z))>0 for those z for which only the inequality (13) is true. Now the inequality (13) holds if, in particular

$$egin{aligned} (1+\lambda) \mid z \mid^{3n} + (4n-2n\lambda+\lambda-1) \mid z \mid^{2n} \ &+ (2n\lambda-4n-\lambda-1) \mid z \mid^{n} + (1-\lambda) > 0 \end{aligned}$$

or,

$$(\mid z \mid^{n} - 1) \left[(1 + \lambda) \mid z \mid^{2n} + (4n - 2n\lambda + 2\lambda) \mid z \mid^{n} + (\lambda - 1) \right] > 0$$

or,

$$(1 + \lambda) |z|^{2n} + (4n - 2n\lambda + 2\lambda) |z|^n + (\lambda - 1) < 0$$
.

The last inequality holds if

(14)
$$|z|^n < \{[(2n + \lambda - n\lambda)^2 + (1 - \lambda^2)]^{1/2} - (2n + \lambda - n\lambda)\}/(1 + \lambda)$$
.

Since f(z) is univalent and starlike for those z for which

Re
$$(zf'(z)/f(z)) > 0$$
,

we have that f(z) is univalent and starlike for $|z| < R^{1/n}$, where R is the right side of (14).

If we put $\lambda = 0$ in Theorem 1 we obtain the following result which, when n = 1, reduces to a result of Ratti [5, Theorem 1].

COROLLARY 1. Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} \cdots$, and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ are analytic and $\operatorname{Re}(g(z)/z) > 0$ for |z| < 1. If $\operatorname{Re}(f(z)/g(z)) > 0$ for |z| < 1 then f(z) is univalent and starlike for $|z| < [(4n^2 + 1)^{1/2} - 2n]^{1/n}$.

The functions $f(z)=z(1-z^n)^2/(1+z^n)^2$ and $g(z)=z(1-z^n)/(1+z^n)$ satisfy the hypothesis of Corollary 1 and it is easy to see that the derivative of f(z) vanishes at $z=[(4n^2+1)^{1/2}-2n]^{1/n}$ and hence $[(4n^2+1)^{1/2}-2n]^{1/n}$ is in fact the radius of univalence for such functions f(z). This shows that Corollary 1 is sharp and hence Theorem 1 is sharp at least for $\lambda=0$.

THEOREM 2. Suppose $f(z) = z + a_2 z^2 + \cdots$, and

$$g(z) = z + b_2 z^2 + \cdots$$

are analytic for |z| < 1 and Re (g(z)/z) > 1/2 for |z| < 1. If

Re
$$(f(z)/[\lambda f(z) + (1 - \lambda)g(z)]) > 0$$
 for $|z| < 1$

then f(z) is univalent and starlike for $|z| < (1 - \lambda)/(3 + \lambda)$.

Proof. Let $h(z)=f(z)/[\lambda f(z)+(1-\lambda)g(z)]=1+c_1z+c_2z^2+\cdots$. Now h(z) is analytic and Re(h(z))>0 for |z|<1 and

(15)
$$f(z) [1 - \lambda h(z)] = (1 - \lambda) h(z) g(z) .$$

If we let g(z) = zp(z), then by applying Lemma 1 with $\alpha = 1/2$ and n = 1 we have that $p(z) = [1 + zu(z)]^{-1}$, where u(z) is analytic and $|u(z)| \le 1$ for |z| < 1. Equation (15) now reduces to

$$f(z) [1 - \lambda h(z)] = (1 - \lambda)zh(z)/[1 + zu(z)]$$
.

Hence

$$\frac{zf'(z)}{f(z)} = \frac{1 - z^2u'(z)}{1 + zu(z)} + \frac{zh'(z)}{h(z)\left[1 - \lambda h(z)\right]}$$

and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \operatorname{Re}\left(\frac{1-z^2u'(z)}{1+zu(z)}\right) - \frac{|zh'(z)/h(z)|}{|1-\lambda h(z)|}$$
.

Using Lemmas 2 and 4 with n=1, we get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \operatorname{Re}\left(\frac{1-z^2u'(z)}{1+zu(z)}\right) - \frac{2\mid z\mid}{(1-\mid z\mid^2)-\lambda(1+\mid z\mid)^2}$$

for $|z| < (1 - \lambda)/(1 + \lambda)$.

Hence Re (zf'(z)/f(z)) > 0 if $|z| < (1 - \lambda)/(1 + \lambda)$ and

$$T(\mid z \mid) \ \mathrm{Re} \left[(1-z^{z}u'(z))(1+\overline{zu(z)}] - 2 \mid z \mid \mathrm{Re} \left[(1+zu(z))(1+\overline{zu(z)})
ight] > 0$$
 ,

where $T(|z|) = (1 - |z|^2) - \lambda(1 + |z|)^2$. The last inequality holds if

$$egin{aligned} T(\mid z\mid) & ext{Re} \ (1+\overline{zu(z)}) - T(\mid z\mid) \ ext{Re} \ [z^2u'(z)(1+\overline{zu(z)}] \ & + 2\mid z\mid ext{Re} \ [(1-zu(z))(1+\overline{zu(z)})] - 4\mid z\mid ext{Re} \ (1+\overline{zu(z)}) > 0 \ , \end{aligned}$$

or if

$$egin{aligned} & [4 \mid z \mid -T(\mid z \mid)] \ ext{Re} \ (1 + \overline{zu(z)}) \ + \ T(\mid z \mid) \ ext{Re} \ [z^z u'(z) (1 + \overline{zu(z)})] \ & < 2 \mid z \mid (1 - \mid z \mid^2 \mid u(z) \mid^2) \end{aligned}$$

$$egin{array}{l} |\ 4\ |\ z\ |\ -\ T(|\ z\ |)\ |\ (1\ +\ |\ z\ |\ |\ u(z)\ |)\ +\ T(|\ z\ |)\ |\ z\ |^2\ |\ u'(z)\ |\ (1\ +\ |\ z\ |\ |\ u(z)\ |)\ <\ (2\ |\ z\ |\ (1\ -\ |\ z\ |^2\ |\ u(z)\ |^2)\ . \end{array}$$

This inequality holds, in view of (5) with n = 1 if

(16)
$$|4|z| - T(|z|)| + T(|z|)|z|^2 (1 - |u(z)|^2)(1 - |z|^2)^{-1} < 2|z|(1 - |z||u(z)|).$$

Two cases arise according as 4|z| - T(|z|) is nonnegative or not.

Case 1. $4 |z| - T(|z|) \ge 0$, i.e. $|z| \ge [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. Since $[(4\lambda + 5)^{1/2} - (\lambda + 2)] < (1 - \lambda)$ for $0 \le \lambda < 1$, it follows, in view of inequality (16), that Re (zf'(z)/f(z)) > 0 for those z for which $[(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda) \le |z| < (1 - \lambda)/(1 + \lambda)$ and

$$egin{array}{lll} 4 \mid z \mid & - T(\mid z \mid) + T(\mid z \mid) \mid z \mid^2 (1 - \mid u(z) \mid^2) (1 - \mid u(z) \mid^2)^{-1} \ & < 2 \mid z \mid (1 - \mid z \mid \mid u(z) \mid) \end{array}.$$

The last inequality holds, because of the original value of T(|z|), if

(17)
$$\frac{2 |z| + 2 |z|^2 - 1 + \lambda(1 + |z|)^2 - \lambda |z|^2 (1 + |z|) / (1 - |z|)}{< |z|^2 |u(z)|^2 - \lambda |z|^2 |u(z)|^2 (1 + |z|) / (1 - |z|) - 2 |z|^2 |u(z)|} .$$

Since $|u(z)| \leq 1$, the right side of inequality (17)

$$\geq |z|^2 |u(z)|^2 - 2|z|^2 |u(z)| - \lambda |z|^2 (1 + |z|)/(1 - |z|)$$
.

Hence inequality (17) holds, if in particular

(18)
$$2|z| + 2|z|^2 - 1 + \lambda(1+|z|)^2 < |z|^2|u(z)|^2 - 2|z|^2|u(z)|$$
.

If we let $F(x) = x^2 |z|^2 - 2x |z|^2$, where x = |u(z)|, $0 \le x \le 1$, then F(x) is a decreasing function of x for $0 \le x \le 1$, and hence

$$F(x) \ge F(1) = -|z|^2$$
 for $0 \le x \le 1$.

Hence inequality (18) holds if $2 |z| + 2 |z|^2 - 1 + \lambda(1 + |z|)^2 < -|z|^2$ or $(3 |z| - 1)(|z| + 1) + \lambda(1 + |z|)^2 < 0$ or $3 |z| - 1 + \lambda(1 + |z|) < 0$ or if $|z| < (1 - \lambda)/(3 + \lambda)$. Since $(1 - \lambda)/(3 + \lambda) < (1 - \lambda)/(1 + \lambda)$, we have shown that

(19)
$$\frac{\text{Re } (zf'(z)/f(z)) > 0}{\text{for } [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda) \le |z| < (1 - \lambda)/(3 + \lambda) . }$$

Case 2. 4 |z| - T(|z|) < 0, i.e. $|z| < [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. We intend to show that Re (zf'(z)/f(z)) > 0 in this case also. Since f(z) and g(z) satisfy, in particular, the hypothesis of Theorem 1 with n = 1, it follows from Theorem 1 that

Re
$$(zf'(z)/f(z)) > 0$$
 for $|z| < [(5 - \lambda^2)^{1/2} - 2]/(1 + \lambda)$.

It is easy to see that

$$[(4\lambda + 5)^{1/2} - (\lambda + 2)] \le (5 - \lambda^2)^{1/2} - 2$$
 for $0 \le \lambda \le 1$

and hence in particular

Re
$$(zf'(z)/f(z)) > 0$$
 for $|z| < [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$.

In view of the above and (19), it now follows that f(z) is univalent and starlike for $|z| < (1 - \lambda)/(3 + \lambda)$ and this completes the proof.

For $\lambda=0$ the above result reduces to a result of Ratti [5, Theorem 2] and improves a result of MacGregor [2, Theorem 4] since Re (g(z)/z)>1/2 does not necessarily imply that g(z) is convex [7]. The functions $f(z)=z(1-z)/(1+z)^z$ and g(z)=z/(1+z) satisfy the hypothesis of Theorem 2 with $\lambda=0$ and f(z) is univalent in no circle |z|< r with r>1/3 since f'(z) vanishes at z=1/3. This shows that Theorem 2 is sharp at least for $\lambda=0$.

A function $f(z)=z+\sum_{k=2}^{\infty}\alpha_kz^k$ is said to be starlike of order α , $0\leq \alpha<1$, for |z|<1 if $\mathrm{Re}\;(zf'(z)/f(z))>\alpha$ for |z|<1, we now prove the following result.

THEOREM 3. Let $f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ be analytic for |z| < 1 and g(z) be starlike of order α , $0 \le \alpha < 1$, for |z| < 1. If Re $(f(z)/[\lambda f(z) + (1 - \lambda)g(z)]) > 0$ for |z| < 1, then f(z) is univalent and starlike for

(i)
$$|z| < [(1-\lambda)/(1+\lambda+2n)]^{1/n}$$
 if $\alpha = 1/2$;

and

(ii)
$$|z| < R^{1/n}$$
 , $if \; lpha
eq 1/2$,

where

$$R=\{[A^z+4(1-\lambda^2)(2lpha-1)]^{1/2}-A\}/[2(1+\lambda)(2lpha-1)]$$
 with $A=2n+\lambda+1-(2lpha-1)(1-\lambda)$.

Proof. Proceeding as in the proof of Theorem 1 we get

$$\operatorname{Re} (zf'(z)/f(z)) \ge \operatorname{Re} (zg'(z)/g(z)) - |zh'(z)/h(z)| |1 - \lambda h(z)|^{-1}$$
.

Applying Lemma 3 (to zg'(z)/g(z)) and Lemmas 2 and 4 we get,

(20)
$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{1 + (2\alpha - 1) |z|^n}{1 + |z|^n} - \frac{2n |z|^n}{(1 - |z|^{2n}) - \lambda(1 + |z|^n)^2}$$

provided that $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$.

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Hence Re (zf'(z)/f(z)) > 0 for those z for which $|z| < [1-\lambda)/(1+\lambda)]^{1/n}$ and the right side of inequality (20) is greater than zero. The latter holds if

(21)
$$G(\mid z\mid^n) \equiv (1+\lambda)(2\alpha-1)\mid z\mid^{2n} + [2n+\lambda+1-(2\alpha-1)(1-\lambda)]\mid z\mid^n-(1-\lambda)<0.$$

Let $|z|^n = t$ and consider the quadratic G(t) for $0 \le t \le 1$. Since $G(0) = \lambda - 1 < 0$, $G[(1 - \lambda)/(1 + \lambda)] = 2n(1 - \lambda)/(1 + \lambda) > 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1 - \lambda)/(1 + \lambda)$ and G(t) < 0 for $0 \le t < t_1$ and G(t) > 0 for $t_1 < t < (1 - \lambda)/(1 + \lambda)$. Hence f(z) is univalent and starlike for those z for which only the inequality (21) holds. Now the inequality (21) holds if

$$|z| < [(1-\lambda)/(1+\lambda+2n)]^{1/n}$$

when $\alpha = 1/2$ and

$$\mid z \mid < \{ [A^2 + 4(1-\lambda^2)(2lpha-1)]^{1/2} - A \}^{1/n} / [2(1+\lambda)(2lpha-1)]^{1/n}$$

when $\alpha \neq 1/2$, where $A = 2n + \lambda + 1 - (2\alpha - 1)(1 - \lambda)$ and this completes the proof.

If we put $\lambda=0$, n=1 and $\alpha=0$ in the above result then we see that $f(z)=z+\sum_{k=2}^\infty a_k z^k$ under the modified hypothesis is univalent and starlike for $|z|<2-\sqrt{3}$, a result obtained by MacGregor [2, Theorem 3]. On the other hand if $\lambda=0$ and n=1, Theorem 3 reduces to a result of Ratti [5, Theorem 3]. The functions

$$f(z) = z(1-z^n)/(1+z^n)^{\frac{2-2\alpha}{n}+1}$$
 and $g(z) = z/(1+z^n)^{\frac{2-2\alpha}{n}}$

show that Theorem 3 is sharp at least for $\lambda = 0$ and arbitrary n, since the derivative of f(z) vanishes at

$$z = \{ [(n+1-\alpha) - ((n+1-\alpha)^2 - (1-2\alpha))^{1/2}]/(1-2\alpha) \}^{1/n}$$

for $\alpha \neq 1/2$ and at z = -1/(2n + 1) when $\alpha = 1/2$.

4. Let S(R) denote the functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic and satisfy |zf'(z)/f(z) - 1| < 1 for |z| < R. Obviously every member of S(R) is univalent and starlike for |z| < R. We now prove the following result.

Theorem 4. Let $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$, and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ be analytic and satisfy Re (g(z)/z) > 0 for |z| < 1. If $|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1| < 1$, $0 \le \lambda < 1$, for |z| < 1, then $f(z) \in S(R^{t+n})$, where R is the smallest positive root of the equation $(2n\lambda + \lambda - n - 1) R^2 - (3n + \lambda - 2n\lambda) R + (1 - \lambda) = 0$.

Proof. Let

(22)
$$h(z) = f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1 = c_n z^n + c_{n+1} z^{n+1} + \cdots$$

By hypothesis, h(z) is analytic and |h(z)| < 1 for |z| < 1 and hence by a result of Goluzin [1] we have that for |z| < 1

$$|h'(z)| \leq n |z|^{n-1} (1 - |h(z)|^2)/(1 - |z|^{2n})$$

and by Schwarz's lemma for |z| < 1

$$|h(z)| \leq |z|^n.$$

If we let g(z) = zp(z), then we have from (22)

$$f(z)[1 - \lambda - \lambda h(z)] = (1 - \lambda)zp(z)[1 + h(z)]$$
.

Hence,

$$\frac{zf'(\mathit{z})}{f(\mathit{z})} = 1 + \frac{zp'(\mathit{z})}{p(\mathit{z})} + \frac{zh'(\mathit{z})}{\left[1 + h(\mathit{z})\right]\left[1 - \lambda - \lambda h(\mathit{z})\right]}$$

and this gives

$$\left| rac{zf'(z)}{f(z)} - 1
ight| \le \left| rac{zp'(z)}{p(z)}
ight| + rac{\left| zh'(z)
ight|}{\left| 1 + h(z)
ight| \left| 1 - \lambda - \lambda h(z)
ight|}$$
 .

Applying Lemma 2, with $\alpha = 0$, we get, in view of (23), for |z| < 1

$$igg|rac{zf'(z)}{f(z)}-1igg| \leq rac{2n\mid z\mid^n}{1-\mid z\mid^{2n}} + rac{n\mid z\mid^n(1-\mid h(z)\mid^2)}{(1-\mid z\mid^{2n})\mid 1+h(z)\mid \mid 1-\lambda-\lambda h(z)\mid} \ \leq rac{2n\mid z\mid^n}{1-\mid z\mid^{2n}} + rac{n\mid z\mid^n(1+\mid h(z)\mid)}{(1-\mid z\mid^{2n})\mid 1-\lambda-\lambda h(z)\mid}$$

by using (24), we have

$$\left| \, rac{z f'(z)}{f(z)} - 1 \,
ight| \, \leq rac{2n \, |z|^n}{1 - |z|^{2n}} + rac{n \, |z|^n}{(1 - |z|^n) \, (1 - \lambda - \lambda \, |z|^n)}$$

valid for $|z| < [(1-\lambda)/\lambda]^{1/n}$. Hence |zf'(z)/f(z) - 1| < 1 if

$$|z| < [(1-\lambda)/\lambda]^{1/n}$$

and

$$2n |z|^n (1-\lambda-\lambda|z|^n) + n |z|^n (1+|z|^n) < (1-|z|^{2n}) (1-\lambda-\lambda|z|^n)$$
 .

The last inequality holds if

(25)
$$G(|z|^n) \equiv \lambda |z|^{3n} + (2n\lambda + \lambda - n - 1) |z|^{2n} - (3n + \lambda - 2n\lambda) |z|^n + (1 - \lambda) > 0.$$

Let $|z|^n = t$ and consider the cubic polynomial G(t) for $0 \le t \le 1$.

G(t) has at most two positive zeros. Since $G(0)=(1-\lambda)>0$ and $G((1-\lambda)/\lambda)=-(n(1-\lambda)/\lambda^2<0$, it follows that $G(t_1)=0$ for some t_1 such that $0< t_1<(1-\lambda)/\lambda$ and G(t)>0 for $0\le t< t_1$ and G(t)<0 for some values of t between t_1 and $(1-\lambda)/\lambda$. Hence

$$|zf'(z)/f(z) - 1| < 1$$

for those values of z for which only the inequality (25) holds. Now inequality (25) holds if, in particular

$$(2n\lambda + \lambda - n - 1) |z|^{2n} - (3n + \lambda - 2n\lambda) |z|^n + (1 - \lambda) > 0$$

and this completes the proof.

If we set $\lambda = 0$ and n = 1 in the above result we have the following.

COROLLARY 2. Suppose $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ are analytic and satisfy Re (g(z)/z) > 0 for |z| < 1. If |f(z)/g(z) - 1| < 1 for |z| < 1, then |zf'(z)/f(z) - 1| < 1 for $|z| < 1/4(\sqrt{17} - 3)$.

It may be noted that Corollary 2 implies, in particular, that f(z) is univalent and starlike for |z| < 1/4 ($\sqrt[3]{17} - 3$) and hence includes a result of Ratti [5, Theorem 4]. If we take $f(z) = z(1-z^n)^2/(1+z^n)$ and $g(z) = z(1-z^n)/(1+z^n)$, it is easy to see that these functions satisfy the hypothesis of Theorem 4 with $\lambda = 0$. We see that f'(z) vanishes at $z_0 = [-3n + (9n^2 + 4n + 4)^{1/2}]/(2n + 2)$ and hence

$$|z_0f'(z_0)/f(z_0)-1|=1$$
.

This shows that Theorem 4 is sharp for at least $\lambda=0$ and also that Corollary 2 is sharp.

THEOREM 5. Let $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ be analytic for |z| < 1 and g(z) be starlike of order α for |z| < 1, $0 \le \alpha < 1$. If

$$|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1 < 1, \ 0 \le \lambda < 1, \ for \ |z| < 1,$$

then f(z) is univalent and starlike for $|z| < R^{1/n}$, where R is the smallest positive root of the equation

(26)
$$(2\alpha - 1)\lambda R^3 - (n + 2\alpha - 1 - \lambda)R^2 + (2\alpha - 2 - 2\alpha\lambda + \lambda - n)R + (1 - \lambda) = 0.$$

Proof. Proceeding as in the proof of Theorem 4 we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{\left[1 + h(z)\right]\left[1 - \lambda - \lambda h(z)\right]} \; .$$

Hence,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) - \frac{\mid zh'(z)\mid}{\mid 1 + h(z)\mid \mid 1 - \lambda - \lambda h(z)\mid}.$$

Since Re $(zg'(z)/g(z)) > \alpha$ and $zg'(z)/g(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$, we have by Lemma 3 and inequalities (23) and (24) that

(27)
$$\operatorname{Re} (zf'(z)/f(z)) \ge [1 + (2\alpha - 1) | z|^n]/(1 + | z|^n) \\ - n | z|^n/[(1 - | z|^n) (1 - \lambda - \lambda | z|^n)]$$

valid for $|z| < [(1-\lambda)/\lambda]^{1/n}$.

Hence Re (zf'(z)/f(z)) > 0 if $|z| < [(1 - \lambda)/\lambda]^{1/n}$ and if (in view of inequality (27))

(28)
$$G(|z|^n) \equiv (2\alpha -) \lambda |z|^{3n} - (n + 2\alpha - 1 - \lambda) |z|^{2n} + (2\alpha - 2 - 2\alpha\lambda + \lambda - n) |z|^n + (1 - \lambda) > 0.$$

Let |z|=t and consider the cubic polynomial G(t) for $0 \le t \le 1$. Since $G(0)=1-\lambda>0$ and $G((1-\lambda)/\lambda)=(-n(1-\lambda))/\lambda^2<0$, it follows that $G(t_1)=0$ for some t_1 such that $0 < t_1 < (1-\lambda)/\lambda$ and G(t)>0 for $0 \le t < t_1$ and G(t)<0 for some t between t_1 and $(1-\lambda)/\lambda$. Hence f(z) is starlike and univalent for $|z|< R^{1/n}$, in view of inequality (28), where R is the smallest positive root of the equation (26).

The case when $\lambda = 0$ in Theorem 5 is of special interest. In this case equation (26) becomes

$$(n + 2\alpha - 1)R^2 - (2\alpha - 2 - n)R - 1 = 0$$

which gives R=1/3 in case $\alpha=0$ and n=1 and

(29)
$$R = \{(2\alpha - 2 - n) + [(2\alpha - 2 - n)^2 + 4(n + 2\alpha - 1)]^{1/2}\}/[2(n + 2\alpha - 1)]$$

if $\alpha \neq 0$. This proves the following result, which includes a result of Ratti [5, Theorem 6].

COROLLARY 3. Suppose $f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\cdots$ and $g(z)=z+b_{n+1}z^{n+1}+b_{n+2}z^{n+2}+\cdots$ are analytic for |z|<1 and g(z) is starlike of order α for |z|<1, $0\leq \alpha<1$. If |f(z)/g(z)-1|<1 for |z|<1 then f(z) is univalent and starlike for

(i)
$$|z| < 1/3$$
 if $\alpha = 0$ and $n = 1$

(ii) $|z| < R^{1/n}$, where R is given by (29) if $\alpha \neq 0$.

It is easy to see that the functions $f(z)=z(1-z^n)/(1+z^n)^{(2-2\alpha)/n}$ and $g(z)=z/(1+z^n)^{(2-2\alpha)/n}$ satisfy the hypothesis of Corollary 3 and also that the derivative of f(z) vanishes at z=1/3 if $\alpha=0$ and n=1, and at $z=\{[(n+2-2\alpha)^2+4(n+2\alpha-1)]^{1/2}-(n+2-2\alpha)\}^{1/n}/[2(n+2\alpha-1)]^{1/n}$ if $\alpha\neq 0$. This shows that Corollary 3 is sharp.

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