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Let T^j denote the compact group which is the Cartesian product of j copies of the circle where j is a positive integer or ω. If $1 \leqq p \leqq \infty$ let $L^p(T^j)$ denote the space of complex valued measurable functions which are integrable with respect to Haar measure on T^j. If j is finite we shall write n instead of j. The subspaces $H^p(T^n)$ of $L^p(T^n)$, i.e. the Hardy spaces of T^n, have many well-known properties. A family of subspaces $H^p(T^\omega)$ of the $L^p(T^\omega)$ is defined and they are shown to have many of the same properties as the $H^p(T^n)$. However a major difference between $H^p(T^\omega)$ and $H^p(T^n)$ is observed. If $1 < p < \infty$ then $H^p(T^n)$ is complemented in $L^p(T^n)$, but $H^p(T^\omega)$ is uncomplemented in $L^p(T^\omega)$ for $1 < p < \infty$ unless $p = 2$.

Special properties of homogeneous functions in $H^1(T^\omega)$. Let j be a positive integer or ω. If j is finite we shall write n in place of j. We shall let T^n denote the compact group which is the Cartesian product of n circles, and T^ω the compact group which is the Cartesian product of countably many circles. The dual of T^n is the direct sum of n copies of the integers, and the dual of T^ω is the direct sum of countably many copies of the integers. If $g \in T^n$, then we write

$$g = (z_1, z_2, \cdots, z_n)$$

where each z_i is a complex number of unit modulus. If $g \in T^\omega$ it has a similar representation, but we must take a countable family, i.e.

$$g = (z_1, z_2, z_3, \cdots) .$$

By abuse of notation if $i \leqq n \leqq \infty$, we let z_i denote that $g \in T^n$ or $g \in T^\omega$ which has the following representation:

$$g = (1, \cdots, 1, z_i, 1, \cdots)$$

where z_i occurs in the ith place. We shall write m_n for the normalized Haar measure on T^n and m for the normalized Haar measure on T^ω.

The dual of T^n can be written as $\sum_{i=1}^n Z$, and if $x \in \sum_{i=1}^n Z$ then we write

$$x = (x_1, x_2, \cdots, x_n)$$

where each x_i is an integer. The dual of T^ω can be written as $\sum_{i=1}^\infty Z$, and if $x \in \sum_{i=1}^\infty Z$, then we write

$$x = (x_1, x_2, x_3, \cdots)$$

where each x_i is an integer, and for any particular x, only finitely many x_i are nonzero.

We define $A_n \subset \sum_{i=1}^{n} Z$ and $A \subset \sum_{i=1}^{\infty} Z$ by

$$A_n = \{x \colon x_i \geqq 0 \text{ for all } i\}$$
$$A = \{x \colon x_i \geqq 0 \text{ for all } i\} .$$

We need the following definitions to define $H^p(T^j)$. Although the definitions could be stated in terms of T^j it is easier to state them in the context of arbitrary compact abelian groups.

DEFINITION 1.1. Suppose G is a compact abelian group with dual group Γ. If $1 \leqq p \leqq \infty$ let $L^p(G)$ denote the space of complex valued measurable functions which are p^{th} power integrable with respect to Haar measure on G. If E is a subset of Γ, f will be called an *E-function* if $f \in L^1(G)$ and $\hat{f}(\gamma) = 0$ if $\gamma \in \Gamma \sim E$, where $\hat{f}(\gamma)$ is the Fourier transform of f evaluated at γ.

DEFINITION 1.2. Suppose $1 \leqq p \leqq \infty$ then $L_E^p(G) = \{f \colon f \in L^p(G)$ and f is an E-function$\}$.

DEFINITION 1.3.

$$H^p(T^n) = L_{A_n}^p(T^n)$$
$$H^p(T^\omega) = L_A^p(T^\omega) .$$

The properties of $H^p(T^n)$ are discussed in [7]. These spaces are related to analytic functions in several complex variables which are defined on the interior of the n-polydisc in C^n, and are subject to certain growth conditions near the distinguished boundary T^n. If $j = \omega$, there is no analogue of the interior of the n-polydisc. However we still have many of the nice properties of $H^p(T^n)$.

It is possible to imbed $H^p(T^n)$ in $H^p(T^\omega)$ in a natural way. We have the following homomorphisms

$$\begin{aligned} \pi_n : T^\omega & \longrightarrow T^n \\ (z_1, z_2, \cdots, z_n, z_{n+1} \cdots) & \longmapsto (z_1, z_2, \cdots z_n) \end{aligned}$$

and π_n induces an isometry I_n.

$$\begin{aligned} I_n : H^p(T^n) & \longrightarrow H^p(T^\omega) \\ f & \longmapsto f \circ \pi_n . \end{aligned} \tag{1}$$

DEFINITION 1.4. Suppose $f \in H^1(T^n)$ and s is a positive integer or

0. Then the *s homogeneous component of* $f = {}_nP_s(f)$, where ${}_nP_s(f)$ is defined by its Fourier transform

$$\widehat{{}_nP_s(f)}(x) = \begin{Bmatrix} \hat{f}(x) \text{ if } \sum x_i = s \\ 0 \quad \text{otherwise} \end{Bmatrix} .$$

That is if f has Fourier series

$$f(g) \sim \sum_{x \in A_n} a_x(g, x) ,$$

then ${}_nP_s(f)$ has the following Fourier series:

$${}_nP_s(f)(g) \sim \sum_{\substack{x \in A_n \\ \Sigma x_i = s}} a_x(g, x) .$$

Then ${}_nP_s(f)$ is a trigonometric polynomial since $\widehat{{}_nP_s(f)}$ has finite support.

DEFINITION 1.5. Suppose $f \in H^1(T^\omega)$ and $f = {}_nP_s(f)$ for some s. Then we say *f is homogeneous of degree s*. The previous definition is motivated by the following fact: If λ is a complex number of unit modulus and we write λ to mean the point $(\lambda, \lambda, \lambda, \cdots, \lambda)$ of T^n, then

$$f(\lambda g) = \lambda^s f(g) \quad \text{for all} \quad g \in T^n$$

if f is homogeneous of degree s. Clearly if f is homogeneous of degree s its Fourier transform has finite support, so f is a trigonometric polynomial and hence $f \in H^p(T^\omega)$ for $1 \leqq p \leqq \infty$. It is easy to show that ${}_nP_s$ is a bounded linear operator from $H^1(T^n)$ into $H^p(T^n)$ for each p. However it is not obvious that we can define an operator P_s on $H^1(T^\omega)$ which is analogous to ${}_nP_s$ on $H^1(T^n)$ because the sum that should define $P_s(f)$ for $f \in H^1(T^\omega)$ is not necessarily finite. The following lemma helps show that P_s can be defined as a bounded linear operator from $H^1(T^\omega)$ into $H^p(T^\omega)$.

LEMMA 1.6. *Suppose s is a positive integer or* 0, *and* $1 \leqq p \leqq \infty$. *Then there exists a projection P_s on $H^p(T^\omega)$ with $\|P_s\| = 1$ satisfying*:

$$\widehat{P_s f}(x) = \begin{Bmatrix} \hat{f}(x) \ \text{if } \Sigma x_i = s \\ 0 \quad \text{otherwise} \end{Bmatrix} , \quad f \in H^p(T^\omega) .$$

That is if f has Fourier series

$$f(g) \sim \sum_{x \in A} a_x(g, x) ,$$

then $P_s(f)$ has the following Fourier series:

$$P_s(f)(g) \sim \sum_{\substack{x \in A \\ \Sigma x_i = s}} a_x(g, x) .$$

Proof. Consider the following subgroup H of $\sum_{i=1}^{\infty} Z$:

$$H = \left\{x : x \in \sum_{i=1}^{\infty} Z \quad \text{and} \quad \Sigma x_i = 0\right\}.$$

But $(\sum_{i=1}^{\infty} Z)/H$ is a quotient group of $\sum_{i=1}^{\infty} Z$ and hence its dual which we shall call D, is a compact subgroup of T^{ω}. Let m_D be normalized Haar measure on D. Since $D \subset T^{\omega}$, we can calculate the Fourier coefficients of m_D with respect to $\sum_{i=1}^{\infty} Z$. It is easy to calculate that

$$\hat{m}_D(x) = \chi_H(x) \quad \text{for all} \quad x \in \sum_{i=1}^{\infty} Z,$$

where $\chi_H(x)$ is the characteristic function of the set H. If s is a positive integer or 0, choose a $y_s \in \sum_{i=1}^{\infty} Z$ so that $\sum_{i=1}^{\infty} (y_s)_i = s$; then for the measure $y_s(g)dm_D(g)$

$$\widehat{y_s m_D}(x) = \hat{m}_D(x - y_s) = \left\{\begin{matrix} 1 & \text{if} \quad \Sigma(x - y_s) = 0 \\ & \text{i.e. } \Sigma(x)_i = s \\ 0 & \text{otherwise} \end{matrix}\right\}.$$

Evidently for all s

$$\int_G | y_s(g)\, dm_D(g) | = 1,$$

so if $f \in H^p(T^{\omega})$ we can consider $f*(y_s dm_D)$ where $*$ denotes the usual convolution of a measure on T^{ω} with a function which is in $H^p(T^{\omega})$, hence in $L^1(T^{\omega})$. We have the following inequalities:

$$(2) \qquad || f*(y_s dm_D) ||_p \leqq || f ||_p \int_G | y_s(g) dm_D(g) | = || f ||_p .$$

If we calculate the Fourier transform of $f*(y_s dm_D)$

$$\widehat{f*(y_s dm_D)}(x) = \hat{f}(x)\widehat{(y_s dm_D)}(x) = \widehat{P_s(f)}(x).$$

Since $f*(y_s dm_D)$ and $P_s(f)$ have the same Fourier transform they are the same element of $H^p(T^{\omega})$, and so from equation (2)

$$|| P_s(f) ||_p = || f*(y_s dm_D) ||_p \leqq || f ||_p$$

and this completes the proof.

DEFINITION 1.7. If $f \in H^p(T^{\omega})$, then the s *homogeneous component of* f *is* $P_s(f)$.

If $f = P_s(f)$ for some s, we say *f is homogeneous of degree s*. This definition is justified by the fact that if f is a homogeneous trigonometric polynomial of degree s on T^{ω}, then we have

$$(3) \qquad f(\lambda g) = \lambda^s f(g) \quad \text{for all} \quad g \in T^{\omega}$$

whenever λ is a complex number of unit modulus and on the left we write λ to mean $(\lambda, \lambda, \cdots)$.

Suppose that f is a homogeneous function and that $f \in H^1(T^j)$, where j is a positive integer or ω. If j is finite, then f is necessarily a trigonometric polynomial and the following lemma and theorem are obvious. However if $j = \omega$, f isn't necessarily a trigonometric polynomial, and the following lemma and theorem require proof.

LEMMA 1.8. *Suppose $f \in H^1(T^\omega)$ and that f is homogeneous of degree s. Then equation* (3) *is satisfied for almost all $g \in T^\omega$ and almost all λ.*

Proof. If f is a trigonometric polynomial there is nothing to prove. Otherwise by using an approximate identity we can find a sequence $\{f_n\}_{n=1}^\infty$ of homogeneous polynomials all of degree s such that

$$\lim_{n\to\infty} f_n = f$$

in the norm of $H^1(T^\omega)$. There exists a subsequence of $\{f_n\}_{n=1}^\infty$ say $\{f_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} f_{n_j}(g) = f(g) \text{ a.e.}$$

where a.e. means for almost all $g \in T^\omega$ with respect to Haar measure on T^ω. $T^\omega \times T$ is the product of the measure spaces T^ω and T, and so $T^\omega \times T$ is a measure space with the product measure.

Let

$$W = \{(g, \lambda) \in T^\omega \times T \text{ such that } f(\lambda g) = \lambda^s f(g)\} .$$

Then W is measurable and we wish to show that the measure of W is 1. Now consider any fixed $\lambda \in T$; we have

$$\lim_{j\to\infty} f_{n_j}(g) = f(g)$$
$$\lim_{j\to\infty} f_{n_j}(\lambda g) = f(\lambda g)$$

except for a null set of g. But for each j

$$f_{n_j}(\lambda g) = \lambda^s f_{n_j}(g) ,$$
$$f(\lambda g) = \lim_{j\to\infty} f_{n_j}(\lambda g) = \lim_{j\to\infty} \lambda^s f_{n_j}(g) = \lambda^s f(g)$$

except for a null set of g. So $m(W) = 1$, which finishes the proof.

The next theorem is an application of a theorem about $\Lambda(p)$ sets. We digress for a moment to define $\Lambda(p)$ set.

DEFINITION 1.9. Let G be a compact abelian group with dual group Γ. If $p > 1$ and $E \subset \Gamma$ we say E is a $\Lambda(p)$ set if $L_E^1(G) = L_E^p(G)$.

DEFINITION 1.10. If A is a subset of Γ and n is a positive integer we define $A^n = \{x \in \Gamma; x = a_1 + a_2 + \cdots + a_n$, where $a_i \in A, 1 \leqq i \leqq n\}$.

THEOREM 1.11. *Suppose G is a compact abelian group with torsion-free dual group Γ. If E is an independent set in Γ, then E^s is a $\Lambda(p)$ set for all $p < \infty$ and all positive integers s.*

Proof. See [3, p. 28, Theorem 4].

THEOREM 1.12. *Suppose $f \in H^1(T^\omega)$ and that f is a homogeneous function of degree s where s is a positive integer or 0. Then $f \in H^p(T^\omega)$ for $1 \leqq p < \infty$.*

Proof. Let $E = \{z_i\}_{i=1}^\infty$. Then E is independent as a set in $\sum_{i=1}^\infty Z$ and so E^s is a $\Lambda(p)$ set for all $p < \infty$, by Theorem 1.11. But since $f \in H^1(T^\omega)$ and f is homogeneous of degree s, f is an E^s-function. By applying Theorem 1.11 we obtain that $f \in H^p(T^\omega)$ for all $p < \infty$, and this completes the proof.

COROLLARY 1.13. *Suppose $f \in H^1(T^\omega)$ and that f is a finite sum of homogeneous functions; then $f \in H^p(T^\omega)$ for $1 \leqq p < \infty$.*

Proof. By assumption f is a finite sum of homogeneous functions so we may write

$$f = \sum_{s=0}^{k} P_s(f) .$$

Since $f \in H^1(T^\omega)$ each $P_s(f) \in H^1(T^\omega)$ for $0 \leqq s \leqq k$. By Theorem 1.12 each $p_s(f) \in H^p(T^\omega)$ for $1 \leqq p < \infty$, so f is a finite sum of functions in $H^p(T^\omega)$ hence $f \in H^p(T^\omega)$.

Theorem 1.12 is really a theorem about $H^1(T^\omega)$ rather than $L^1(T^\omega)$. In that context Theorem 1.12 is false. In fact Theorem 1.12 is false even for $L^1(T^2)$ and hence for $L^1(T^\omega)$.

If j is a positive integer or ∞, we define homogeneity for arbitrary functions in $L^1(T^j)$ as follows: If $f \in L^1(T^j)$, we say f is homogeneous of degree s if

$$\hat{f}(x) = 0 \text{ if } x \in \sum_{i=1}^{j} Z \text{ and } \Sigma x_i \neq s .$$

To show that Theorem 1.12 can't be extended to $L^1(T^2)$, we shall construct for every $p > 1$ and for every positive integer N, a homogeneous polynomial f of degree 0 on T^2 such that

$$\|f\|_1 = 1$$
$$\|f\|_p \geqq N .$$

For given $p > 1$, find a trigonometric polynomial b defined on T such that

$$\|b\|_1 = 1$$
$$\|b\|_p \geqq N$$

where $b(z_1)$ has Fourier series

$$b(z_1) = \sum_{k=0}^{t} a_k z_1^k .$$

Define the polynomial f by

$$f(z_1, z_2) = \sum_{k=0}^{t} a_k z_1^k z_2^{-k} .$$

We wish to compute the norm of f in $L^1(T^2)$ and in $L^p(T^2)$:

$$\begin{aligned}\|f\|_1 &= \int_{T^2} |f(z_1, z_2)|\, dm_1(z_1) dm_2(z_2)\\ &= \int_{T^2} \left|\sum_{k=0}^{t} a_k (z_1 z_2^{-1})^k\right| dm_1(z_1) dm_2(z_2)\\ &= \int_{T^2} \left|\sum_{k=0}^{t} a_k (z_1)^k\right| dm_1(z_1) dm_2(z_2) = \int_T \|b\|_1 dm_2(z_2) = \int_T 1\, dm_2(z_2) = 1 .\end{aligned}$$

The crucial equality in equation (4) is justified by the translation invariance of $dm_1(z_1)$. By a similar computation we have

$$\|f\|_p = \|b\|_p \geqq N$$

and this provides the desired counterexample.

2. A convergence theorem for $H^p(T^\omega)$. By the M. Riesz theorem on conjugate functions [8], if $1 < p < \infty$ and $f \in H^p(T)$, then

$$f = \lim_{n\to\infty} \sum_{s=0}^{n} a_s z_1^s , \qquad a_s = \hat{f}(s)$$

in the norm of $H^p(T)$. In our terminology this can be written

$$f = \lim_{n\to\infty} \sum_{s=0}^{n} {}_1P_s(f) .$$

The next theorem gives an analogous result for $H^p(T^\omega)$. The proof uses a theorem about ordered groups so we digress for a moment to define the relevant terms.

Suppose Γ is a discrete abelian group and P is a subset of Γ with the following properties:

1. If $\gamma_1 \in P$ and $\gamma_2 \in P$ then $\gamma_1 + \gamma_2 \in P$.

If $-P$ denotes the set whose elements are the inverses of the elements of P then we have

2. $P \cap (-P) = \{0\}$
3. $P \cup (-P) = \Gamma$.

Under these conditions P induces an order in Γ as follows: For γ_1 and γ_2 elements of Γ, say $\gamma_1 \geqq \gamma_2$ if $\gamma_1 - \gamma_2 \in P$. It is easy to check that this is a linear order. A given group may have many different orders corresponding to different choices of P with the three properties above.

DEFINITION 2.1. Suppose G is a compact abelian group whose dual group Γ is ordered. Let f be a trigonometric polynomial on G with Fourier series

$$f(g) \sim \sum_{\gamma \in \Gamma} a_\gamma (g, \gamma) .$$

Define $\Phi(f)$ by

$$\Phi(f)(g) \sim \sum_{\substack{\gamma \in \Gamma \\ \gamma \geqq 0}} a_\gamma (g, \gamma) .$$

We shall need the following generalization of the M. Riesz theorem on conjugate functions. It is due to Bochner [1].

THEOREM 2.2. *Suppose $1 < p < \infty$. Then there exists a constant A_p, independent of G or the particular order in Γ such that if f is a trigonometric polynomial on G, then*

$$\| \Phi(f) \|_p \leqq A_p \| f \|_p .$$

THEOREM 2.3. *Let $1 < p < \infty$. Then if $f \in H^p(T^\omega)$*

$$\lim_{n \to \infty} \sum_{s=0}^{n} P_s(f) = f$$

in the norm of $H^p(T^\omega)$.

Proof. Fix p. Define Y_n by

$$Y_n(f) = \sum_{s=0}^{n} P_s(f) \text{ if } f \in H^p(T^\omega) .$$

Clearly trigonometric polynomials are dense in $H^p(T^\omega)$ and

$$\lim_{n \to \infty} Y_n(f) = f$$

whenever f is a trigonometric polynomial. It remains to show that the family $\{Y_n\}_{n=1}^{\infty}$ is uniformly bounded on trigonometric polynomials, i.e.

$$\| Y_n(f) \|_p \leqq K \| f \|_p$$

f a trigonometric polynomial where K is a positive constant independent of n and f. Then by a standard argument in functional analysis, the proof is complete. I shall show that the norm of Y_n is majorized by A_p, where A_p is the constant of Theorem 2.2.

Our first task is to induce an order in $\sum_{i=1}^{\infty} Z$ so that we can apply Theorem 2.2. First choose a family $\{d_i\}_{i=1}^{\infty}$ of real numbers which satisfies the following properties:

1. $d_1 = -1$, $-1 < d_i < -n/(n+1)$ for $i \neq 1$.
2. The set $\{d_i\}$ is independent in the group sense as a subset of the reals.

We define a homomorphism from $\sum_{i=1}^{\infty} Z$ into the reals by

$$\pi\colon \sum_{i=1}^{\infty} \longrightarrow R$$

$$x \longmapsto \sum_{i=1}^{\infty} d_i x_i \ .$$

π is clearly a homomorphism; since the d_i are linearly independent, it has a trivial kernel, i.e. if $\pi(x) = 0$ then $x = 0$. Define

$$P = \left\{x\colon x \in \sum_{i=1}^{\infty} Z \text{ and } \pi(x) \geqq 0\right\} .$$

Then P satisfies the necessary properties to induce an order in $\sum_{i=1}^{\infty} Z$. If $f(g)$ is an arbitrary trigonometric polynomial on T^{ω} define a trigonometric polynomial $f_1(g)$ as follows:

$$f_1(g) = z_1^{-n}(g) f(g) \ .$$

Let $f(g) = \Sigma a_x(g, x)$. Then

$$f_1(g) = z_1^{-n}(g) f(g) = \Sigma a_x(g, -nz_1)(g, x) = \Sigma a_x(g, x - nz_1)$$

and

$$\phi(f_1) = \sum_{\pi(x - nz_1) \geqq 0} a_x(g, x - nz_1) \ .$$

If $\pi(x - nz_1) \geqq 0$, then

$$0 \leqq \pi(x - nz_1) = \pi(x) + \pi(-nz_1) = \pi(x) - n\pi(z_1) = \pi(x) + n$$

and $\pi(x) \geqq -n$. But $\pi(x) = \Sigma d_i x_i$, and by using property 1 of $\{d_i\}$ it is clear that $\pi(x) \geqq -n$ if and only if $\Sigma x_i \leqq n$. So $\phi(f_1) = \Sigma a_x(g, x - nz_1)$.

Then it is easy to compute that $\Sigma x_i \leqq n$

$$z_1^n \Phi(f_1) = \sum_{i=1}^{n} P_i(f) = Y_n(f) .$$

By Theorem 2.2 we have that

$$|| \Phi(f_1) ||_p \leqq A_p || f_1 ||_p .$$

So we have

$$\begin{aligned} || Y_n(f) ||_p &= || z_1^n \Phi(f_1) ||_p = || \Phi f_1 ||_p \leqq A_p || f_1 ||_p \\ &= A_p || z_1^{-n} f ||_p = A_p || f ||_p , \end{aligned}$$

so the norm of Y_n is less than or equal to A_p and the proof is complete.

3. The complementation problem. The next theorem shows that $H^p(T^\omega)$ is uncomplemented as a subspace of $L^p(T^\omega)$ if $p \neq 2$. This is in contrast to $H^p(T^n)$ which is complemented in $L^p(T^n)$ except when $p = 1$ or $p = \infty$. Although other examples of uncomplemented subspaces of an L^p space are known, $H^p(T^\omega)$ has the advantage of being defined in a concrete way.

DEFINITION 3.1. Let G be a compact abelian group. If $f \in L^1(G)$ let f_{g_0} denote the *g_0-translate of f* where

$$f_{g_0}(g) = f(g_0 + g) .$$

LEMMA 3.2. *Let G be a compact abelian group with dual group Γ. Suppose $1 \leqq p < \infty$ and that T is a bounded projection from $L^p(G)$ onto $L^p_E(G)$. Then a linear operator Q can be defined by*

$$Q(f) = \int_G [T(f_g)]_{-g} dm(g) \qquad f \in L^p(G) ,$$

where the integral is the Bochner integral.

Q is the natural projection from $L^p(G)$ onto $L^p_E(G)$, i.e., if $f \in L^p(G)$ then $Q(f)$ is defined by its Fourier transform as follows:

$$\widehat{G(f)}(x) = \begin{Bmatrix} \hat{f}(x) & x \in E \\ 0 & \text{otherwise} \end{Bmatrix} .$$

Proof. The proof for the case $G = T, \Gamma = Z, E = Z^+, p = 1$ is given [4, page 154]. The proof in the general case is analogous.

THEOREM 3.3. *Suppose $p \neq 2$, then $H^p(T^\omega)$ is uncomplemented as subspace of $L^p(T^\omega)$.*

Proof. If $p=1$ or $p=\infty$, there is really nothing to prove. There is a theorem in [4, pp. 154–155] which proves that $H^1(T)$ is uncomplemented in $L^1(T)$, and that $H^\infty(T)$ is uncomplemented in $L^\infty(T)$. Then since $H^i(T)$ and $L^i(T)$ can be isometrically embedded into $H^i(T^\omega)$ and $L^i(T^\omega)$ respectively for $i=1, \infty$, the theorem is proved for $p=1$ or $p=\infty$. In any case the argument which follows is valid for $p=1$, and with slight modifications for $p=\infty$.

Let S be the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is defined on trigonometric polynomials by

$$\begin{aligned} S\colon L^p(T^\omega) &\longrightarrow H^p(T^\omega) \\ f \quad &\longmapsto \quad S(f) \end{aligned}$$

where

$$\widehat{S(f)}(x) = \begin{Bmatrix} \hat{f}(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{Bmatrix}.$$

We wish to show that S can't be extended to a bounded operator defined on all of $L^p(T^\omega)$. To do this it is sufficient to find trigonometric polynomials f_n on T^ω such that

$$\|f_n\|_p = 1 \tag{5}$$

$$\|S(f_n)\|_p = (1+\varepsilon)^n \quad \text{where} \quad \varepsilon > 0 . \tag{6}$$

By [8, p. 295, Ex. 2] we can find a trigonometric polynomial h defined on T so that

$$h(z_1) = \sum_{k=-n}^{n} a_k z_1^k \qquad \|h\|_p = 1$$

and if

$$h_+(z_1) = \sum_{k=0}^{n} a_k z_1^k$$

then we have

$$\|h_+\|_p = 1 + \varepsilon$$

where ε is some positive number which depends upon p. Consider the trigonometric polynomial r defined on T^2 by

$$r(z_1, z_2) = h(z_1)h(z_2) = \left(\sum_{k=-n}^{n} a_k z_1^k\right)\left(\sum_{k=-n}^{n} a_k z_2^k\right).$$

Define r_+ by

$$r_+(z_1, z_2) = h_+(z_1)h_+(z_2) = \left(\sum_{k=0}^{n} a_k z_1^k\right)\left(\sum_{k=0}^{n} a_k z_2^k\right).$$

Then it is easy to compute that

$$\|r\|_p = \|h\|_p^2 = 1$$
$$\|r_+\|_p = (\|h_+\|_p)^2 = (1+\varepsilon)^2 .$$

We define trigonometric polynomials on T^ω by

$$f_1 = I_1(h) \qquad f_2 = I_2(r)$$

where I_1 and I_2 were defined in equation (1). It is easy to check that

$$S(f_1) = I_1(h_+) \qquad S(f_2) = I_2(r_+)$$

and since I_1 and I_2 are isometries we have

$$\|f_1\|_p = \|I_1(h)\|_p = \|h\|_p = 1$$
$$\|S(f_1)\|_p = \|I_1(h_+)\|_p = \|h_+\|_p = 1+\varepsilon$$
$$\|f_2\|_p = \|I_2(r)\|_p = \|r\|_p = 1$$
$$\|S(f_2)\|_p = \|I_2(r_+)\|_p = \|r_+\|_p = (1+\varepsilon)^2 .$$

By a similar argument we can construct trigonometric polynomials $f_3, f_4, \cdots$ and hence f_n for any n and f_n will satisfy equations (5) and (6). This shows that the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$ isn't bounded. To finish the proof we must show there is no bounded projection of any kind from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is the identity when restricted to $H^p(T^\omega)$.
Suppose there exists $\tilde{S}$ a linear transformation from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is the identity when restricted to $H^p(T^\omega)$. Define a linear operator Q by

$$Q(f) = \int_{T^\omega} [\tilde{S}(f_g)]_{-g}\, dm(g)$$

where the integral is the Bochner integral. Then Q is a bounded linear operator from $L^p(T^\omega)$ into $H^p(T^\omega)$ and by Lemma 3.2 we have that $Q = S$, where S is the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$. But we know that S isn't a bounded projection and this provides the contradiction which finishes the proof.

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