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**ON SPACES OF DISTRIBUTIONS STRONGLY REGULAR WITH
RESPECT TO PARTIAL DIFFERENTIAL OPERATORS**

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A distribution T in Ω is said to be strongly regular with respect to the differential operator $P(D)$, if $P^k(D)T$, $k = 0, 1, \dots$, are of bounded order in any open set $\Omega' \subset \subset \Omega$. Necessary and sufficient conditions on the polynomials P and Q are established in order that a distribution T strongly regular with respect to $P(D)$ be strongly regular with respect to $Q(D)$.

Let $P(D)$ be a partial differential operator in R^n with constant coefficients and $P^k(D)$, $k = 1, 2, \dots$, its successive iterations. The following result is due to L. Hörmander ([3], Theorem 3.6 and Remark on p. 233):

If $P(D)$ is hypoelliptic and T is a distribution such that $P^k(D)T$, $k = 1, 2, \dots$, have a bounded order in any relatively compact open subset of R^n , then T is a C^∞ -function.

In other words, the space \mathcal{E}_P of distributions in R^n "strongly regular with respect to $P(D)$ " is contained in the space \mathcal{E} of C^∞ -functions; in this case $\mathcal{E}_P = \mathcal{E}$. The concept of strong regularity with respect to $P(D)$ coincides with that of strong regularity in some variables (see [6], p. 453), when $P(D)$ is the Laplace operator in those variables.

Suppose now that given are two arbitrary partial differential operators $P(D)$ and $Q(D)$. Then the question arises: Under what conditions on P and Q is $\mathcal{E}_P \subset \mathcal{E}_Q$? In particular, if $P(D)$ is "Q-hypoelliptic," i.e. all solutions $U \in \mathcal{D}'$ of the equation

$$P(D)U = 0$$

are in \mathcal{E}_Q , must then be $\mathcal{E}_P \subset \mathcal{E}_Q$? The Q-hypoelliptic operators were studied (in a slightly different but equivalent version) and characterized by E. A. Gorin and V. V. Grušin [2].

In this paper we give necessary and sufficient conditions for the inclusion $\mathcal{E}_P(\Omega) \subset \mathcal{E}_Q(\Omega)$, where $\mathcal{E}_P(\Omega)$ and $\mathcal{E}_Q(\Omega)$ are the spaces of "strongly regular" distributions on an arbitrary open set $\Omega \subset R^n$. These conditions are, in general, stronger than the Q-hypoellipticity of $P(D)$. If the inclusion in question holds for every Q-hypoelliptic operator $P(D)$, then $Q(D)$ must be hypoelliptic and the problem reduces to that in Hörmander's theorem stated above.

1. The spaces $\mathcal{E}_P(\Omega)$ and $C_P^{\prime\prime, \infty}(\Omega)$.

Let Ω be a nonempty open subset of R^n . A distribution $T \in \mathcal{D}'(\Omega)$ will be called strongly regular with respect to the differential operator $P(D)$, if to every open set Ω' having compact closure contained in Ω (we express this by writing $\Omega' \subset\subset \Omega$) there exists an integer $m \geq 0$ such that $P^k(D)T, k = 0, 1, \dots$, are all of order $\leq m$ in Ω' , i.e. the restrictions of $P^k(D)T$ to Ω' are all in $\mathcal{D}'^m(\Omega')$ ¹. We denote by $\mathcal{E}_P(\Omega)$ the space of all distributions in Ω , which are strongly regular with respect to $P(D)$. We also denote by $C_P^{\mu, \infty}(\Omega)$, where μ is an integer ≥ 0 , the space of all C^μ -functions in Ω such that $P^k(D)D^\alpha f, |\alpha| \leq \mu, k = 0, 1, \dots$, are continuous functions; here $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Consider now the spaces $\mathcal{E}_P(\Omega)$ and $\mathcal{E}_Q(\Omega)$ corresponding to the differential operators $P(D)$ and $Q(D)$ respectively.

THEOREM 1. *If $\mathcal{E}_P(\Omega) \subset \mathcal{E}_Q(\Omega)$, then to any open set $\Omega' \subset\subset \Omega$ there exists an integer $\mu \geq 0$ such that the restriction mapping $f \rightarrow f|_{\Omega'}$ maps $C_P^{\mu, \infty}(\Omega)$ into $C_Q^{\mu, \infty}(\Omega')$.*

Proof. Let Ω' be an open set satisfying the assumption $\Omega' \subset\subset \Omega$. We first prove the existence of nonnegative integers ν and m such that

$$(1) \quad \{Q^k(D)f|_{\Omega'} : f \in C_P^{\nu, \infty}(\Omega), k = 0, 1, \dots\} \subset \mathcal{D}'^m(\Omega').$$

Suppose that inclusion (1) does not hold for any ν and m . Then to every ν and m there exist a function $f \in C_P^{\nu, \infty}(\Omega)$ and a k such that $Q^k(D)f|_{\Omega'} \notin \mathcal{D}'^m(\Omega')$. Thus we can find strictly increasing sequences of positive integers ν_i, m_i and k_i , and a sequence of functions f_i with the following properties:

- (2) $f_i \in C_P^{\nu_i, \infty}(\Omega)$,
- (3) $Q^{k_i}(D)f_i|_{\Omega'} \in \mathcal{D}'^{m_i}(\Omega'), k = 0, 1, \dots$,
- (4) $Q^{k_i}(D)f_i|_{\Omega'}$ is of order m_i ,
- (5) $qk_i < \nu_{i+1}$,

where $i = 1, 2, \dots$, and q is the order of the operator $Q(D)$.

We denote by $\Omega_i, i = 1, 2, \dots$, open subsets of Ω such that

$$(6) \quad \Omega_i \subset\subset \Omega_{i+1} \text{ and } \bigcup_{i=1}^{\infty} \Omega_i = \Omega.$$

Next we set

$$a_1 = 1 \text{ and } a_i = 2^{-i}M_i^{-1}, \quad i = 2, 3, \dots,$$

¹ $P^0(D)$ is the identity operator, i.e. $P^0(D)T = T$.

where

$$M_i = \sup \{ |P^k(D)f_i(x)| + |Q^l(D)f_i(x)| + 1 \}$$

and the supremum is taken over all $x \in \Omega_i$ and $k, l = 0, 1, \dots, k_{i-1}$. Note that $Q^l(D)f_i, l = 0, 1, \dots, k_{i-1}$, are continuous functions in Ω , because of (5).

The function

$$f = \sum_{i=1}^{\infty} a_i f_i$$

is defined and continuous in Ω , since the f_i 's are continuous in Ω and the series converges there almost uniformly. Moreover, for any k we have (distributionally)

$$(7) \quad P^k(D)f = \sum_{i=1}^{\infty} a_i P^k(D)f_i .$$

But each term of the last series is a continuous function in Ω , by (1). Also

$$a_i \sup_{x \in \Omega_j} |P^k(D)f_i(x)| \leq 2^{-i}$$

whenever $k < i$ and $j \leq i$, by the definition of a_i . Hence it follows that the series (7) converges almost uniformly in Ω , for any k . Consequently $f \in C_P^{0,\infty}(\Omega) \subset \mathcal{E}_P(\Omega)$.

We now show that f is not in $\mathcal{E}_Q(\Omega)$, which is a contradiction to our hypothesis. We write

$$g_j = \sum_{i=1}^j a_i f_i \text{ and } h_j = \sum_{i=j+1}^{\infty} a_i f_i .$$

In view of (3) and (4), the restriction of $Q^{k_j}(D)g_j$ to Ω' is a distribution of order m_j . On the other hand, $Q^{k_j}(D)f_i, i = j + 1, j + 2, \dots$, are continuous functions in Ω , because of (2) and (5). Furthermore, by the definition of the a_i 's, the series

$$\sum_{i=j+1}^{\infty} a_i Q^{k_j}(D)f_i$$

converges almost uniformly in Ω , and so $Q^{k_j}(D)h_j$ is in Ω a continuous function. Thus

$$Q^{k_j}(D)f = Q^{k_j}(D)g_j + Q^{k_j}(D)h_j$$

is in Ω' a distribution of order m_j . Since $m_j \rightarrow \infty, f$ is not in $\mathcal{E}_Q(\Omega)$. This contradiction proves (1).

Consider now the fundamental solution E of the iterated Laplace equation, i.e.

$$\Delta^r E = \delta .$$

For sufficiently large γ , E is m times continuously differentiable. Therefore every distribution T on Ω' such that $\Delta^r T \in \mathcal{D}'^m(\Omega')$ is, in fact, a continuous function (see [5], vol. 2, p. 47). We choose $\mu = 2\gamma + \nu$, where ν is the integer occurring in (1). Then, if $f \in C_P^{\mu, \infty}(\Omega)$, it follows that $\Delta^r f \in C_P^{\nu, \infty}(\Omega)$ whence, in view of (1), $Q^k(D)\Delta^r f|_{\Omega'} = \Delta^r Q^k(D)f|_{\Omega'} \in \mathcal{D}'^m(\Omega')$. Thus, by what we said before, $Q^k(D)f|_{\Omega'}$ is a continuous function, for every $k = 0, 1, \dots$, i.e. $f|_{\Omega'} \in C_Q^{0, \infty}(\Omega')$. The proof is complete.

2. **Necessary conditions.** We proceed to derive necessary conditions for the inclusion $\mathcal{E}_P(\Omega) \subset \mathcal{E}_Q(\Omega)$. In view of Theorem 1 it suffices to find necessary conditions for the inclusion

$$(8) \quad \{f|_{\Omega'}: f \in C_P^{\mu, \infty}(\Omega)\} \subset C_Q^{0, \infty}(\Omega') .$$

We accomplish this by means of the standard argument based on the closed graph theorem and the Seidenberg-Tarski theorem (see [1]).

Let Ω_j , $j = 1, 2, \dots$, be open sets satisfying conditions (6). We define the topology in $C_P^{\mu, \infty}(\Omega)$ by means of the semi-norms

$$v_j(f) = \sup |P^k(D)D^\alpha f(x)| ,$$

where the supremum is taken over all $x \in \Omega_j$, $|\alpha| \leq \mu$ and $k \leq j$. Similarly, if Ω'_j , $j = 1, 2, \dots$, are open sets satisfying conditions analogous to (6) with Ω replaced by Ω' , we define the topology in $C_Q^{0, \infty}(\Omega')$ by means of the semi-norms

$$w_j(f) = \sup_{x \in \Omega'_j, k \leq j} |Q^k(D)f(x)| .$$

Then $C_P^{\mu, \infty}(\Omega)$ and $C_Q^{0, \infty}(\Omega')$ become Fréchet spaces. Moreover, the restriction mapping $C_P^{\mu, \infty}(\Omega) \rightarrow C_Q^{0, \infty}(\Omega')$ is closed and therefore continuous, by the closed graph theorem. Hence, to every integer $l > 0$, there exists an integer $k > 0$ and a constant $C > 0$ such that

$$(9) \quad w_l(f) \leq C \max_{1 \leq j \leq k} v_j(f) ,$$

for every $f \in C_P^{\mu, \infty}(\Omega)$. Applying condition (9) to the function

$$f(x) = e^{i\langle x, \zeta \rangle} ,$$

where $\zeta = \xi + i\eta$ and $\xi, \eta \in R^n$, we obtain the following lemma².

LEMMA 1. *If the inclusion (8) holds then, for every integer $l > 0$, we can find an integer $k > 0$ and constants $C, c > 0$ such that*

² We assume that $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$, where $D_j = -i(\partial/\partial x_j)$.

$$(10) \quad |Q^l(\zeta)| \leq C(1 + |\xi|^\mu)(1 + |P^k(\zeta)|)e^{e^{|\eta|}}.$$

We denote by $N(P, a)$, V_a and W_a the sets of all $\zeta = \xi + i\eta \in C^n$ such that $|P(\zeta)| \leq a$, $|\eta| \leq a$ and $|\xi| \leq a$, respectively.

LEMMA 2. *If condition (10) is satisfied, then $Q(\zeta)$ is bounded on every set $N(P, a) \cap V_b$, $a, b \geq 0$.*

Proof. Suppose there are $a, b \geq 0$ such that $Q(\zeta)$ is not bounded on $N(P, a) \cap V_b$. Then the function

$$s(t) = \sup_{\zeta \in N(P, a) \cap V_b \cap W_t} |Q(\zeta)|$$

is defined and continuous for sufficiently large t , and

$$(11) \quad s(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty.$$

But, for a given t , $s(t)$ is the largest of all s such that the equations and inequalities

$$(12) \quad \begin{aligned} |P(\xi + i\eta)|^2 &\leq a^2, |\eta|^2 \leq b^2, \\ |Q(\xi + i\eta)|^2 &= s^2, |\xi|^2 \leq t^2, s \geq 0, t \geq 0, \end{aligned}$$

have a solution $\xi, \eta \in R^n$. Applying to (12) the Seidenberg-Tarski theorem and next a well-known argument (see [4], p. 276, or [6], p. 317) one shows easily that, for sufficiently large t , $s(t)$ is an algebraic function. We now expand $s(t)$ in a Puiseux series in a neighborhood of infinity and make use of (11). It follows that

$$s(t) > t^h$$

for some $h > 0$ and all t sufficiently large. On the other hand, $s(t)$ is assumed for some $\xi = \xi(t), \eta = \eta(t)$, and

$$|\xi(t)| \leq t.$$

Choosing in (10) $l > \mu h^{-1}$ we obtain a contradiction, which proves the lemma.

THEOREM 2. *If $\mathcal{E}_P(\Omega) \subset \mathcal{E}_Q(\Omega)$, then the following equivalent conditions are satisfied:*

- (I₁) $Q(\zeta)$ is bounded on every set $N(P, a) \cap V_b$.
- (I₂) For any $a \geq 0$ there are constants $C, h > 0$ such that

$$|Q(\zeta)|^h \leq C(1 + |\eta|), \text{ for all } \zeta \in N(P, a).$$

- (I₃) For any $b \geq 0$ there are constants $C', h' > 0$ such that

$$|Q(\zeta)|^{h'} \leq C'(1 + |P(\zeta)|), \text{ for all } \zeta \in V_b.$$

Proof. In view of Theorem 1, Lemma 1 and Lemma 2, we need only to show that conditions (I₁)-(I₃) are equivalent. Also the implications (I₂) ⇒ (I₁) and (I₃) ⇒ (I₁) are obvious. We prove that (I₁) ⇒ (I₂).

Consider the real polynomial

$$W(\xi, \eta, r, s, t) = (\alpha^2 - |P(\xi + i\eta)|^2 - r^2)^2 + (s^2 - |\eta|^2)^2 + (t^2 - |Q(\xi + i\eta)|^2)^2$$

of $2n + 3$ real variables. If $\xi, \eta \in R^n$ lie on the surface

$$(13) \quad W(\xi, \eta, r, s, t) = 0,$$

then $\zeta = \xi + i\eta \in N(P, a)$. Moreover, by condition (I₁), the surface (13) is contained in a domain defined by an inequality

$$|s| > \varphi(|t|),$$

where $\varphi(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. Applying now a theorem of Gorin ([1], Theorem 4.1) we conclude that there exist constants $C, h > 0$ satisfying condition (I₂). Thus (I₁) ⇒ (I₂). The proof of the implication (I₁) ⇒ (I₃) is similar.

3. Sufficient conditions. We now prove that conditions (I₁)-(I₃) are sufficient for the inclusion under consideration. Our first goal is to construct a sequence of suitable fundamental solutions for the operators $P^k(D), k = 1, 2, \dots$. We achieve this by modifying the construction of a fundamental solution for $P(D)$ given in [2].

In what follows p and q denote the orders of the differential operators $P(D)$ and $Q(D)$, respectively.

LEMMA 3. *Suppose that conditions (I₁)-(I₃) are satisfied. Then there exist continuous functions $F_k, k = 1, 2, \dots$, in R^n with the following properties:*

(a) For $\nu = p + q + n$ and any k ,

$$E_k = (\lambda - \Delta)^\nu F_k$$

is a fundamental solution for $P^k(D)$, i.e.

$$P^k(D)E_k = \delta$$

(b) $P^j(D)F_k = F_{k-j}$, for $j = 1, 2, \dots, k - 1$.

(c) $Q^l(D)F_k, k, l = 1, 2, \dots$, are continuous functions in $R^n \setminus \{0\}$.

(d) For any l there is a k such that $Q^l(D)F_k$ is a continuous function in R^n .

Proof. For any $\xi' = (\xi_1, \dots, \xi_{n-1}) \in R^{n-1}$, consider the subset of the complex ζ_n -plane

$$U(\xi') = \{ \zeta_n \in C : |P(\xi', \zeta_n)| \leq 1 \text{ or } |\lambda + |\xi'|^2 + \zeta_n^2| \leq 1 \},$$

where $\lambda > 2p$. There exist constants $C, h > 0$ such that

$$(14) \quad |Q(\xi', \zeta_n)| \leq C(1 + |\eta_n|^h),$$

for all $\xi' \in R^{n-1}$ and $\zeta_n = \xi_n + i\eta \in U(\xi')$. This follows from (I₂), when $|P(\xi', \zeta_n)| \leq 1$ and can be easily verified in the other case.

Let $U^-(\xi')$ be the union of all connected components of $U(\xi')$ having nonempty intersections with $C^- = \{ \zeta_n \in C : \eta_n < 0 \}$. We denote by $L(\xi')$ the boundary of $C^- \cup U^-(\xi')$.

If $\zeta_n \in L(\xi')$, we have

$$(15) \quad |P(\xi', \zeta_n)| \geq 1;$$

also there are constants $C', h' > 0$ (independent of ξ') such that

$$(16) \quad |Q(\xi', \zeta_n)| \leq C' |P(\xi', \zeta_n)|^{h'}.$$

Inequality (16) is implied by (I₃) and (15), since $(\xi', \zeta_n) \in V_{2p}$, when $\zeta_n \in L(\xi')$.

For $k = 1, 2, \dots$, we set

$$F_k(x) = \frac{1}{(2\pi)^n} \int_{R^{n-1}} \left\{ \int_{L(\xi')} \frac{e^{i\langle x, \zeta \rangle}}{(\lambda + |\xi'|^2 + \zeta_n^2)^p P^k(\zeta)} d\zeta_n \right\} d\xi'.$$

The functions F_k are obviously continuous, because of (15). We claim that they satisfy the conditions (a)-(d).

Conditions (a) and (b) follow from general properties of the Fourier transforms of distributions.

The verification of condition (c) can be carried out in the same way as in [2] (see the proof of Lemma 4). We give a brief sketch of the argument.

Suppose first that, for a given k , $F_k^{(j)}$ is a function obtained by a construction as above, where the contour of integration (corresponding to $L(\xi')$) lies in the complex ζ_j -plane; in particular $F_k^{(n)} = F_k$. Then

$$Q^l(D)[F_k - F_k^{(j)}], j = 1, \dots, n - 1; l = 1, 2, \dots,$$

are continuous functions in R^n ; we omit the easy proof of this fact. Thus condition (c) will be verified, if we show that $Q^l(D)F_k^{(j)}$, $l = 1, 2, \dots$, are continuous for $x_j \neq 0$ ($j = 1, \dots, n$).

Consider, for example, the function F_k and let $x_n < 0$. In this case the contour $L(\xi')$ can be replaced by the boundary $V^-(\xi')$ of $U^-(\xi')$. By (14), there are positive constants C_1 and C_2 such that

$$\eta_n \leq -C_1 |Q(\xi', \zeta_n)|^{1/h} + C_2$$

for all $\xi' \in R^{n-1}$ and $\zeta_n \in V^-(\xi')$. Hence, if $\zeta = (\xi', \zeta_n)$, we have

$$|Q^l(\zeta)e^{i\langle x, \zeta \rangle}| \leq |Q(\zeta)|^l \exp \{x_n(C_1|Q(\zeta)|^{1/h} - C_2)\}.$$

It follows that the integral

$$\int_{R^{n-1}} \left\{ \int_{V^-(\xi')} \frac{Q^l(\zeta)e^{i\langle x, \zeta \rangle}}{(\lambda + |\xi'|^2 + \zeta_n^2)^l P^k(\zeta)} d\zeta_n \right\} d\xi'$$

converges absolutely and coincides with $Q^l(D)F_k(x)$, for every l .

In case $x_n > 0$ we can reason similarly, replacing $L(\xi')$ by a contour $V^+(\xi')$ lying entirely in the half-plane $\eta_n \geq 0$.

Condition (d) is a consequence of inequality (16). In fact,

$$\frac{Q^l(\xi', \zeta_n)}{P^k(\xi', \zeta_n)}$$

is bounded for $\xi' \in R^{n-1}, \zeta_n \in L(\xi')$, whenever $k \geq h'l$.

Lemma 3 is now established.

THEOREM 3. *If conditions (I₁) – (I₃) are satisfied, the $\mathcal{E}_P(\Omega) \subset \mathcal{E}_Q(\Omega)$, for any open set $\Omega \subset R^n$.*

Proof. Assume that $T \in \mathcal{E}_P(\Omega)$ and fix an arbitrary open set $\Omega' \subset \subset \Omega$. We have to show that the restrictions of $Q^l(D)T$, $l = 0, 1, \dots$, to Ω' are all in a space $\mathcal{D}'^m(\Omega')$.

By Lemma 3, there are fundamental solutions E_k for the operators $P^k(D)$, $k = 1, 2, \dots$, representable according to (a) with the functions F_k satisfying conditions (b) – (d). Let l be given and let k be the integer corresponding to l in condition (d).

There are open sets $\Omega_j, j = 0, 1, \dots, k + 1$, such that

$$(17) \quad \Omega' \subset \subset \Omega_{k+1} \subset \subset \Omega_k \subset \subset \dots \subset \subset \Omega_0 \subset \subset \Omega.$$

Since $T \in \mathcal{E}_P(\Omega)$, the restrictions of $P^j(D)T, j = 0, 1, \dots$, to Ω_0 are all of order $\leq m_0$, say. For every $j = 1, 2, \dots, k + 1$, we now choose a function $\varphi_j \in \mathcal{D}(\Omega_{j-1})$ such that $\varphi = 1$ on Ω_j . Then the distributions

$$S_1 = \varphi_1 T, S_j = \varphi_j P(D)S_{j-1}, j = 2, 3, \dots, k + 1,$$

are all of order $\leq m_0$. Moreover

$$(18) \quad S_1 = T \text{ on } \Omega_1$$

and

$$(19) \quad P(D)S_j - S_{j+1} = 0 \text{ on } \Omega_{j+1}, \quad j = 1, \dots, k.$$

Making use of (a) we may write

$$S_1 = \sum_{j=1}^k [P(D)S_j - S_{j+1}] * E_j + S_{k+1} * E_k,$$

whence

$$(20) \quad Q^l(D)S_1 = \sum_{j=1}^k [P(D)S_j - S_{j+1}] * Q^l(D)E_j + S_{k+1} * Q^l(D)E_k ;$$

here $*$ denotes the convolution. By (19), the "values" on Ω' of each convolution

$$[P(D)S_j - S_{j+1}] * Q^l(D)E_j$$

depend on the values of $Q^l(D)E_j$ outside a neighborhood of the origin (see [5], Chapter VI, Theorem III). Therefore the restriction to Ω' of the sum in (20) is a distribution of order $\leq m_0 + p + 2\nu$. On the other hand, the last term in (20) is of order $\leq m_0 + p + 2\nu$, because of (a) and (d). Hence the restriction of $Q^l(D)S_1$ to Ω' is of order $\leq m = m_0 + p + 2\nu$ and m_0 can be chosen the same for all l . Since, by (18), the restrictions of $Q^l(D)S_1$ and $Q^l(D)T$ to Ω' coincide, the theorem is proved.

Combining Theorem 2 with Theorem 3 we obtain the following corollary.

COROLLARY. *Each of the conditions (I₁) – (I₃) is necessary and sufficient for the inclusion $\mathcal{E}_p(\Omega) \subset \mathcal{E}_q(\Omega)$, where Ω is any nonempty open set.*

REMARK. Suppose that

$$Q(\zeta) = P(\zeta) \sum_{j=1}^n \zeta_j^2$$

where $P(\zeta)$ is an arbitrary polynomial. Then the operator $P(D)$ is Q -hypoelliptic (see [2], Theorem 1), but condition (I₃) is not satisfied, unless $P(D)$ (and consequently $Q(D)$) is hypoelliptic.

REFERENCES

1. E. A. Gorin, *Asymptotic properties of polynomials and algebraic functions of several variables*, Uspehi Mat. Nauk, **16** (1961), 91-118.
2. E. A. Gorin and V. V. Grušin, *Local theorems for partial differential equations with constant coefficients*, Trudy Moskow. Mat. Obsč., **14** (1965), 200-210.
3. L. Hörmander, *On the theory of general partial differential operators*, Acta Math., **94** (1955), 161-248.
4. ———, *Linear Partial Differential Operators*, New York, 1969.
5. L. Schwartz, *Théorie des Distributions I-II*, Paris, 1957-1959.
6. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, New York-London-Paris, 1966.

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Alexander (Smbat) Abian, <i>The use of mitotic ordinals in cardinal arithmetic</i>	1
Helen Elizabeth Adams, <i>Filtrations and valuations on rings</i>	7
Benno Artmann, <i>Geometric aspects of primary lattices</i>	15
Marilyn Breen, <i>Determining a polytope by Radon partitions</i>	27
David S. Browder, <i>Derived algebras in L_1 of a compact group</i>	39
Aiden A. Bruen, <i>Unimbeddable nets of small deficiency</i>	51
Michael Howard Clapp and Raymond Frank Dickman, <i>Unicoherent compactifications</i>	55
Heron S. Collins and Robert A. Fontenot, <i>Approximate identities and the strict topology</i>	63
R. J. Gazik, <i>Convergence in spaces of subsets</i>	81
Joan Geramita, <i>Automorphisms on cylindrical semigroups</i>	93
Kenneth R. Goodearl, <i>Distributing tensor product over direct product</i>	107
Julien O. Hennefeld, <i>The non-conjugacy of certain algebras of operators</i>	111
C. Ward Henson, <i>The nonstandard hulls of a uniform space</i>	115
M. Jeanette Huebener, <i>Complementation in the lattice of regular topologies</i>	139
Dennis Lee Johnson, <i>The diophantine problem $Y^2 - X^3 = A$ in a polynomial ring</i>	151
Albert Joseph Karam, <i>Strong Lie ideals</i>	157
Soon-Kyu Kim, <i>On low dimensional minimal sets</i>	171
Thomas Latimer Kriete, III and Marvin Rosenblum, <i>A Phragmén-Lindelöf theorem with applications to $\mathcal{M}(u, v)$ functions</i>	175
William A. Lampe, <i>Notes on related structures of a universal algebra</i>	189
Theodore Windle Palmer, <i>The reducing ideal is a radical</i>	207
Kulumani M. Rangaswamy and N. Vanaja, <i>Quasi projectives in abelian and module categories</i>	221
Ghulam M. Shah, <i>On the univalence of some analytic functions</i>	239
Joseph Earl Valentine and Stanley G. Wayment, <i>Criteria for Banach spaces</i>	251
Jerry Eugene Vaughan, <i>Linearly stratifiable spaces</i>	253
Zbigniew Zielezny, <i>On spaces of distributions strongly regular with respect to partial differential operators</i>	267