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DAVID FREMLIN

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D. H. FREMLIN

Let E be a Riesz space ($=$ vector lattice), with a locally solid Hausdorff linear space topology. Then its completion also has a Riesz space structure. In this paper it is shown how a pair of important properties which may be possessed by E are inherited by its completion.

In general this article will rest on the foundations of [4] and [5]. A linear space topology on a Riesz space E is *locally solid* if 0 has a neighbourhood basis consisting of solid sets. In this case, the lattice operations are uniformly continuous; consequently (assuming that the topology is Hausdorff) they can be extended to the linear topological space completion $\hat{E}$ of E, and $\hat{E}$ will also be a locally solid topological Riesz space ([5, p. 235; 4, p. 108]). E is now a *Riesz subspace* of $\hat{E}$, i.e. a linear subspace which is also a sublattice.

My object is to show how two important and common properties are preserved by the process of completion. Unfortunately, although these properties have been studied by various authors (see e.g. [3]), no satisfactory terminology has been devised. I hope that my use of the words "Fatou" (§1) and "Lebesgue" (§5), suggested by the famous convergence theorems, will prove acceptable.

1. Fatou topologies.

Let E be a Riesz space and $\mathfrak{T}$ a topology on E. I will call $\mathfrak{T}$ *Fatou* if (i) it is a linear space topology (ii) 0 has a base consisting of sets U which are solid and such that if $\varnothing \subset A \subseteqq U$ and $A \uparrow x$ in E (i.e. if A is nonempty, directed upwards, and has x for its least upper bound), then $x \in U$.

This property is exceedingly common. Consider, for example, $C(X)$ for any compact space X; the basic neighbourhoods of 0 are of the form $\{x: \|x\|_\infty \leqq \varepsilon\}$, and these all have the property described above. Similarly, in all the L^p spaces, for $0 \leqq p \leqq \infty$, the usual topologies are Fatou.

The most striking thing about Fatou topologies is Nakano's theorem (see [2]). For its full strength this requires a further concept. Let us call a linear space topology on a Riesz space E a *Levi* topology if every topologically bounded set $A \subseteqq E$ which is directed upwards has an upper bound in E. (For example, all the spaces adduced above have Levi topologies. Also, the weak topology associated with a locally convex Hausdorff Levi topology will always be Levi). Then: A Levi Fatou Hausdorff topology on a Dedekind complete Riesz space is complete. For a proof of this theorem, see [4], Proposition IV. 1.5. ([4] uses the phrases "locally order complete" and "boundedly complete" for Fatou and Levi topologies respectively in Dedekind complete spaces).

2. **Extensions of Riesz spaces; the spaces $C_\infty(X)$.** Let E be a Riesz space. I shall call a Riesz subspace F of E *orderdense* if, for every $x \geqq 0$ in E,

$$x = \sup \{y: y \in F, 0 \leqq y \leqq x\} .$$

An important consequence of this is that if A is a nonempty subset of F and $x = \sup A$ in F, that is, if x is the least member of F which is an upper bound of A, then $x = \sup A$ in E. It follows that if F is orderdense in E, and G is orderdense in F, then G is orderdense in E.

Let X be a compact extremally disconnected Hausdorff topological space. Let $C_\infty(X)$ be the set of all those continuous functions x from X to the extended real line $[-\infty, \infty]$ such that $\{t: -\infty < x(t) < \infty\}$ is dense in X. Because every continuous real-valued function defined on a dense open subset of X has a unique extension to a member of $C_\infty(X)$ ([6, Lemma V. 2.1]), $C_\infty(X)$ has a natural Riesz space structure under which it is Dedekind complete ([6, Theorem V. 2.2]). The point is that every Archimedean Riesz space can be embedded as an orderdense Riesz subspace of some $C_\infty(X)$ ([6, Theorems IV. 11.1 and V. 4.2]).

[6] gives several properties of the space $C_\infty(X)$, but not the one we shall need; so I set it out here.

PROPOSITION 1. *Let X and $C_\infty(X)$ be as above. Let $A \subseteqq C_\infty(X)^+$ be a nonempty set such that for every $x > 0$ in $C_\infty(X)$ there is an $n \in N$ such that*

$$nx \neq \sup_{y \in A} y \wedge nx .$$

Then A is bounded above in $C_\infty(X)$.

Proof. Define $w : X \rightarrow [0, \infty]$ by

$$w(t) = \sup_{y \in A} y(t) \ \forall\, t \in X .$$

Then w is lower semi-continuous. Define $v : X \rightarrow [0, \infty]$ by

$$v(t) = \inf \left\{ \sup_{u \in U} w(u) : U \text{ a } nhd \text{ of } t \right\}$$

for every $t \in X$. Then v is continuous ([6, Theorem V. 1.1]). My aim is to prove that $v \in C_\infty(X)$, i.e. that v is finite on a dense set.

Suppose that $G \subseteqq X$ is open and not empty. As X is compact and Hausdorff, there is a continuous function x on X such that $x > 0$ but $x(t) = 0 \; \forall \; t \in X \backslash G$. Now $x \in C_\infty(X)$, so there is an $n \in N$ such that

$$nx \neq \sup_{y \in A} y \wedge nx \, ,$$

that is, there is a $z > 0$ in $C_\infty(X)$ such that

$$y \wedge nx \leqq nx - z \, \forall \, y \in A \, .$$

Of course $z \leqq nx$, so z is finite everywhere and $z(t) = 0 \; \forall \; t \in X \backslash G$. Let $H = \{t: z(t) > 0\}$; then H is not empty and $H \subseteqq G$.

But if $t \in H$, $y(t) \leqq nx(t) - z(t) \; \forall \; y \in A$, so $w(t) \leqq nx(t) - z(t)$; and as $nx - z$ is continuous, $v(t) \leqq nx(t) - z(t) < \infty \; \forall \; t \in H$.

Consequently, $\{t: v(t) < \infty\}$ meets G. As G is arbitrary, $v \in C_\infty(X)$ and is the required upper bound for A.

3. THEOREM 1. *Let E be an Archimedean Riesz space with a Hausdorff Fatou topology. Let $\hat{E}$ be its linear topological space completion with its natural Riesz space structure. Then* (i) *E is an orderdense Riesz subspace of $\hat{E}$* (ii) *the topology on $\hat{E}$ is Fatou.*

Proof. My method is to find a complete Riesz space extending E which has the required properties.

(a) Let X be a compact extremally disconnected Hausdorff topological space such that E can be embedded as an orderdense Riesz subspace of $C_\infty(X)$ (§2 above). Let $\mathscr{B}$ be the set of all neighbourhoods U of 0 in E satisfying the Fatou property in §1, i.e. such that U is solid and if $\varnothing \subset A \subseteqq U$ and $A \uparrow x$ in E then $x \in U$. Then $\mathscr{B}$ is a base of neighbourhoods of 0. For each $U \in \mathscr{B}$, set

$$\tilde{U} = \{w: w \in C_\infty(X), \forall \, x \in E, |x| \leqq |w| \Rightarrow x \in U\} \, .$$

Then $\tilde{U}$ is a solid subset of $C_\infty(X)$. Note that $\tilde{U} \cap E = U$.

(b) Suppose that U and V belong to $\mathscr{B}$ and that $U + U \subseteqq V$. Then $\tilde{U} + \tilde{U} \subseteqq \tilde{V}$. For suppose that $w_1, w_2 \in \tilde{U}$ and that $x \in E$ is such that $|x| \leqq |w_1 + w_2|$. Set $v_1 = |w_1| \wedge |x|$ and $v_2 = |x| - v_1 \leqq |w_2|$. Then $A_i = \{y: y \in E, 0 \leqq y \leqq v_i\} \uparrow v_i$ for $i = 1, 2$, so $A_1 + A_2 \uparrow v_1 + v_2 = |x|$ in E. But $A_1 + A_2 \subseteqq U + U \subseteqq V$, so $|x| \in V$ and $x \in V$. As x is arbitrary, $w_1 + w_2 \in \tilde{V}$; as w_1 and w_2 are arbitrary, $\tilde{U} + \tilde{U} \subseteqq \tilde{V}$.

(c) It follows that if we set

$$H = \bigcap_{U \in \mathscr{B}} \bigcup_{\alpha \in \mathscr{R}} \alpha \tilde{U} ,$$

then H is a solid linear subspace of $C_\infty(X)$, including E, and $\{\tilde{U} \cap H: U \in \mathscr{B}\}$ is a neighbourhood basis at 0 for a linear space topology $\mathfrak{T}$ on H. As every $\tilde{U} \cap H$ is solid, $\mathfrak{T}$ is locally solid; as $\tilde{U} \cap E = U$ for every $U \in \mathscr{B}$, $\mathfrak{T}$ induces the original topology on E. Also, $\mathfrak{T}$ is Hausdorff, for if $w \in H$ and $w \neq 0$, there is an $x \in E$ such that $0 < x \leqq |w|$; now if $U \in \mathscr{B}$ is such that $x \notin U$, $w \notin \tilde{U}$.

(d) If $U \in \mathscr{B}$, $\varnothing \subset A \subseteqq \tilde{U}$, and $A \uparrow w$ in $C_\infty(X)$, then $w \in \tilde{U}$. For suppose that $x \in E$ and that $|x| \leqq |w|$. Then

$$\{y^+ + w^-: y \in A\} \uparrow w^+ + w^- = |w| \geqq |x| ,$$

so

$$\{|x| \wedge (y^+ + w^-): y \in A\} \uparrow |x| .$$

Now set

$$B = \{z: z \in E, \exists\, y \in A, 0 \leqq z \leqq |x| \wedge (y^+ + w^-)\} .$$

Then $B \uparrow$, and as E is orderdense in $C_\infty(X)$, $B \uparrow |x|$. But if $z \in B$ there is a $y \in A$ such that

$$z \leqq y^+ + w^- \leqq y^+ + y^- = |y| ,$$

so, as $y \in \tilde{U}$, $z \in U$. Because $U \in \mathscr{B}$, $x \in U$. As x is arbitrary, $w \in \tilde{U}$.

(e) Consequently the sets $\tilde{U} \cap H$ all satisfy the Fatou condition, and $\mathfrak{T}$ is Fatou. (Here we have used the fact that H is orderdense in $C_\infty(X)$, so that if $A \uparrow w$ in H, then $A \uparrow w$ in $C_\infty(X)$).

(f) It also follows that $\mathfrak{T}$ is Levi. For suppose that $A \subseteqq H$ is directed upwards, is not empty, and is bounded. Then of course $B = \{y^+: y \in A\}$ is directed upwards, and it is bounded because $\mathfrak{T}$ is locally solid. Now suppose that $x > 0$ in $C_\infty(X)$. Let $U \in \mathscr{B}$ be such that $x \notin \tilde{U}$. Let $n > 0$ be such that $A \subseteqq nU$. Now

$$\{n^{-1}y \wedge x: y \in B\}$$

is a subset of $\tilde{U}$, directed upwards; so its supremum belongs to $\tilde{U}$ and cannot be x. Thus $\sup_{y \in B} y \wedge nx$ is not nx, and B satisfies the condition of Proposition 1; so B, and therefore A, is bounded above in $C_\infty(X)$. Let $z_0 = \sup A$ in $C_\infty(X)$; this exists as $C_\infty(X)$ is Dedekind complete. If $V \in \mathscr{B}$, there is an $m > 0$ such that $m^{-1}A \subseteqq \tilde{V}$, so by (d) again $m^{-1}z_0 \in \tilde{V}$ i.e. $z_0 \in m\tilde{V}$. As V is arbitrary, $z_0 \in H$, and is the required upper bound for A in H.

(g) Thus $\mathfrak{T}$ satisfies the conditions of Nakano's theorem, and H is complete. So $\hat{E}$ may be regarded as the closure of E in H. Because E is orderdense in H, it is orderdense in $\hat{E}$. Finally, it is easy to see that the topology on $\hat{E}$ induced by $\mathfrak{T}$ is Fatou, because $\mathfrak{T}$ itself is Fatou and $\hat{E}$ is orderdense in H.

REMARK. Of course the condition "Archimedean" in the hypotheses of the theorem is redundant, because any Riesz space with a Hausdorff locally solid linear space topology must be Archimedean. The same applies to Theorem 2 below.

4. Counter-example. Suppose that $E = C([0, 1])$, the space of real-valued continuous functions on the unit interval. Give E the topology induced by $\| \|_1$ where

$$\|x\|_1 = \int |x| d\mu_L \; \forall \; x \in E ,$$

μ_L being Lebesgue measure. Then $\| \|_1$ is a Riesz norm so the topology is locally solid. But it is not Fatou and E is not orderdense in its completion $L^1(\mu_L)$.

5. Lebesgue topologies. I should now like to proceed to a stronger condition, also fulfilled by many examples. Because it is of great interest in many contexts, I give as general a definition as I can. Let E be any partially ordered set. A topology $\mathfrak{T}$ on E is *Lebesgue* if, whenever A is a non-empty subset of E and either $A \uparrow x$ or $A \downarrow x$ in E, then x belongs to the closure $\bar{A}$ of A. We shall be interested, of course, in linear space topologies on Riesz spaces; in this case, $\mathfrak{T}$ is Lebesgue iff $0 \in \bar{A}$ whenever $\emptyset \subset A \downarrow 0$.

Now the ordinary topologies on the L^p spaces, for $0 \leqq p < \infty$, are Lebesgue; so is the norm topology on $c_0(N)$. We note that the exceptions are the L^∞ and $C(X)$ spaces. However, the weak topology $\mathfrak{T}_s(L^\infty, L^1)$ is Lebesgue; in fact it is the case that the Mackey topology $\mathfrak{T}_k(L^\infty, L^1)$ is Lebesgue. Of course, if $\mathfrak{T}$ is Lebesgue and $\mathfrak{S}$ is weaker than $\mathfrak{T}$, then $\mathfrak{S}$ is Lebesgue.

Lebesgue topologies have many remarkable properties. I give one of the simplest.

LEMMA 1. *A Lebesgue locally solid linear space topology on a Riesz space is Fatou.*

Proof. Let U be any neighbourhood of 0; let V be a closed neighbourhood of 0 included in U; let W be a solid neighbourhood of 0 included in V. The point is that $\bar{W}$ is solid ([4, Proposition IV.

4.8]). But now $\bar{W} \subseteqq U$ and $\bar{W}$ satisfies the Fatou condition because the topology is Lebesgue.

6. THEOREM 2. *Let E be an Archimedean Riesz space with a Lebesgue locally solid Hausdorff linear space topology. Then the completion $\hat{E}$ of E also has a Lebesgue topology.*

Proof. We know by Lemma 1 and Theorem 1 that E is order-dense in $\hat{E}$. Suppose, if possible, that $A \downarrow 0$ in $\hat{E}$, A is not empty, but that $0 \notin \bar{A}$. Let U be a solid neighbourhood of 0 in $\hat{E}$ such that A does not meet U. Let V be a solid neighbourhood of 0 in $\hat{E}$ such that $V + V + V \subseteqq U$. Fix $x_0 \in A$ and find a $y_0 \in E$ such that $x_0 - y_0 \in V$; without loss of generality, I may suppose that $y_0 \geqq 0$. Now

$$\{y_0 \wedge (x_0 - x)^+; x \in A\} \uparrow y_0 \wedge x_0 ,$$

so if

$$B = \{z: z \in E, \exists\, x \in A, 0 \leqq z \leqq y_0 \wedge (x_0 - x)^+\} ,$$

$B \uparrow x_0 \wedge y_0$ in $\hat{E}$. Similarly,

$$C = \{w: w \in E, 0 \leqq w \leqq (y_0 - x_0)^+\} \uparrow (y_0 - x_0)^+ ,$$

and so $B + C \uparrow y_0$ in E. As the topology on E is Lebesgue, there exist $z \in B$ and $w \in C$ such that

$$y_0 - w - z \in V .$$

But as V is solid, $w \in V$, so $y_0 - z \in V + V$, and

$$x_0 - z = y_0 - z + (x_0 - y_0) \in V + V + V \subseteqq U .$$

However, there is an $x \in A$ such that $0 \leqq z \leqq (x_0 - x)^+$, and there is an $x_1 \in A$ such that $x_1 \leqq x \wedge x_0 \leqq x_0 - z$. But U is solid, so $x_1 \in U$; which is the contradiction we require.

7. **Conclusion.** I think that Theorem 1 is more surprising than Theorem 2. Both Fatou and Lebesgue topologies are frequently mysterious; but when we require a topology to be both locally solid and Lebesgue we are imposing such a powerful condition that we expect agreeable results to follow quickly. The Fatou property is harder to tackle. Its actual applications in Theorem 1, while certainly essential (see § 4), are buried too deep in the argument to be readily disentangled; so it's not clear just what it is about Fatou topologies that makes the theorem true.

Theorem 1 is reminiscent of the result in [1] that if E is any Riesz space, then the canonical image of E in $E^{\times\times}$ or $(E_n^{\sim})_n^{\sim}$ is orderdense.

In fact this can be deduced from Theorem 1, though (as far as I know) only by an extremely involved route. But there may be some hope that the techniques of [1] could be adapted to give a simpler proof of Theorem 1.

Theorem 2 is more straightforward, and can be proved independently of Theorem 1 without much difficulty. If in Theorem 2 we know that E is locally convex, there is a proof direct from the result in [1] quoted above. But the hypothesis of local convexity doesn't seem to help in Theorem 1.

Theorem 2 recalls the construction of the ordinary function spaces. If the spaces L^1, L^2 etc. are thought of as completions of the space S of equivalence classes of simple functions under the appropriate norms, their properties can be deduced from the fact that each of these norms induces a Lebesgue locally solid topology on S.

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UNIVERSITY OF ESSEX
COLCHESTER, ENGLAND

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