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ON THE UNIVALENCE OF SOME ANALYTIC FUNCTIONS

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1.

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Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

be analytic and satisfy

(a)
$$\operatorname{Re}\left(f(z)/[\lambda f(z) + (1-\lambda) g(z)]\right) > 0$$

or

(b)
$$|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1| < 1$$

for $|z| < 1, 0 \le \lambda <$

We propose to determine the values of R such that f(z) is univalent and starlike for |z| < R under the assumption (i) Re(g(z)/z) > 0, or (ii) Re $(zg'(z)/g(z)) > \alpha$, $0 \le \alpha < 1$.

We also consider the case when n = 1 and $\operatorname{Re}(g(z)/z) > 1/2$ and show that under condition (a) f(z) is univalent and starlike for $|z| < (1 - \lambda)/(3 + \lambda)$.

2. LEMMA 1. If $p(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ is analytic and satisfies $\operatorname{Re}(p(z)) > \alpha, 0 \leq \alpha < 1$, for |z| < 1, then

(1)
$$p(z) = [1 + (2\alpha - 1)z^n u(z)]/[1 + z^n u(z)]$$
, for $|z| < 1$,

where u(z) is analytic and $|u(z)| \leq 1$ for |z| < 1.

Proof. Let

(2)
$$F(z) = [p(z) - \alpha]/(1 - \alpha) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

F(z) is analytic and Re(F(z)) > 0 for |z| < 1 and hence

 $(3) h(z) = [1 - F(z)]/[1 + F(z)] = d_n z^n + d_{n+1} z^{n+1} + \cdots,$

is analytic and |h(z)| < 1 for |z| < 1. Thus, by Schwarz's lemma

$$(4) h(z) = z^n u(z) ,$$

where u(z) is analytic and $|u(z)| \leq 1$ for |z| < 1. Now equations (2), (3) and (4) prove (1).

LEMMA 2. Under the hypothesis of Lemma 1 we have for |z| < 1

 $|zp'(z)/p(z)| \leq 2nz^n(1-lpha)/\{(1-|z|^n) [1+(1-2lpha)|z|^n]\}.$

Proof. Proceeding as in the proof of Lemma 1, we have in view of (3) and a result of Goluzin [1] that for |z| < 1

$$(5) \qquad |h'(z)| \leq n |z|^{n-1} (1 - |h(z)|^2)/(1 - |z|^{2n}) .$$

Using (3), the inequality (5) takes the form

$$|F'(z)| \leq 2n |z|^{n-1} \operatorname{Re} (F(z))/(1 - |z|^{2n})$$
 .

Hence, in view of (2),

(6)
$$|p'(z)| \leq 2n |z|^{n-1} [\operatorname{Re} (p(z)) - \alpha]/(1 - |z|^{2n})$$

or,

$$(7) |zp'(z)/p(z)| \leq 2n |z|^n (1 - \alpha/(|p(z)|)/(1 - |z|^{2n}).$$

Equation (4) gives

(8) $|h(z)| \leq |z|^n$ for |z| < 1,

and hence, by virtue of (3),

(9)
$$|F(z)| \leq (1 + |z|^n)/(1 - |z|^n)$$
 for $|z| < 1$.

From (2) and (9),

$$egin{array}{ll} | \ p(z) \ | \ = \ | \ lpha \ + \ (1 - lpha) F(z) \ | \ & \leq lpha \ + \ (1 - lpha) \ | \ F(z) \ | \ & \leq [1 + (1 - 2 lpha) \ | \ z \ |^n] / (1 - | \ z \ |^n) \;. \end{array}$$

The inequality (7), because of the last inequality, reduces to

$$||zp'(z)/p(z)| \leq 2n ||z|^n (1-lpha)/\{(1-||z|^n) [1+(1-2lpha) ||z|^n]\} ext{ for } ||z| < 1$$

and this completes the proof.

We remark that in the case $\alpha = 0$, the above lemma reduces to a result of MacGregor [2; Lemma 1] and the inequality (6) with $\alpha = 0$, n = 1, gives another result of MacGregor [2, Lemma 2].

LEMMA 3. Under the hypothesis of Lemma 1 we have for |z| < 1Re $(p(z)) \ge [1 + (2\alpha - 1) |z|^n]/(1 + |z|^n)$.

Proof. We have from equation (3), F(z) = [1 - h(z)]/[1 + h(z)]and also from (8), $|h(z)| \leq |z|^n$ for |z| < 1. Hence the image of |z| < r (0 < r < 1) under F(z) lies in the interior of the circle with the line segment joining the points $(1 - r^n)/(1 + r^n)$ and $(1 + r^n)/(1 - r^n)$ as a diameter. Consequently Re $(F(z)) \geq (1 - |z|^n)/(1 + |z|^n)$ for |z| < 1. The result now follows from the last inequality involving F(z) and equation (2).

LEMMA 4. ([6]). If $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$ is analytic and Re (h(z)) > 0 for |z| < 1, then

$$[1-\lambda \mid h(z) \mid]^{-1} \leq (1-\mid z \mid^n)/[(1-\mid z \mid^n) - \lambda(1+\mid z \mid^n)]$$

for $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$, where $0 \leq \lambda < 1$.

3. THEOREM 1. Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$, and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ are analytic and $\operatorname{Re}(g(z)/z) > 0$ for |z| < 1. If $\operatorname{Re}(f(z)/[\lambda f(z) + (1 - \lambda)g(z)]) > 0$, $0 \leq \lambda < 1$, for |z| < 1, then f(z) is univalent and starlike for $|z| < R^{1/n}$, where $R = \{[(2n + \lambda - n\lambda)^2 + (1 - \lambda^2)]^{1/2} - (2n + \lambda - n\lambda)\}/(1 + \lambda).$

Proof. Let

$$h(z) = f(z)/[\lambda f(z) + (1 - \lambda)g(z)] = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

then h(z) is analytic and $\operatorname{Re}(h(z)) > 0$ for |z| < 1. Now

(10)
$$f(z) [1 - \lambda h(z)] = (1 - \lambda) h(z) z p(z)$$

where $p(z) = g(z)/z = 1 + b_{n+1}z^n + b_{n+2}z^{n+1} + \cdots$. Multiplying the logarithmic derivative of both sides of equation (10) by z we have

(11)
$$zf'(z)/f(z) = 1 + zp'(z)/p(z) + zh'(z)/\{h(z)[1 - \lambda h(z)]\}$$
.

Equation (11) is valid for those z for which $1 - \lambda h(z) \neq 0$ and |z| < 1. Since $|h(z)| \leq (1 + |z|^n)/(1 - |z|^n)$, $1 - \lambda h(z) \neq 0$ in particular if $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$. Now from equation (11), we have

$$| \, z f'(z) / f(z) \, - \, 1 \, | \, \leq | \, z p'(z) / p(z) \, | \, + \, | \, z h'(z) / h(z) \, | \, | \, 1 \, - \, \lambda h(z) \, |^{-1}$$

and by using Lemma 2 with $\alpha = 0$ and Lemma 4, this gives

$$(12) \quad |zf'(z)/f(z) - 1| \leq \frac{2n |z|^n}{1 - |z|^{2n}} + \frac{2n |z|^n}{(1 - |z|^{2n}) - \lambda(1 + |z|^n)^2}, \\ = \frac{2n |z|^n [(1 - |z|^n) - \lambda(1 + |z|^n) + (1 - |z|^n)]}{(1 - |z|^{2n}) [(1 - |z|^n) - \lambda(1 + |z|^n)]}$$

provided that $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$.

The fact that |zf'(z)/f(z) - 1| < 1 implies that $\operatorname{Re}(zf'(z)/f(z)) > 0$, it follows from the inequality (12) that $\operatorname{Re}(zf'(z)/f(z) > 0)$ if

 $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$

and if

(13)
$$G(|z|^{n}) \equiv (1 + \lambda) |z|^{3n} + (4n + 2n\lambda + \lambda - 1) |z|^{2n} + (2n\lambda - 4n - \lambda - 1) |z|^{n} + (1 - \lambda) > 0.$$

Let $|z|^n = t$ and consider the cubic polynomial G(t) for $0 \leq t \leq 1$. G(t) has at most two positive zeros. Since $G(0) = (1 - \lambda) > 0$, $G[(1 - \lambda)/(1 + \lambda)] = -4\lambda n(1 - \lambda)/(1 + \lambda)^2 < 0$ and $G(1) = 4\lambda n > 0$, it follows that $G(t_i) = 0$ for some t_i such that $0 < t_i < (1 - \lambda)/(1 + \lambda)$ and G(t) > 0 for $0 \leq t < t_i$ and G(t) < 0 for $t_i < t < (1 - \lambda)/(1 + \lambda)$. Hence Re (zf'(z)/f(z)) > 0 for those z for which only the inequality (13) is true. Now the inequality (13) holds if, in particular

or,

$$(\mid z \mid^n - 1) \left[(1 + \lambda) \mid z \mid^{2n} + (4n - 2n\lambda + 2\lambda) \mid z \mid^n + (\lambda - 1)
ight] > 0$$

or,

$$(1+\lambda) \, |\, z\,|^{{ extsf{2n}}} + \, (4n\, - \, 2n\lambda \, + \, 2\lambda) \, |\, z\,|^n + \, (\lambda \, - \, 1) < 0$$
 .

The last inequality holds if

$$(14) \quad |z|^n < \{[(2n+\lambda-n\lambda)^2+(1-\lambda^2)]^{1/2}-(2n+\lambda-n\lambda)\}/(1+\lambda)$$
 .

Since f(z) is univalent and starlike for those z for which

$${
m Re} \left(z f'(z) / f(z)
ight) > 0$$
 ,

we have that f(z) is univalent and starlike for $|z| < R^{1/n}$, where R is the right side of (14).

If we put $\lambda = 0$ in Theorem 1 we obtain the following result which, when n = 1, reduces to a result of Ratti [5, Theorem 1].

COROLLARY 1. Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} \cdots$, and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ are analytic and $\operatorname{Re}(g(z)/z) > 0$ for |z| < 1. If $\operatorname{Re}(f(z)/g(z)) > 0$ for |z| < 1 then f(z) is univalent and starlike for $|z| < [(4n^2 + 1)^{1/2} - 2n]^{1/n}$.

The functions $f(z) = z(1-z^n)^2/(1+z^n)^2$ and $g(z) = z(1-z^n)/(1+z^n)$ satisfy the hypothesis of Corollary 1 and it is easy to see that the derivative of f(z) vanishes at $z = [(4n^2 + 1)^{1/2} - 2n]^{1/n}$ and hence $[(4n^2 + 1)^{1/2} - 2n]^{1/n}$ is in fact the radius of univalence for such functions f(z). This shows that Corollary 1 is sharp and hence Theorem 1 is sharp at least for $\lambda = 0$.

THEOREM 2. Suppose
$$f(z) = z + a_2 z^2 + \cdots$$
, and

$$g(z) = z + b_2 z^2 + \cdots$$

are analytic for |z| < 1 and Re (g(z)/z) > 1/2 for |z| < 1. If

$$\operatorname{Re}\left(f(z)/[\lambda f(z) + (1-\lambda)g(z)]
ight) > 0 \qquad \quad for \mid z \mid < 1$$

then f(z) is univalent and starlike for $|z| < (1 - \lambda)/(3 + \lambda)$.

Proof. Let $h(z) = f(z)/[\lambda f(z) + (1-\lambda)g(z)] = 1 + c_1 z + c_2 z^2 + \cdots$. Now h(z) is analytic and Re(h(z)) > 0 for |z| < 1 and

(15)
$$f(z) [1 - \lambda h(z)] = (1 - \lambda) h(z) g(z) .$$

If we let g(z) = zp(z), then by applying Lemma 1 with $\alpha = 1/2$ and n = 1 we have that $p(z) = [1 + zu(z)]^{-1}$, where u(z) is analytic and $|u(z)| \leq 1$ for |z| < 1. Equation (15) now reduces to

$$f(z) [1 - \lambda h(z)] = (1 - \lambda)zh(z)/[1 + zu(z)]$$
.

Hence

$$rac{zf'(z)}{f(z)} = rac{1-z^2 u'(z)}{1+z u(z)} + rac{zh'(z)}{h(z) \left[1-\lambda h(z)
ight]}$$

and

$$\operatorname{Re}\left(rac{zf'(z)}{f(z)}
ight) \geqq \operatorname{Re}\left(rac{1-z^2u'(z)}{1+zu(z)}
ight) - rac{\mid zh'(z)/h(z)\mid}{\mid 1-\lambda h(z)\mid} \,.$$

Using Lemmas 2 and 4 with n = 1, we get

$$\mathrm{Re}\left(rac{zf'(z)}{f(z)}
ight) \geqq \mathrm{Re}\left(rac{1-z^2u'(z)}{1+zu(z)}
ight) - rac{2\mid z\mid}{(1-\mid z\mid^2) - \lambda(1+\mid z\mid)^2}$$

for $|z| < (1 - \lambda)/(1 + \lambda)$. Hence Re (zf'(z)/f(z)) > 0 if $|z| < (1 - \lambda)/(1 + \lambda)$ and T(|z|) Re $[(1 - z^2u'(z))(1 + \overline{zu(z)}] - 2 |z|$ Re $[(1 + zu(z))(1 + \overline{zu(z)}] > 0$, where $T(|z|) = (1 - |z|^2) - \lambda(1 + |z|)^2$. The last inequality holds if

$$egin{aligned} T(\mid z \mid) & ext{Re} \left(1 + \overline{zu(z)}
ight) - T(\mid z \mid) & ext{Re} \left[z^2 u'(z)(1 + \overline{zu(z)})
ight] \ &+ 2 \mid z \mid ext{Re} \left[(1 - zu(z))(1 + \overline{zu(z)})
ight] - 4 \mid z \mid ext{Re} \left(1 + \overline{zu(z)}
ight) > 0 ext{ ,} \end{aligned}$$

or if

or

This inequality holds, in view of (5) with n = 1 if

(16)
$$|4|z| - T(|z|) | + T(|z|) | z|^2 (1 - |u(z)|^2) (1 - |z|^2)^{-1} \\ < 2|z| (1 - |z||u(z)|).$$

Two cases arise according as 4 |z| - T(|z|) is nonnegative or not.

Case 1. $4 |z| - T(|z|) \ge 0$, i.e. $|z| \ge [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. Since $[(4\lambda + 5)^{1/2} - (\lambda + 2)] < (1 - \lambda)$ for $0 \le \lambda < 1$, it follows, in view of inequality (16), that Re (zf'(z)/f(z)) > 0 for those z for which $[(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda) \le |z| < (1 - \lambda)/(1 + \lambda)$ and

The last inequality holds, because of the original value of T(|z|), if

$$(17) \quad \frac{2 |z| + 2 |z|^2 - 1 + \lambda(1 + |z|)^2 - \lambda |z|^2(1 + |z|)/(1 - |z|)}{<|z|^2 |u(z)|^2 - \lambda |z|^2 |u(z)|^2 (1 + |z|)/(1 - |z|) - 2 |z|^2 |u(z)|}.$$

Since $|u(z)| \leq 1$, the right side of inequality (17)

$$\geq \mid z \mid^{\scriptscriptstyle 2} \mid u(z) \mid^{\scriptscriptstyle 2} - 2 \mid z \mid^{\scriptscriptstyle 2} \mid u(z) \mid - \lambda \mid z \mid^{\scriptscriptstyle 2} (1 + \mid z \mid) / (1 - \mid z \mid)$$
 .

Hence inequality (17) holds, if in particular

$$(18) \qquad 2 |z| + 2 |z|^2 - 1 + \lambda (1 + |z|)^2 < |z|^2 |u(z)|^2 - 2 |z|^2 |u(z)|.$$

If we let $F(x) = x^2 |z|^2 - 2x |z|^2$, where x = |u(z)|, $0 \le x \le 1$, then F(x) is a decreasing function of x for $0 \le x \le 1$, and hence

$$F(x) \ge F(1) = -|z|^2$$
 for $0 \le x \le 1$.

Hence inequality (18) holds if $2 |z| + 2 |z|^2 - 1 + \lambda(1 + |z|)^2 < -|z|^2$ or $(3 |z| - 1)(|z| + 1) + \lambda(1 + |z|)^2 < 0$ or $3 |z| - 1 + \lambda(1 + |z|) < 0$ or if $|z| < (1 - \lambda)/(3 + \lambda)$. Since $(1 - \lambda)/(3 + \lambda) < (1 - \lambda)/(1 + \lambda)$, we have shown that

(19)
$$\begin{array}{l} \operatorname{Re}\,(zf'(z)/f(z))>0\\ \text{for}\,\,[(4\lambda+5)^{1/2}-(\lambda+2)]/(1+\lambda)\leq |\,z\,|<(1-\lambda)/(3+\lambda)\;.\end{array}$$

Case 2. 4 |z| - T(|z|) < 0, i.e. $|z| < [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. We intend to show that Re(zf'(z)/f(z)) > 0 in this case also. Since f(z) and g(z) satisfy, in particular, the hypothesis of Theorem 1 with n = 1, it follows from Theorem 1 that ${
m Re}\;(zf'(z)/f(z))>0\;\;{
m for}\;\;|\;z\;|<[(5-\lambda^2)^{1/2}-2]/(1+\lambda)$.

It is easy to see that

$$\left[(4\lambda+5)^{1/2}-(\lambda+2)\right] \leq (5-\lambda^2)^{1/2}-2 \quad \text{for } 0 \leq \lambda \leq 1$$

and hence in particular

$${
m Re}\;(zf'(z)/f(z))>0\;\;{
m for}\;\;|\;z\,|<[(4\lambda\,+\,5)^{_{1/2}}-(\lambda\,+\,2)]/(1\,+\,\lambda)$$
 .

In view of the above and (19), it now follows that f(z) is univalent and starlike for $|z| < (1 - \lambda)/(3 + \lambda)$ and this completes the proof.

For $\lambda = 0$ the above result reduces to a result of Ratti [5, Theorem 2] and improves a result of MacGregor [2, Theorem 4] since Re (g(z)/z) > 1/2 does not necessarily imply that g(z) is convex [7]. The functions $f(z) = z(1-z)/(1+z)^2$ and g(z) = z/(1+z) satisfy the hypothesis of Theorem 2 with $\lambda = 0$ and f(z) is univalent in no circle |z| < r with r > 1/3 since f'(z) vanishes at z = 1/3. This shows that Theorem 2 is sharp at least for $\lambda = 0$.

A function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is said to be starlike of order α , $0 \leq \alpha < 1$, for |z| < 1 if Re $(zf'(z)/f(z)) > \alpha$ for |z| < 1, we now prove the following result.

THEOREM 3. Let $f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ be analytic for |z| < 1 and g(z) be starlike of order α , $0 \leq \alpha < 1$, for |z| < 1. If Re $(f(z)/[\lambda f(z) + (1 - \lambda)g(z)]) > 0$ for |z| < 1, then f(z) is univalent and starlike for

(i)
$$|z| < [(1 - \lambda)/(1 + \lambda + 2n)]^{1/n}$$
 if $\alpha = 1/2$;

and

 $|\, z\,| < R^{\scriptscriptstyle 1/n} \;, \qquad \qquad if \; \alpha \neq 1/2 \;,$

where

with

$$egin{aligned} R &= \{ [A^2 + 4(1-\lambda^2)(2lpha-1)]^{1/2} - A \} / [2(1+\lambda)(2lpha-1)] \ A &= 2n+\lambda+1-(2lpha-1)(1-\lambda). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem 1 we get

$${
m Re}\;(zf'(z)/f(z)) \ge {
m Re}\;(zg'(z)/g(z)) \,-\, |\,zh'(z)/h(z)\;|\;|\,1\,-\,\lambda h(z)\;|^{-_1}$$
 .

Applying Lemma 3 (to zg'(z)/g(z)) and Lemmas 2 and 4 we get,

(20)
$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{1+(2\alpha-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{(1-|z|^{2n})-\lambda(1+|z|^n)^2}$$

provided that $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$.

Hence Re (zf'(z)/f(z)) > 0 for those z for which $|z| < [1-\lambda)/(1+\lambda)]^{1/n}$ and the right side of inequality (20) is greater than zero. The latter holds if

(21)
$$G(|z|^n) \equiv (1+\lambda)(2\alpha-1) |z|^{2n} + [2n+\lambda+1-(2\alpha-1)(1-\lambda)] |z|^n - (1-\lambda) < 0.$$

Let $|z|^n = t$ and consider the quadratic G(t) for $0 \leq t \leq 1$. Since $G(0) = \lambda - 1 < 0$, $G[(1 - \lambda)/(1 + \lambda)] = 2n(1 - \lambda)/(1 + \lambda) > 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1 - \lambda)/(1 + \lambda)$ and G(t) < 0 for $0 \leq t < t_1$ and G(t) > 0 for $t_1 < t < (1 - \lambda)/(1 + \lambda)$. Hence f(z) is univalent and starlike for those z for which only the inequality (21) holds. Now the inequality (21) holds if

$$|\,z\,| < [(1-\lambda)/(1+\lambda+2n)]^{1/n}$$

when $\alpha = 1/2$ and

$$|z| < \{[A^2 + 4(1 - \lambda^2)(2lpha - 1)]^{1/2} - A\}^{1/n} / [2(1 + \lambda)(2lpha - 1)]^{1/n}$$

when $\alpha \neq 1/2$, where $A = 2n + \lambda + 1 - (2\alpha - 1)(1 - \lambda)$ and this completes the proof.

If we put $\lambda = 0$, n = 1 and $\alpha = 0$ in the above result then we see that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ under the modified hypothesis is univalent and starlike for $|z| < 2 - \sqrt{3}$, a result obtained by MacGregor [2, Theorem 3]. On the other hand if $\lambda = 0$ and n = 1, Theorem 3 reduces to a result of Ratti [5, Theorem 3]. The functions

$$f(z) = z(1-z^n)/(1+z^n)^{\frac{2-2\alpha}{n}+1}$$
 and $g(z) = z/(1+z^n)^{\frac{2-2\alpha}{n}}$

show that Theorem 3 is sharp at least for $\lambda = 0$ and arbitrary *n*, since the derivative of f(z) vanishes at

$$z = \{[(n + 1 - lpha) - ((n + 1 - lpha)^2 - (1 - 2lpha))^{1/2}]/(1 - 2lpha)\}^{1/n}$$

for $\alpha \neq 1/2$ and at z = -1/(2n + 1) when $\alpha = 1/2$.

4. Let S(R) denote the functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic and satisfy |zf'(z)/f(z) - 1| < 1 for |z| < R. Obviously every member of S(R) is univalent and starlike for |z| < R. We now prove the following result.

THEOREM 4. Let $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$, and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ be analytic and satisfy $\operatorname{Re}(g(z)/z) > 0$ for |z| < 1. If $|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1| < 1$, $0 \leq \lambda < 1$, for |z| < 1, then $f(z) \in S(\mathbb{R}^{1/n})$, where \mathbb{R} is the smallest positive root of the equation $(2n\lambda + \lambda - n - 1) \mathbb{R}^2 - (3n + \lambda - 2n\lambda) \mathbb{R} + (1 - \lambda) = 0$.

Proof. Let

(22)
$$h(z) = f(z)/[\lambda f(z) + (1 - \lambda)g(z)] - 1 = c_n z^n + c_{n+1} z^{n+1} + \cdots$$

By hypothesis, h(z) is analytic and |h(z)| < 1 for |z| < 1 and hence by a result of Goluzin [1] we have that for |z| < 1

(23)
$$|h'(z)| \leq n |z|^{n-1} (1 - |h(z)|^2)/(1 - |z|^{2n})$$

and by Schwarz's lemma for |z| < 1

$$|h(z)| \leq |z|^n.$$

If we let g(z) = zp(z), then we have from (22)

$$f(z)[1-\lambda-\lambda h(z)] = (1-\lambda)zp(z)[1+h(z)].$$

Hence,

$$rac{zf'(z)}{f(z)} = 1 + rac{zp'(z)}{p(z)} + rac{zh'(z)}{[1 + h(z)][1 - \lambda - \lambda h(z)]}$$

and this gives

$$\Big|rac{zf'(z)}{f(z)}-1\Big| \leq \Big|rac{zp'(z)}{p(z)}\Big| + rac{|zh'(z)|}{|1+h(z)||1-\lambda-\lambda h(z)|}\,.$$

Applying Lemma 2, with $\alpha = 0$, we get, in view of (23), for |z| < 1

$$ig| rac{zf'(z)}{f(z)} - 1 ig| \le rac{2n \mid z \mid^n}{1 - \mid z \mid^{2n}} + rac{n \mid z \mid^n (1 - \mid h(z) \mid^2)}{(1 - \mid z \mid^{2n}) \mid 1 + h(z) \mid \mid 1 - \lambda - \lambda h(z) \mid} \ \le rac{2n \mid z \mid^n}{1 - \mid z \mid^{2n}} + rac{n \mid z \mid^n (1 + \mid h(z) \mid)}{(1 - \mid z \mid^{2n}) \mid 1 - \lambda - \lambda h(z) \mid}$$

by using (24), we have

$$\Big| \, rac{z f'(z)}{f(z)} - 1 \, \Big| \, \leq rac{2n \, | \, z \, |^n}{1 \, - \, | \, z \, |^{2n}} + rac{n \, | \, z \, |^n}{(1 \, - \, | \, z \, |^n) \, (1 - \lambda - \lambda \, | \, z \, |^n)}$$

valid for $|z| < [(1 - \lambda)/\lambda]^{1/n}$. Hence |zf'(z)/f(z) - 1| < 1 if $|z| < [(1 - \lambda)/\lambda]^{1/n}$

and

$$2n \mid z \mid^n (1-\lambda-\lambda \mid z \mid^n) + n \mid z \mid^n (1+\mid z \mid^n) < (1-\mid z \mid^{2n})(1-\lambda-\lambda \mid z \mid^n)$$
 .

The last inequality holds if

(25)
$$G(|z|^n) \equiv \lambda |z|^{3n} + (2n\lambda + \lambda - n - 1) |z|^{2n} - (3n + \lambda - 2n\lambda) |z|^n + (1 - \lambda) > 0.$$

Let $|z|^n = t$ and consider the cubic polynomial G(t) for $0 \leq t \leq 1$.

G(t) has at most two positive zeros. Since $G(0) = (1 - \lambda) > 0$ and $G((1 - \lambda)/\lambda) = -(n(1 - \lambda)/\lambda^2 < 0)$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1 - \lambda)/\lambda$ and G(t) > 0 for $0 \le t < t_1$ and G(t) < 0 for some values of t between t_1 and $(1 - \lambda)/\lambda$. Hence

$$|zf'(z)/f(z) - 1| < 1$$

for those values of z for which only the inequality (25) holds. Now inequality (25) holds if, in particular

$$(2n\lambda+\lambda-n-1)\mid z\mid^{_{2n}}-(3n+\lambda-2n\lambda)\mid z\mid^{_{n}}+(1-\lambda)>0$$

and this completes the proof.

If we set $\lambda = 0$ and n = 1 in the above result we have the following.

COROLLARY 2. Suppose $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ are analytic and satisfy Re(g(z)/z) > 0 for |z| < 1. If |f(z)/g(z) - 1| < 1 for |z| < 1, then |zf'(z)/f(z) - 1| < 1 for $|z| < 1/4(\sqrt{17} - 3)$.

It may be noted that Corollary 2 implies, in particular, that f(z) is univalent and starlike for $|z| < 1/4 (\sqrt{17} - 3)$ and hence includes a result of Ratti [5, Theorem 4]. If we take $f(z) = z(1-z^n)^2/(1+z^n)$ and $g(z) = z(1-z^n)/(1+z^n)$, it is easy to see that these functions satisfy the hypothesis of Theorem 4 with $\lambda = 0$. We see that f'(z) vanishes at $z_0 = [-3n + (9n^2 + 4n + 4)^{1/2}]/(2n + 2)$ and hence

$$|z_{\scriptscriptstyle 0} f'(z_{\scriptscriptstyle 0})/f(z_{\scriptscriptstyle 0})-1|=1$$
 .

This shows that Theorem 4 is sharp for at least $\lambda = 0$ and also that Corollary 2 is sharp.

THEOREM 5. Let $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ be analytic for |z| < 1 and g(z) be starlike of order α for |z| < 1, $0 \leq \alpha < 1$. If

$$| f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1 < 1$$
 , $0 \leq \lambda < 1$, for $| z | < 1$,

then f(z) is univalent and starlike for $|z| < R^{1/n}$, where R is the smallest positive root of the equation

(26)
$$\begin{array}{l} (2\alpha-1)\lambda R^3-(n+2\alpha-1-\lambda)R^2\\ +(2\alpha-2-2\alpha\lambda+\lambda-n)R+(1-\lambda)=0 \end{array} . \end{array}$$

Proof. Proceeding as in the proof of Theorem 4 we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{\left[1 + h(z)\right]\left[1 - \lambda - \lambda h(z)\right]}$$

Hence,

$$\operatorname{Re}\left(rac{zf'(z)}{f(z)}
ight) \geqq \operatorname{Re}\left(rac{zg'(z)}{g(z)}
ight) - rac{\mid zh'(z)\mid}{\mid 1 + h(z)\mid \mid 1 - \lambda - \lambda h(z)\mid} \, .$$

Since Re $(zg'(z)/g(z)) > \alpha$ and $zg'(z)/g(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$, we have by Lemma 3 and inequalities (23) and (24) that

(27)
$$\operatorname{Re} \left(zf'(z)/f(z) \right) \ge \left[1 + (2\alpha - 1) \mid z \mid^{n} \right]/(1 + \mid z \mid^{n}) \\ - n \mid z \mid^{n} / \left[(1 - \mid z \mid^{n}) (1 - \lambda - \lambda \mid z \mid^{n}) \right]$$

valid for $|z| < [(1 - \lambda)/\lambda]^{1/n}$.

Hence Re (zf'(z)/f(z)) > 0 if $|z| < [(1 - \lambda)/\lambda]^{1/n}$ and if (in view of inequality (27))

(28)

$$G(|z|^{n}) \equiv (2\alpha -)\lambda |z|^{3n} - (n + 2\alpha - 1 - \lambda) |z|^{2n} + (2\alpha - 2 - 2\alpha\lambda + \lambda - n) |z|^{n} + (1 - \lambda) > 0.$$

Let |z| = t and consider the cubic polynomial G(t) for $0 \leq t \leq 1$. Since $G(0) = 1 - \lambda > 0$ and $G((1 - \lambda)/\lambda) = (-n(1 - \lambda))/\lambda^2 < 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1 - \lambda)/\lambda$ and G(t) > 0 for $0 \leq t < t_1$ and G(t) < 0 for some t between t_1 and $(1 - \lambda)/\lambda$. Hence f(z) is starlike and univalent for $|z| < R^{1/n}$, in view of inequality (28), where R is the smallest positive root of the equation (26).

The case when $\lambda = 0$ in Theorem 5 is of special interest. In this case equation (26) becomes

$$(n+2\alpha-1)R^2 - (2\alpha-2-n)R - 1 = 0$$

which gives R = 1/3 in case $\alpha = 0$ and n = 1 and

(29)
$$R = \{(2\alpha - 2 - n) + [(2\alpha - 2 - n)^2 + 4(n + 2\alpha - 1)]^{1/2}\}/[2(n + 2\alpha - 1)]$$

if $\alpha \neq 0$. This proves the following result, which includes a result of Ratti [5, Theorem 6].

COROLLARY 3. Suppose $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ and $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ are analytic for |z| < 1 and g(z) is starlike of order α for |z| < 1, $0 \le \alpha < 1$. If |f(z)/g(z) - 1| < 1 for |z| < 1 then f(z) is univalent and starlike for

(i) |z| < 1/3 if $\alpha = 0$ and n = 1

(ii) $|z| < R^{1/n}$, where R is given by (29) if $\alpha \neq 0$.

It is easy to see that the functions $f(z) = z(1-z^n)/(1+z^n)^{(2-2\alpha)/n}$ and $g(z) = z/(1+z^n)^{(2-2\alpha)/n}$ satisfy the hypothesis of Corollary 3 and also that the derivative of f(z) vanishes at z = 1/3 if $\alpha = 0$ and n = 1, and at $z = \{[(n+2-2\alpha)^2 + 4(n+2\alpha-1)]^{1/2} - (n+2-2\alpha)\}^{1/n}/$ $[2(n+2\alpha-1)]^{1/n}$ if $\alpha \neq 0$. This shows that Corollary 3 is sharp.

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