

Pacific Journal of Mathematics

THE DISAPPEARING CLOSED SET PROPERTY

V. M. KLASSEN

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A topological space X is said to have the disappearing closed set (DCS) property or to be a DCS space, if for every proper closed subset C there is a family of open sets $\{U_i\}_{i=1}^{\infty}$ such that $U_{i+1} \subseteq U_i$ and $\bigcap_{i=1}^{\infty} U_i = \emptyset$, and there is also a sequence $\{h_i\}$ of homeomorphisms on X onto X such that $h_i(C) \subseteq U_i$, for all i . Properties of DCS spaces are studied as are connections between this and other related definitions.

I. Simple examples of sets with the DCS property are the n -sphere, $n > 0$, and the open n -cell, $n > 0$. This definition was formulated in an attempt to generalize the definition of invertible set which has been extensively studied by Doyle, Hocking and others [1, 2, 3, 4, 6]. A space X is said to be invertible if for every proper closed subset C of X there is a homeomorphism h on X onto X such that $h(C) \subseteq X - C$. Neither of these definitions implies the other. For example, an open n -cell is not invertible, and on the other hand, the 0-sphere is invertible but does not satisfy the DCS property. However, both definitions require that closed sets can be made "small" or "thin."

It is proved in [5] that compact n -manifolds have the DCS property. It is the purpose of this paper to investigate some other topological properties of DCS spaces.

II. THEOREM 1. *Any disconnected DCS space X must have an infinite number of components.*

Proof. Suppose X has a finite number of components, A_j , $j = 1, \dots, n$. Each A_j is both open and closed. Consider the DCS property applied to $\bigcup_{j=2}^n A_j = B$, a closed set. There are open sets $\{U_i\}_{i=1}^{\infty}$ and homeomorphisms $\{h_i\}_{i=1}^{\infty}$ such that $h_i(B) \subseteq U_i$, $U_{i+1} \subseteq U_i$, and $\bigcap_{i=1}^{\infty} U_i = \emptyset$. Since there are at most a finite number of components A_i and since the U_i form a decreasing sequence of open sets whose intersection is empty, there must be an m such that for each $j = 1, \dots, n$, there are $x_j \in A_j$ such that $x_j \notin U_m$. But $X - U_m \subseteq h_m(A_1)$, since $h_m(B) \subseteq U_m$ and $X = A_1 \cup B$, $A_1 \cap B = \emptyset$. Thus $x_j \in h_m(A_1)$, $j = 1, \dots, n$. But this is a contradiction unless $n = 1$, since $h_m(A_1)$ is connected, but intersects all components of X .

An example of a DCS space which is not connected is the product space obtained by crossing the real numbers with the rationals.

One method of constructing DCS spaces is given by the following:

THEOREM 2. *If X and Y are DCS spaces, so is $X \times Y$.*

Proof. Let C be a proper closed subset of $X \times Y$, and let $P \subseteq X$, $Q \subseteq Y$ be open sets in X and Y , respectively, such that $P \times Q \subseteq X \times Y - C$. Let $\{U_i\}_{i=1}^\infty$, $\{h_i\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$, $\{k_i\}_{i=1}^\infty$ be the open sets and homeomorphisms for $X - P$ and $Y - Q$ in X and Y , respectively. If $(x, y) \in X \times Y$, define $\phi_i(x, y) = \{h_i(x), k_i(y)\}$. Now $\{W_i\}_{i=1}^\infty = \{(U_i \times Y) \cup (X \times V_i)\}_{i=1}^\infty$ is a decreasing sequence of open sets in $X \times Y$, with empty intersection. Also, $\phi_i(C) \subseteq W_i$. Thus, $X \times Y$ has the DCS property.

The relation between invertible spaces and spaces with the DCS property can be seen more clearly in the following analysis.

If an invertible T_1 space X has the property that the intersection of all neighborhoods of any point is that point, and if any closed set C in an open set U may be "moved" so as to miss any given $x \in U$, without moving outside U , then X has the DCS property. (If U is open, $U - \{x\}$ is also.)

III. This suggests a relationship with another concept, also studied by Doyle and Hocking. A space X is near-homogeneous if for any $x \in X$ and any open set U such that $x \in U$, for every $y \in X$ there is a homeomorphism on X onto X such that $h(y) \in U$.

Once again, the 0-sphere is a space that does not satisfy the DCS property, but is near-homogeneous. However, the following converse is true:

THEOREM 3. *Every DCS space X is near-homogeneous.*

Proof. Let $x \in X$ and U an open set containing x . Let $y \in X$. Consider $C = X - U$, a proper closed subset of X . Since X has the DCS property, there is a sequence of homeomorphisms $\{h_i\}_{i=1}^\infty$ on X onto X such that $\bigcap_{i=1}^\infty h_i(C) = \emptyset$, a somewhat weaker statement than the DCS property allows. There is some j such that $y \notin h_j(C)$. But then $y \in h_j(U)$, so $h_j^{-1}(y) \in U$. Thus, X is near-homogeneous.

In the preceding proof, it is seen that near-homogeneity does not require that closed sets get "thin," but that they move around enough. An equivalent form of the definition of near-homogeneity, related to the DCS property, is of interest here.

THEOREM 4. *Let $H(X)$ be the family of all homeomorphisms on X onto X . X is near-homogeneous iff, for every proper closed set $C \subseteq X$, $\bigcap_{h \in H(X)} h(C) = \emptyset$.*

Proof. If X is near-homogeneous, let C be a closed subset of X ,

and let $U = X - C$. Let $y \in C$. Then there is an $h \in H(X)$ such that $h(y) \in U$, by near-homogeneity and thus $\bigcap_{h \in H(X)} h(C) = \emptyset$.

Conversely, let $x, y \in X$, and let U be an open set such that $x \in U$. Let $C = X - U$. If $y \notin C$, there is nothing to show, so suppose $y \in C$. Then there is an $h \in H(X)$ such that $h(y) \notin C$. Otherwise $\bigcap_{h \in H(X)} h(C)$ would not be empty. But this is the desired homeomorphism.

IV. Another definition relating to invertibility that has been studied is that of local invertibility. A space X is said to be invertible at a point $x \in X$ if for every open set U containing x there is a homeomorphism h on X onto X such that $h(X - U) \subseteq U$. In [2] it was proved that for such a space certain local properties become global properties. For example, if X is invertible and locally compact at x , then X is compact. The corresponding definition here is the following. A space X has the DCS/x property for all closed sets which miss x . It is evident that a space X has the DCS property, iff it has the DCS/x property for each $x \in X$. Examples of spaces with the DCS/x property include the closed n -cell, the n -leafed rose and, in fact any space that is invertible at x in such a way that the inverting homeomorphism may be taken to fix x . A space that is not invertible at any point but which does have the DCS/x property is the "half-open" annulus $[0, 1) \times S_1$. It will have the DCS/x property for every point of $\{0\} \times S_1$.

Since the DCS/x definition cannot guarantee that any part of the closed set will be carried close to x under any of the homeomorphisms, theorems as sweeping as those of local invertibility cannot be obtained. However, the following is true:

THEOREM 5. *Let X be a space that has the DCS/x property at x and suppose X is locally T_i , $i = 0, 1, 2$, in a neighborhood P of x . Then X is T_i .*

Proof. Let $y, z \in X$, $y \neq z$ (perhaps one is x). Let $\{U_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ be the open sets and homeomorphisms given by the DCS/x property for the closed set $X - P$. There is a j such that $y, z \in U_j$. Then $y, z \in h_j(X - P)$, so $y, z \in h_j(P)$. But then $h_j(y)$ and $h_j(z)$ have the separation property required and thus y and z do also.

Note that this kind of argument is an improvement on near-homogeneity, since it makes it possible to bring two points (or any finite number of points) into a neighborhood of x at once.

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Received August 11, 1971.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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