SOME $H^p$ SPACES WHICH ARE UNCOMPLEMENTED IN $L^p$  

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Let $T^j$ denote the compact group which is the Cartesian product of $j$ copies of the circle where $j$ is a positive integer or $\omega$. If $1 \leq p \leq \infty$ let $L^p(T^j)$ denote the space of complex valued measurable functions which are integrable with respect to Haar measure on $T^j$. If $j$ is finite we shall write $n$ instead of $j$. The subspaces $H^p(T^n)$ of $L^p(T^n)$, i.e. the Hardy spaces of $T^n$, have many well-known properties. A family of subspaces $H^p(T^\omega)$ of the $L^p(T^\omega)$ is defined and they are shown to have many of the same properties as the $H^p(T^n)$. However a major difference between $H^p(T^\omega)$ and $H^p(T^n)$ is observed. If $1 < p < \infty$ then $H^p(T^n)$ is complemented in $L^p(T^n)$, but $H^p(T^\omega)$ is uncomplemented in $L^p(T^\omega)$ for $1 < p < \infty$ unless $p = 2$.

Special properties of homogeneous functions in $H^1(T^\omega)$. Let $j$ be a positive integer or $\omega$. If $j$ is finite we shall write $n$ in place of $j$. We shall let $T^n$ denote the compact group which is the Cartesian product of $n$ circles, and $T^\omega$ the compact group which is the Cartesian product of countably many circles. The dual of $T^n$ is the direct sum of $n$ copies of the integers, and the dual of $T^\omega$ is the direct sum of countably many copies of the integers. If $g \in T^n$, then we write

$$g = (z_1, z_2, \ldots, z_n)$$

where each $z_i$ is a complex number of unit modulus. If $g \in T^\omega$ it has a similar representation, but we must take a countable family, i.e.

$$g = (z_1, z_2, z_3, \ldots).$$

By abuse of notation if $i \leq n \leq \infty$, we let $z_i$ denote that $g \in T^n$ or $g \in T^\omega$ which has the following representation:

$$g = (1, \ldots, 1, z_i, 1, \ldots)$$

where $z_i$ occurs in the $i$th place. We shall write $m_n$ for the normalized Haar measure on $T^n$ and $m$ for the normalized Haar measure on $T^\omega$.

The dual of $T^n$ can be written as $\sum_{i=1}^n Z$, and if $x \in \sum_{i=1}^n Z$ then we write

$$x = (x_1, x_2, \ldots, x_n)$$

where each $x_i$ is an integer. The dual of $T^\omega$ can be written as $\sum_{i=1}^\infty Z$, and if $x \in \sum_{i=1}^\infty Z$, then we write
where each $x_i$ is an integer, and for any particular $x$, only finitely many $x_i$ are nonzero.

We define $A_n \subset \sum_{i=1}^n \mathbb{Z}$ and $A \subset \sum_{i=1}^\infty \mathbb{Z}$ by

$$A_n = \{x: x_i \geq 0 \text{ for all } i\}$$

$$A = \{x: x_i \geq 0 \text{ for all } i\}.$$ 

We need the following definitions to define $H^p(T^\omega)$. Although the definitions could be stated in terms of $T^\omega$ it is easier to state them in the context of arbitrary compact abelian groups.

**Definition 1.1.** Suppose $G$ is a compact abelian group with dual group $\Gamma$. If $1 \leq p \leq \infty$ let $L^p(G)$ denote the space of complex valued measurable functions which are $p^{th}$ power integrable with respect to Haar measure on $G$. If $E$ is a subset of $\Gamma$, $f$ will be called an $E$-function if $f \in L^1(G)$ and $\hat{f}(\gamma) = 0$ if $\gamma \in \Gamma \sim E$, where $\hat{f}(\gamma)$ is the Fourier transform of $f$ evaluated at $\gamma$.

**Definition 1.2.** Suppose $1 \leq p \leq \infty$ then $L^p(G) = \{f: f \in L^p(G) \text{ and } f \text{ is an } E\text{-function}\}$.

**Definition 1.3.**

$$H^p(T^\omega) = L^p_{\pi_n}(T^n)$$

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The properties of $H^p(T^n)$ are discussed in [7]. These spaces are related to analytic functions in several complex variables which are defined on the interior of the $n$-polydisc in $\mathbb{C}^n$, and are subject to certain growth conditions near the distinguished boundary $T^n$. If $j = \omega$, there is no analogue of the interior of the $n$-polydisc. However we still have many of the nice properties of $H^p(T^n)$.

It is possible to imbed $H^p(T^n)$ in $H^p(T^\omega)$ in a natural way. We have the following homomorphisms

$$\pi_n: T^\omega \longrightarrow T^n \quad (z_1, z_2, \ldots, z_n, z_{n+1}, \ldots) \longmapsto (z_1, z_2, \ldots, z_n)$$

and $\pi_n$ induces an isometry $I_n$.

$$I_n: H^p(T^n) \longrightarrow H^p(T^\omega) \quad f \longmapsto f \circ \pi_n \quad \text{(1)}$$

**Definition 1.4.** Suppose $f \in H^1(T^n)$ and $s$ is a positive integer or
0. Then the \( s \) homogeneous component of \( f = \pi P_s(f) \), where \( \pi P_s(f) \) is defined by its Fourier transform

\[
\pi P_s(f)(x) = \begin{cases} \hat{f}(x) & \text{if } \sum x_i = s \\ 0 & \text{otherwise} \end{cases}
\]

That is if \( f \) has Fourier series

\[
f(g) \sim \sum_{x \in A^n} a_x(g, x),
\]

then \( \pi P_s(f) \) has the following Fourier series:

\[
\pi P_s(f)(g) \sim \sum_{x \in A^n \atop \sum x_i = s} a_x(g, x).
\]

Then \( \pi P_s(f) \) is a trigonometric polynomial since \( \pi P_s(f) \) has finite support.

**Definition 1.5.** Suppose \( f \in H^1(T^n) \) and \( f = \pi P_s(f) \) for some \( s \). Then we say \( f \) is homogeneous of degree \( s \). The previous definition is motivated by the following fact: If \( \lambda \) is a complex number of unit modulus and we write \( \lambda \) to mean the point \((\lambda, \lambda, \lambda, \cdots, \lambda)\) of \( T^n \), then

\[
f(\lambda g) = \lambda^s f(g) \quad \text{for all } g \in T^n
\]

if \( f \) is homogeneous of degree \( s \). Clearly if \( f \) is homogeneous of degree \( s \) its Fourier transform has finite support, so \( f \) is a trigonometric polynomial and hence \( f \in H^p(T^n) \) for \( 1 \leq p \leq \infty \). It is easy to show that \( \pi P_s \) is a bounded linear operator from \( H^1(T^n) \) into \( H^p(T^n) \) for each \( p \). However it is not obvious that we can define an operator \( \pi P_s \) on \( H^1(T^n) \) which is analogous to \( \pi P_s \) on \( H^1(T^n) \) because the sum that should define \( \pi P_s(f) \) for \( f \in H^1(T^n) \) is not necessarily finite. The following lemma helps show that \( \pi P_s \) can be defined as a bounded linear operator from \( H^1(T^n) \) into \( H^p(T^n) \).

**Lemma 1.6.** Suppose \( s \) is a positive integer or 0, and \( 1 \leq p \leq \infty \). Then there exists a projection \( \pi P_s \) on \( H^p(T^n) \) with \( \| \pi P_s \| = 1 \) satisfying:

\[
\pi P_s(f)(x) = \begin{cases} \hat{f}(x) & \text{if } \sum x_i = s \\ 0 & \text{otherwise} \end{cases}, \quad f \in H^p(T^n).
\]

That is if \( f \) has Fourier series

\[
f(g) \sim \sum_{x \in A} a_x(g, x),
\]

then \( \pi P_s(f) \) has the following Fourier series:

\[
\pi P_s(f)(g) \sim \sum_{x \in A \atop \sum x_i = s} a_x(g, x).
\]
Proof. Consider the following subgroup $H$ of $\sum_{i=1}^{\infty} \mathbb{Z}$:

$$H = \left\{ x : x \in \sum_{i=1}^{\infty} \mathbb{Z} \text{ and } \sum x_i = 0 \right\}.$$  

But $(\sum_{i=1}^{\infty} \mathbb{Z})/H$ is a quotient group of $\sum_{i=1}^{\infty} \mathbb{Z}$ and hence its dual which we shall call $D$, is a compact subgroup of $T^\omega$. Let $m_D$ be normalized Haar measure on $D$. Since $D \subset T^\omega$, we can calculate the Fourier coefficients of $m_D$ with respect to $\sum_{i=1}^{\infty} \mathbb{Z}$. It is easy to calculate that

$$\hat{m}_D(x) = \chi_H(x) \quad \text{for all } x \in \sum_{i=1}^{\infty} \mathbb{Z},$$

where $\chi_H(x)$ is the characteristic function of the set $H$. If $s$ is a positive integer or 0, choose a $y_s \in \sum_{i=1}^{\infty} \mathbb{Z}$ so that $\sum_{i=1}^{\infty} (y_s)_i = s$; then for the measure $y_s(g)dm_D(g)$

$$\hat{y_s}m_D(x) = \hat{m}_D(x - y_s) = \begin{cases} 1 & \text{if } \Sigma(x - y_s) = 0 \\ \text{i.e. } \Sigma(x)_i = s & \\ 0 & \text{otherwise} \end{cases}.$$  

Evidently for all $s$

$$\int_T |y_s(g) dm_D(g)| = 1,$$

so if $f \in H^p(T^\omega)$ we can consider $f*(y_s dm_D)$ where $*$ denotes the usual convolution of a measure on $T^\omega$ with a function which is in $H^p(T^\omega)$, hence in $L^1(T^\omega)$. We have the following inequalities:

$$\left\| f*(y_s dm_D) \right\|_p \leq \left\| f \right\|_p \int_T |y_s(g) dm_D(g)| = \left\| f \right\|_p.$$  

If we calculate the Fourier transform of $f*(y_s dm_D)$

$$\hat{f*(y_s dm_D)}(x) = \hat{f(x)}(y_s dm_D)(x) = \hat{P_s}(f)(x).$$

Since $f*(y_s dm_D)$ and $P_s(f)$ have the same Fourier transform they are the same element of $H^p(T^\omega)$, and so from equation (2)

$$\left\| P_s(f) \right\|_p = \left\| f*(y_s dm_D) \right\|_p \leq \left\| f \right\|_p$$

and this completes the proof.

**Definition 1.7.** If $f \in H^p(T^\omega)$, then the $s$ homogeneous component of $f$ is $P_s(f)$.

If $f = P_s(f)$ for some $s$, we say $f$ is homogeneous of degree $s$. This definition is justified by the fact that if $f$ is a homogeneous trigonometric polynomial of degree $s$ on $T^\omega$, then we have

$$f(\lambda g) = \lambda^s f(g) \quad \text{for all } g \in T^\omega.$$  


whenever λ is a complex number of unit modulus and on the left we write λ to mean (λ, λ, •••).

Suppose that f is a homogeneous function and that \( f \in H^1(T^j) \), where \( j \) is a positive integer or \( \omega \). If \( j \) is finite, then \( f \) is necessarily a trigonometric polynomial and the following lemma and theorem are obvious. However if \( j = \omega \), \( f \) isn't necessarily a trigonometric polynomial, and the following lemma and theorem require proof.

**Lemma 1.8.** Suppose \( f \in H^1(T^\omega) \) and that \( f \) is homogeneous of degree \( s \). Then equation (3) is satisfied for almost all \( g \in T^\omega \) and almost all \( \lambda \).

**Proof.** If \( f \) is a trigonometric polynomial there is nothing to prove. Otherwise by using an approximate identity we can find a sequence \( \{f_n\}_{n=1}^\infty \) of homogeneous polynomials all of degree \( s \) such that

\[
\lim_{n \to \infty} f_n = f
\]

in the norm of \( H^1(T^\omega) \). There exists a subsequence of \( \{f_n\}_{n=1}^\infty \) say \( \{f_{n_j}\}_{j=1}^\infty \) such that

\[
\lim_{j \to \infty} f_{n_j}(g) = f(g) \text{ a.e.}
\]

where a.e. means for almost all \( g \in T^\omega \) with respect to Haar measure on \( T^\omega \). \( T^\omega \times T \) is the product of the measure spaces \( T^\omega \) and \( T \), and so \( T^\omega \times T \) is a measure space with the product measure.

Let

\[
W = \{(g, \lambda) \in T^\omega \times T \text{ such that } f(\lambda g) = \lambda^s f(g)\}.
\]

Then \( W \) is measurable and we wish to show that the measure of \( W \) is 1. Now consider any fixed \( \lambda \in T \); we have

\[
\lim_{j \to \infty} f_{n_j}(g) = f(g)
\]

\[
\lim_{j \to \infty} f_{n_j}(\lambda g) = f(\lambda g)
\]

except for a null set of \( g \). But for each \( j \)

\[
f_{n_j}(\lambda g) = \lambda^s f_{n_j}(g),
\]

\[
f(\lambda g) = \lim_{j \to \infty} f_{n_j}(\lambda g) = \lim_{j \to \infty} \lambda^s f_{n_j}(g) = \lambda^s f(g)
\]

except for a null set of \( g \). So \( m(W) = 1 \), which finishes the proof.

The next theorem is an application of a theorem about \( \Lambda(p) \) sets. We digress for a moment to define \( \Lambda(p) \) set.
DEFINITION 1.9. Let $G$ be a compact abelian group with dual group $\Gamma$. If $p > 1$ and $E \subset \Gamma$ we say $E$ is a $\Lambda(p)$ set if $L_p^{\ast}(G) = L_p^{\ast}(G)$.

DEFINITION 1.10. If $A$ is a subset of $\Gamma$ and $n$ is a positive integer we define $A^n = \{x \in \Gamma; x = a_1 + a_2 + \cdots + a_n, \text{where } a_i \in A, 1 \leq i \leq n\}$.

THEOREM 1.11. Suppose $G$ is a compact abelian group with torsion-free dual group $\Gamma$. If $E$ is an independent set in $\Gamma$, then $E^s$ is a $\Lambda(p)$ set for all $p < \infty$ and all positive integers $s$.

Proof. See [3, p. 28, Theorem 4].

THEOREM 1.12. Suppose $f \in H^1(T^\omega)$ and that $f$ is a homogeneous function of degree $s$ where $s$ is a positive integer or 0. Then $f \in H^p(T^\omega)$ for $1 \leq p < \infty$.

Proof. Let $E = \{z_i\}_{i=1}^\infty$. Then $E$ is independent as a set in $\sum_{i=1}^\infty Z$ and so $E^s$ is a $\Lambda(p)$ set for all $p < \infty$, by Theorem 1.11. But since $f \in H^1(T^\omega)$ and $f$ is homogeneous of degree $s$, $f$ is an $E^s$-function. By applying Theorem 1.11 we obtain that $f \in H^p(T^\omega)$ for all $p < \infty$, and this completes the proof.

COROLLARY 1.13. Suppose $f \in H^1(T^\omega)$ and that $f$ is a finite sum of homogeneous functions; then $f \in H^p(T^\omega)$ for $1 \leq p < \infty$.

Proof. By assumption $f$ is a finite sum of homogeneous functions so we may write

$$f = \sum_{s=0}^k P_s(f).$$

Since $f \in H^1(T^\omega)$ each $P_s(f) \in H^1(T^\omega)$ for $0 \leq s \leq k$. By Theorem 1.12 each $P_s(f) \in H^p(T^\omega)$ for $1 \leq p < \infty$, so $f$ is a finite sum of functions in $H^p(T^\omega)$ hence $f \in H^p(T^\omega)$.

Theorem 1.12 is really a theorem about $H^1(T^\omega)$ rather than $L^1(T^\omega)$. In that context Theorem 1.12 is false. In fact Theorem 1.12 is false even for $L^1(T^\omega)$ and hence for $L^p(T^\omega)$.

If $j$ is a positive integer or $\infty$, we define homogeneity for arbitrary functions in $L^1(T^i)$ as follows: If $f \in L^1(T^i)$, we say $f$ is homogeneous of degree $s$ if

$$\hat{f}(x) = 0 \text{ if } x \in \sum_{i=1}^j Z \text{ and } \Sigma x_i \neq s.$$ 

To show that Theorem 1.12 can't be extended to $L^1(T^\omega)$, we shall construct for every $p > 1$ and for every positive integer $N$, a homo-
geneous polynomial $f$ of degree 0 on $T^n$ such that
\[
\|f\|_1 = 1 \\
\|f\|_p \geq N.
\]

For given $p > 1$, find a trigonometric polynomial $b$ defined on $T$ such that
\[
\|b\|_1 = 1 \\
\|b\|_p \geq N
\]
where $b(z_i)$ has Fourier series
\[
b(z_i) = \sum_{k=0}^t a_k z_i^k.
\]

Define the polynomial $f$ by
\[
f(z_1, z_2) = \sum_{k=0}^t a_k z_1^k z_2^{-k}.
\]

We wish to compute the norm of $f$ in $L^p(T^n)$ and in $L^p(T^n)$:
\[
\|f\|_1 = \int_{T^n}|f(z_1, z_2)|dm_1(z_1)dm_2(z_2)
\]
\[
= \int_{T^n} \left| \sum_{k=0}^t a_k (z_1 z_2^{-1})^k \right| dm_1(z_1)dm_2(z_2)
\]
\[
= \int_{T^n} \left| \sum_{k=0}^t a_k (z_1)^k \right| dm_1(z_1)dm_2(z_2) = \int_{T^n} \|b\|_1 dm_2(z_2) = \int_{T^n} 1 dm_2(z_2) = 1.
\]

The crucial equality in equation (4) is justified by the translation invariance of $dm_i(z_i)$. By a similar computation we have
\[
\|f\|_p = \|b\|_p \geq N
\]
and this provides the desired counterexample.

2. A convergence theorem for $H^p(T^n)$. By the M. Riesz theorem on conjugate functions [8], if $1 < p < \infty$ and $f \in H^p(T)$, then
\[
f = \lim_{n \to \infty} \sum_{s=0}^n a_s z_i^s,
\]
in the norm of $H^p(T)$. In our terminology this can be written
\[
f = \lim_{n \to \infty} \sum_{s=0}^n P_s(f).
\]
The next theorem gives an analogous result for $H^p(T^n)$. The proof uses a theorem about ordered groups so we digress for a moment to define the relevant terms.
Suppose $\Gamma$ is a discrete abelian group and $P$ is a subset of $\Gamma$ with the following properties:

1. If $\gamma_1 \in P$ and $\gamma_2 \in P$ then $\gamma_1 + \gamma_2 \in P$.
2. $P \cap (-P) = \{0\}$
3. $P \cup (-P) = \Gamma$.

Under these conditions $P$ induces an order in $\Gamma$ as follows: For $\gamma_1$ and $\gamma_2$ elements of $\Gamma$, say $\gamma_1 \geq \gamma_2$ if $\gamma_1 - \gamma_2 \in P$. It is easy to check that this is a linear order. A given group may have many different orders corresponding to different choices of $P$ with the three properties above.

**Definition 2.1.** Suppose $G$ is a compact abelian group whose dual group $\Gamma$ is ordered. Let $f$ be a trigonometric polynomial on $G$ with Fourier series

$$f(g) \sim \sum_{\gamma \in \Gamma} a_{\gamma}(g, \gamma).$$

Define $\Phi(f)$ by

$$\Phi(f)(g) \sim \sum_{\gamma \geq 0} a_{\gamma}(g, \gamma).$$

We shall need the following generalization of the M. Riesz theorem on conjugate functions. It is due to Bochner [1].

**Theorem 2.2.** Suppose $1 < p < \infty$. Then there exists a constant $A_p$, independent of $G$ or the particular order in $\Gamma$ such that if $f$ is a trigonometric polynomial on $G$, then

$$\|\Phi(f)\|_p \leq A_p \|f\|_p.$$

**Theorem 2.3.** Let $1 < p < \infty$. Then if $f \in H^p(T^\omega)$

$$\lim_{n \to \infty} \sum_{\theta = 0}^{n} P_n(f) = f$$

in the norm of $H^p(T^\omega)$.

**Proof.** Fix $p$. Define $Y_n$ by

$$Y_n(f) = \sum_{\theta = 0}^{n} P_n(f) \text{ if } f \in H^p(T^\omega).$$

Clearly trigonometric polynomials are dense in $H^p(T^\omega)$ and

$$\lim_{n \to \infty} Y_n(f) = f.$$
whenever $f$ is a trigonometric polynomial. It remains to show that the family $\{Y_n\}_{n=1}^\infty$ is uniformly bounded on trigonometric polynomials, i.e.

$$\| Y_n(f) \|_p \leq K \| f \|_p$$

$f$ a trigonometric polynomial where $K$ is a positive constant independent of $n$ and $f$. Then by a standard argument in functional analysis, the proof is complete. I shall show that the norm of $Y_n$ is majorized by $A_p$, where $A_p$ is the constant of Theorem 2.2.

Our first task is to induce an order in $\sum_{i=1}^\infty Z$ so that we can apply Theorem 2.2. First choose a family $\{d_i\}_{i=1}^\infty$ of real numbers which satisfies the following properties:

1. $d_i = -1, -1 < d_i < -n/(n + 1)$ for $i \neq 1$.
2. The set $\{d_i\}$ is independent in the group sense as a subset of the reals.

We define a homomorphism from $\sum_{i=1}^\infty Z$ into the reals by

$$\pi: \sum_{i=1}^\infty \longrightarrow R$$

$$x \longmapsto \sum_{i=1}^\infty d_i x_i .$$

$\pi$ is clearly a homomorphism; since the $d_i$ are linearly independent, it has a trivial kernel, i.e. if $\pi(x) = 0$ then $x = 0$. Define

$$P = \left\{ x: x \in \sum_{i=1}^\infty Z \text{ and } \pi(x) \geq 0 \right\} .$$

Then $P$ satisfies the necessary properties to induce an order in $\sum_{i=1}^\infty Z$.

If $f(g)$ is an arbitrary trigonometric polynomial on $T^w$ define a trigonometric polynomial $f_i(g)$ as follows:

$$f_i(g) = z_i^{-n}(g) f(g) .$$

Let $f(g) = \Sigma a_x(g, x)$. Then

$$f_i(g) = z_i^{-n}(g) f(g) = \Sigma a_x(g, -nz_i)(g, x) = \Sigma a_x(g, x - nz_i)$$

and

$$\phi(f_i) = \sum_{\pi(x - nz_i) \geq 0} a_x(g, x - nz_i) .$$

If $\pi(x - nz_i) \geq 0$, then

$$0 \leq \pi(x - nz_i) = \pi(x) + \pi(-nz_i) = \pi(x) - n\pi(z_i) = \pi(x) + n$$

and $\pi(x) \geq -n$. But $\pi(x) = \Sigma d_i x_i$, and by using property 1 of $\{d_i\}$ it is clear that $\pi(x) \leq n$ if and only if $\Sigma x_i \leq n$. So $\phi(f_i) = \Sigma a_x(g, x - nz_i)$. 

Then it is easy to compute that \( \sum x_i \leq n \)

\[
\sum_{i=1}^{n} P_i(f) = Y_n(f).
\]

By Theorem 2.2 we have that

\[
\| \Phi(f_i) \|_p \leq A_p \| f_i \|_p.
\]

So we have

\[
\| Y_n(f) \|_p = \| \sum_{i=1}^{n} \Phi(f_i) \|_p = \| \Phi f_i \|_p \leq A_p \| f_i \|_p
\]

\[
= A_p \| z^{-n} f \|_p = A_p \| f \|_p,
\]

so the norm of \( Y_n \) is less than or equal to \( A_p \) and the proof is complete.

### 3. The complementation problem.

The next theorem shows that \( H^p(T^\omega) \) is uncomplemented as a subspace of \( L^p(T^\omega) \) if \( p \neq 2 \). This is in contrast to \( H^p(T^n) \) which is complemented in \( L^p(T^n) \) except when \( p = 1 \) or \( p = \infty \). Although other examples of uncomplemented subspaces of an \( L^p \) space are known, \( H^p(T^\omega) \) has the advantage of being defined in a concrete way.

**Definition 3.1.** Let \( G \) be a compact abelian group. If \( f \in L^i(G) \) let \( f_{g_0} \) denote the \( g_0 \)-translate of \( f \) where

\[
f_{g_0}(g) = f(g_0 + g).
\]

**Lemma 3.2.** Let \( G \) be a compact abelian group with dual group \( \Gamma \). Suppose \( 1 \leq p < \infty \) and that \( T \) is a bounded projection from \( L^p(G) \) onto \( L^p_{\Gamma}(G) \). Then a linear operator \( Q \) can be defined by

\[
Q(f) = \int_G [T(f_g)]_g d\mu(g) \quad f \in L^p(G),
\]

where the integral is the Bochner integral.

\( Q \) is the natural projection from \( L^p(G) \) onto \( L^p_{\Gamma}(G) \), i.e., if \( f \in L^p(G) \) then \( Q(f) \) is defined by its Fourier transform as follows:

\[
G(f)(x) = \begin{cases} \hat{f}(x) & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}.
\]

**Proof.** The proof for the case \( G = T, \Gamma = Z, E = Z^1, p = 1 \) is given [4, page 154]. The proof in the general case is analogous.

**Theorem 3.3.** Suppose \( p \neq 2 \), then \( H^p(T^\omega) \) is uncomplemented as subspace of \( L^p(T^\omega) \).
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Proof. If $p = 1$ or $p = \infty$, there is really nothing to prove. There is a theorem in [4, pp. 154-155] which proves that $H^i(T)$ is uncomplemented in $L^i(T)$, and that $H^\infty(T)$ is uncomplemented in $L^\infty(T)$. Then since $H^i(T)$ and $L^i(T)$ can be isometrically embedded into $H^i(T')$ and $L^i(T')$ respectively for $i = 1, \infty$, the theorem is proved for $p = 1$ or $p = \infty$. In any case the argument which follows is valid for $p = 1$, and with slight modifications for $p = \infty$.

Let $S$ be the natural projection from $L^p(T')$ into $H^p(T')$ which is defined on trigonometric polynomials by

$$S: L^p(T') \longrightarrow H^p(T')$$

$$f \longmapsto S(f)$$

where

$$S(\hat{f})(x) = \begin{cases} \hat{f}(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

We wish to show that $S$ can’t be extended to a bounded operator defined on all of $L^p(T')$. To do this it is sufficient to find trigonometric polynomials $f_n$ on $T'$ such that

(5) $\|f_n\|_p = 1$

(6) $\|S(f_n)\|_p = (1 + \epsilon)^n$ where $\epsilon > 0$.

By [8, p. 295, Ex. 2] we can find a trigonometric polynomial $h$ defined on $T$ so that

$$h(z_1) = \sum_{k=-n}^n a_k z_1^k \quad \|h\|_p = 1$$

and if

$$h_+(z_1) = \sum_{k=0}^n a_k z_1^k$$

then we have

$$\|h_+\|_p = 1 + \epsilon$$

where $\epsilon$ is some positive number which depends upon $p$. Consider the trigonometric polynomial $r$ defined on $T'$ by

$$r(z_1, z_2) = h(z_1)h(z_2) = \left( \sum_{k=-n}^n a_k z_1^k \right) \left( \sum_{k=-n}^n a_k z_2^k \right).$$

Define $r_+$ by

$$r_+(z_1, z_2) = h_+(z_1)h_+(z_2) = \left( \sum_{k=0}^n a_k z_1^k \right) \left( \sum_{k=0}^n a_k z_2^k \right).$$
Then it is easy to compute that
\[ ||r'||_p = ||h||_p^p = 1 \]
\[ ||r_+||_p = (||h_+||_p)^p = (1 + \varepsilon)^2. \]
We define trigonometric polynomials on \( T^ω \) by
\[ f_1 = I_1(h) \quad f_2 = I_2(r) \]
where \( I_1 \) and \( I_2 \) were defined in equation (1). It is easy to check that
\[ S(f_1) = I_1(h_+) \quad S(f_2) = I_2(r_+) \]
and since \( I_1 \) and \( I_2 \) are isometries we have
\[ \begin{align*}
||f_1||_p &= ||I_1(h)||_p = ||h||_p = 1 \\
||S(f_1)||_p &= ||I_1(h_+)||_p = ||h_+||_p = 1 + \varepsilon \\
||f_2||_p &= ||I_2(r)||_p = ||r||_p = 1 \\
||S(f_2)||_p &= ||I_2(r_+)||_p = ||r_+||_p = (1 + \varepsilon)^2. 
\end{align*} \]
By a similar argument we can construct trigonometric polynomials \( f_3, f_4, \cdots \) and hence \( f_n \) for any \( n \) and \( f_n \) will satisfy equations (5) and (6). This shows that the natural projection from \( L^p(T^ω) \) into \( H^p(T^ω) \) isn’t bounded. To finish the proof we must show there is no bounded projection of any kind from \( L^p(T^ω) \) into \( H^p(T^ω) \) which is the identity when restricted to \( H^p(T^ω) \).
Suppose there exists \( \tilde{S} \) a linear transformation from \( L^p(T^ω) \) into \( H^p(T^ω) \) which is the identity when restricted to \( H^p(T^ω) \). Define a linear operator \( Q \) by
\[ Q(f) = \int_{T^ω} [\tilde{S}(f)]_{-} dm(g) \]
where the integral is the Bochner integral. Then \( Q \) is a bounded linear operator from \( L^p(T^ω) \) into \( H^p(T^ω) \) and by Lemma 3.2 we have that \( Q = S \), where \( S \) is the natural projection from \( L^p(T^ω) \) into \( H^p(T^ω) \). But we know that \( S \) isn’t a bounded projection and this provides the contradiction which finishes the proof.

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References


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