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MULTIPLIERS OF TYPE (p, p)

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It will be shown in this paper that the Banach algebra of all continuous multipliers on $L_p(G)$ (G a locally compact group, $p \in [0, \infty[$) may be viewed as the set of all multipliers on a natural Banach algebra with minimal approximate left identity.

Let G be an arbitrary locally compact group, λ its left Haar measure, and p a number in $[1, \infty[$. Write \mathfrak{B}_p for the Banach algebra of all bounded linear operators on L_p and write \mathfrak{M}_p for the subset of \mathfrak{B}_p consisting of those operators which commute with all left translation operators; elements of \mathfrak{M}_p are called *multipliers of type (p, p)* . If A is a Banach algebra, then a bounded linear operator T on A such that $T(a \cdot b) = T(a) \cdot b$ for all $a, b \in A$ is called a *multiplier on A* ; write $\mathfrak{m}(A)$ for the set of all such. By C_0 will be meant the set of all continuous complex-valued functions on G which have compact support. A function f in L_p such that for each g in L_p , the function $g * f(x) = \int g(t)f(t^{-1}x)d\lambda(t)$ exists λ -almost everywhere, $g * f$ is in L_p , and $\|g * f\|_p \leq \|g\|_p \cdot k$ where k is a positive number independent of g , is said to be *p -tempered*; write L_p^t for the set of all such. Evidently L_p^t is closed under convolution and C_0 is a subset of L_p^t . Thus, for each f in L_p^t and h in C_0 , there is precisely one operator W in \mathfrak{B}_p such that $W(g) = g * f * h$ for all g in L_p ; write \mathfrak{A}_p for the norm closure in \mathfrak{B}_p of the linear span of all such W . The principal result of this paper is that \mathfrak{A}_p is a Banach algebra with minimal approximate left identity and that $\mathfrak{m}(\mathfrak{A}_p)$ and \mathfrak{M}_p are isomorphic isometric Banach algebras.

THEOREM 1. *Let f be a function in L_p and k a positive number such that $\|g * f\|_p \leq \|g\|_p \cdot k$ for all g in C_0 . Then f is in L_p^t .*

Proof. First of all, suppose that h is a function in $L_1 \cap L_p$. As is well known, $h * f$ is in L_p and $\|h * f\|_p \leq \|h\|_1 \cdot \|f\|_p$. Let $\{h_n\}$ be a sequence in C_0 which converges to h in the L_p and L_1 norms both. It follows from the above that $\{h_n * f\}$ converges to $h * f$ in L_p . This fact and the hypothesis for f imply

$$\|h * f\|_p = \lim_n \|h_n * f\|_p \leq \overline{\lim}_n \|h_n\|_p \cdot k = \|h\|_p \cdot k.$$

Let h be now an arbitrary function from L_p . We may assume that h vanishes off some σ -finite set A . Let $\{A_n\}$ be an increasing nest of λ -finite and λ -measurable subsets of G such that their union

is A . Let for each $n \in N$, h_n be the product of h with the characteristic function of A_n . Let π_j ($j = 0, 1, 2, 3$) be the minimal non-negative functions on the complex field K such that $z = \sum_{j=0}^3 i^j \pi_j(z)$ for each $z \in K$.

Fix j in $\{0, 1, 2, 3\}$. For each $x \in G$, define the measurable function w^x in $[0, \infty]^G$ by letting $w^x(t) = \pi_j[h(t) \cdot f(t^{-1}x)]$ for all $t \in G$. For each $x \in G$ and $n \in N$, define the measurable function w_n^x in $[0, \infty]^G$ by letting $w_n^x(t) = \pi_j[h_n(t) \cdot f(t^{-1}x)]$ for all $t \in G$. Since the sequence $\{w_n^x\}$ converges upwards to w^x for each $x \in G$, it follows from the monotone convergence theorem that $\lim_n \int w_n^x d\lambda = \int w^x d\lambda$. Define the function F in $[0, \infty]^G$ by letting $F(x) = \int w^x d\lambda$ for all $x \in G$. For each $n \in N$, define the function F_n in $[0, \infty]^G$ by letting $F_n(x) = \int w_n^x d\lambda$ for all $x \in G$. Thus, $\{F_n\}$ converges upwards to F at each point $x \in G$.

For each $n \in N$, h_n is in $L_1 \cap L_p$; it follows that $\pi_j[h_n * f]$ is in L_p , and so equals F_n almost everywhere. Hence, each F_n is measurable whence F is measurable. Further, by the monotone convergence theorem and the inequality which concludes the initial paragraph of this proof,

$$\begin{aligned} \|F\|_p &= \lim_n \|F_n\|_p \\ &= \lim_n \|\pi_j[h_n * f]\|_p \leq \overline{\lim}_n \|h_n * f\|_p \leq \overline{\lim}_n \|h_n\|_p \cdot k = \|h\|_p \cdot k. \end{aligned}$$

Recalling that $F(x) = \int \pi_j[h(t) \cdot f(t^{-1}x)] dt$ almost everywhere and j was arbitrary, we see that $h * f$ exists almost everywhere, is in L_p and $\|h * f\|_p \leq \|h\|_p \cdot 4k$. This proves that f is p -tempered.

The condition given in Theorem 1 for a function in L_p to be in L_p^t is clearly necessary as well as sufficient. Another such condition was proved in [4], Theorem 1.3:

THEOREM 2. *Let f be a function in L_p such that $g * f$ is defined and in L_p for all g in L_p . Then f is in L_p^t .*

For each $f \in L_p^t$, there is precisely one operator $W_f \in \mathfrak{B}_p$ such that

$$(1) \quad W_f(g) = g * f$$

for all $g \in L_p$. For $f \in C_{00}$, we have as well (see [1] 20.13)

$$(2) \quad \|W_f\| \leq \int \Delta^{-(p-1)/p} |f| d\lambda.$$

It is easy to check that

$$(3) \quad W_{f^{*h}} = W_h \circ W_f$$

for all f and h in L_p^t .

THEOREM 3. *The set \mathfrak{A}_p is a complete subalgebra of \mathfrak{M}_p and it possesses a minimal left approximate identity (i.e., a net $\{T_\alpha\}$ such that $\overline{\lim}_\alpha \|T_\alpha\| \leq 1$ and $\lim \|T_\alpha \circ T - T\| = 0$ for all $T \in \mathfrak{A}_p$).*

Proof. A simple calculation shows that, when f is in L_p^t , then W_f is in \mathfrak{M}_p . Evidently, \mathfrak{M}_p is a Banach algebra; hence, \mathfrak{A}_p is a subset of \mathfrak{M}_p . That \mathfrak{A}_p is a Banach space is an elementary consequence of its definition. That \mathfrak{A}_p is a Banach algebra is a consequence of the fact that $L_p^t * C_{00}$ is closed under convolution.

For each compact neighborhood E of the identity of G , let f_E be a nonnegative function in C_∞ which vanishes outside E and such that $\int f_E d\lambda = 1$. Directing the family of compact neighborhoods of the identity by letting $E > F$ when $E \subset F$, we obtain a net $\{f_E\}$ which is a minimal approximate identity for L_1 . If $\{h_\gamma\}$ denotes the product net of $\{f_E\}$ with itself, then $\{h_\gamma\}$ is again a minimal approximate identity for L_1 and the net $\{W_{h_\gamma}\}$ is in \mathfrak{A}_p . Since Δ is unity and continuous at the identity of G , we have by (2),

$$\overline{\lim}_\gamma \|W_{h_\gamma}\| \leq \overline{\lim}_\gamma \int \Delta^{-(p-1)/p} h_\gamma d\lambda \leq 1.$$

For $f \in L_p^t$ and $g \in C_{00}$, (3) and (2) imply

$$\begin{aligned} \overline{\lim}_\gamma \|W_{h_\gamma} \circ W_{f^{*g}} - W_{f^{*g}}\| &= \overline{\lim}_\gamma \|(W_{g^{*h_\gamma}} - W_g) \circ W_f\| \\ &\leq \overline{\lim}_\gamma \|W_{g^{*h_\gamma}} - W_g\| \cdot \|W_f\| \leq \left(\overline{\lim}_\gamma \int |g^{*h_\gamma} - g| \cdot \Delta^{-(p-1)/p} d\lambda \right) \cdot \|W_f\| \\ &\leq \overline{\lim}_\gamma \|g^{*h_\gamma} - g\|_1 \cdot \sup \{ \Delta^{-(p-1)/p}(x) : g^{*h_\gamma}(x) \neq g(x) \} \cdot \|W_f\| = 0 \end{aligned}$$

since $\overline{\lim}_\gamma \|g^{*h_\gamma} - g\|_1 = 0$ and since the net of sets $\{x \in G : g^{*h_\gamma}(x) \neq g(x)\}$ is eventually contained in some fixed compact set. Since $L_p^t * C_{00}$ generates a dense subset of \mathfrak{A}_p , we have $\lim \|W_{h_\gamma} \circ T - T\| = 0$ for all $T \in \mathfrak{A}_p$. Thus, $\{W_{h_\gamma}\}$ is a minimal left approximate identity for \mathfrak{A}_p .

We now turn to \mathfrak{M}_p . We shall need a theorem proved in [3] 4.2.

THEOREM 4. *Let μ and the elements of a net $\{\mu_\alpha\}$ be bounded, complex, regular Borel measures on G such that*

$$(a) \quad \lim_\alpha \|\mu_\alpha\| = \|\mu\|$$

and

$$(b) \quad \lim_{\alpha} \int f d\mu_{\alpha} = \int f d\mu \quad \text{for each } f \in C_{00} .$$

Then, for each $g \in L_p$ ($p \in [1, \infty]$), $\lim_{\alpha} \| \mu_{\alpha} * g - \mu * g \|_p = 0$.

COROLLARY. For each multiplier T in \mathfrak{M}_p and each bounded, complex, regular Borel measure μ , we have

$$(i) \quad T(\mu * g) = \mu * T(g)$$

for all $g \in L_p$. In particular, for $f \in L_1$, we have

$$(ii) \quad T(f * g) = f * T(g) .$$

Proof. Since T commutes with left translation operators, it is evident that (i) holds when μ is a linear combination of Dirac measures. Now let μ be arbitrary. Since the extreme points of the unit ball of the conjugate space C_{00}^* (where C_{00} bears the uniform or supremum norm) are Dirac measures, and since Alaoglu's Theorem implies that the unit ball of C_{00}^* is $\sigma(C_{00}^*, C_{00})$ -compact, it follows by the Krein-Milman Theorem that there exists a net $\{\mu_{\alpha}\}$ consisting of linear combinations of Dirac measures such that the hypotheses (a) and (b) of Theorem 4 are satisfied. By Theorem 4, we have $\lim_{\alpha} \| \mu_{\alpha} * g - \mu * g \|_p = 0$ for all $g \in L_p$. This implies that $\lim_{\alpha} \| T(\mu_{\alpha} * g) - T(\mu * g) \|_p = 0$ for all $g \in L_p$. Consequently,

$$\begin{aligned} \| T(\mu * g) - \mu * T(g) \|_p &\leq \overline{\lim}_{\alpha} \| T(\mu * g) - T(\mu_{\alpha} * g) \|_p \\ &+ \overline{\lim}_{\alpha} \| T(\mu_{\alpha} * g) - \mu * T(g) \|_p = 0 + \overline{\lim}_{\alpha} \| \mu_{\alpha} * T(g) - \mu * T(g) \|_p = 0 . \end{aligned}$$

This proves part (i). Part (ii) is a special case of (i).

THEOREM 5. For each multiplier T in \mathfrak{M}_p and each function f in C_{00} , the function $T(f)$ is in L_p^t and $W_{T(f)} = T \circ W_f$.

Proof. Because f is in L_p , it follows from the corollary to Theorem 4 and (1) that $g * T(f) = T(g * f) = T \circ W_f(g)$ for all $g \in C_{00}$. This implies that $\| g * T(f) \|_p \leq \| T \| \cdot \| W_f \| \cdot \| g \|_p$ for all $g \in C_{00}$. Thus, by Theorem 1, $T(f)$ is in L_p^t . Since C_{00} is dense in L_p , we have that $W_{T(f)} = T \circ W_f$.

We purpose to identify the multipliers on \mathfrak{A}_p . To accomplish this, we shall set down a general multiplier identification theorem.

Let B be a normed algebra with identity and let A be any sub-algebra of B which is $\| \cdot \|_B$ -complete and which has a minimal left approximate identity. Define $\mathfrak{R}(B, A)$ to be the coarsest topology with respect to which each of the seminorms ${}^a \| \cdot \|$ ($a \in A$) is continuous where ${}^a \| b \| = \| b \cdot a \|_B$ for all $b \in B$. It is known (see [3] 1.4. (ii)) that

(4) the map $(a, b) \longrightarrow a \cdot b$ is $\mathfrak{R}(B, A)$ -continuous

when a and b run through any $\| \cdot \|_B$ -bounded subset of B .

THEOREM 6. *Let A and B be as above and suppose that the following hold:*

(i) *the unit ball A_1 of A is $\mathfrak{R}(B, A)$ -dense in the unit ball B_1 of B ;*

(ii) $\|b\|_B = \sup\{\|b \cdot a\|_B : a \in A_1\}$ *for each $b \in B_1$;*

(iii) B_1 *is $\mathfrak{R}(B, A)$ -complete.*

Then $m(A)$ is isomorphic to B .

Proof. By [3] 1.8. (iv), A is a left ideal in B . Define the map $T| \rightarrow m(A)$ by letting $T_b(a) = b \cdot a$ for all $b \in B$ and $a \in A$. That T is an algebra homomorphism of B into $m(A)$ is easy to check. That T is an isometry follows from (ii). That T is onto is a consequence of [3] 1.12.

LEMMA 1. *The unit ball of \mathfrak{A}_p is $\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)$ -dense in the unit ball of \mathfrak{M}_p .*

Proof. Let T be any operator in the unit ball of \mathfrak{M}_p . Let $\{W_{h_\gamma}\}$ be the minimal left approximate identity for \mathfrak{A}_p chosen in Theorem 3. For each index γ , we know from Theorem 5 and (3) that $T(h_\gamma)$ is in L_p^t and $W_{h_\gamma} \circ T \circ W_{h_\gamma} = W_{h_\gamma} \circ W_{T(h_\gamma)} = W_{T(h_\gamma) * h_\gamma}$. From (4), we see that $\{W_{h_\gamma} \circ T \circ W_{h_\gamma}\}$ converges to $I \circ T \circ I = T$ in $\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)$: in other words, $\lim W_{T(h_\gamma) * h_\gamma} = T$ in $\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)$.

Thus, we must have $\underline{\lim}_\gamma \|W_{T(h_\gamma) * h_\gamma}\| \geq \|T\|$, as is easily seen. But $\overline{\lim}_\gamma \|W_{T(h_\gamma) * h_\gamma}\| = \overline{\lim}_\gamma \|W_{h_\gamma} \circ T \circ W_{h_\gamma}\| \leq \overline{\lim}_\gamma \|W_{h_\gamma}\|^2 \cdot \|T\| \leq \|T\|$. Thus, we have $\lim_\gamma \|W_{T(h_\gamma) * h_\gamma}\| = \|T\|$. It follows that $\lim_\gamma \|W_{T(h_\gamma) * h_\gamma}\|^{-1} \cdot W_{T(h_\gamma) * h_\gamma} = T$ in $\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)$. We have shown that T is the $\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)$ -limit of operators in the unit ball of \mathfrak{A}_p .

LEMMA 2. *Let $\{T_\alpha\}$ be any $\mathfrak{R}(\mathfrak{B}_p, \mathfrak{A}_p)$ -Cauchy net in \mathfrak{B}_p such that $\sup_\alpha \|T_\alpha\| < \infty$. Then there is an operator T in \mathfrak{B}_p such that $\lim_\alpha T_\alpha = T$ in both the strong operator topology and the topology $\mathfrak{R}(\mathfrak{B}_p, \mathfrak{A}_p)$.*

Proof. Let S be the subspace of L_p spanned by the set $L_p * L_p^t * C_{00}$. If g is in L_p and $\{h_\gamma\}$ is the net in $L_p^t * C_\infty$ constructed in the proof of Theorem 3, then $\lim_\gamma \|g * h_\gamma - g\|_p = 0$ (see [1] 20.15. ii). It follows that S is dense in L_p .

Let $\sum_{j=1}^m f_j * h_j * g_j$ be a typical element of S where $f_j \in L_p$, $h_j \in L_p^t$, and $g_j \in C_{00}$ ($j = 1, 2, \dots, m$). Then $W_{h_j * g_j}$ is in \mathfrak{A}_p ($j = 1, 2, \dots, m$) so that, by hypothesis, the net $\{T_\alpha \circ W_{h_j * g_j}\}$ is $\| \cdot \|$ -Cauchy in \mathfrak{B}_p . Since

$T_\alpha(f_j * h_j * g_j) = T_\alpha \circ W_{h_j * g_j}(f_j)$ for each $j = 1, 2, \dots, m$ and each index α , it follows that the net $\{T_\alpha(f_j * h_j * g_j)\}$ is $\|\cdot\|_p$ -Cauchy for each $j = 1, 2, \dots, m$. Thus, $\{T_\alpha(\sum_{j=1}^m f_j * h_j * g_j)\}$ is $\|\cdot\|_p$ -Cauchy and so has some limit in L_p which we shall write as $T_0(\sum_{j=1}^m f_j * h_j * g_j)$. The operator $T_0|S \rightarrow L_p$ thus defined is clearly linear and, by the hypothesis $\sup_\alpha \|T_\alpha\| < \infty$, is also bounded. Since S is dense in L_p , T_0 is the restriction to S of a unique operator T in \mathfrak{B}_p . Since the net $\{T_\alpha\}$ converges to T on the dense subspace S of L_p , and since $\sup_\alpha \|T_\alpha\| < \infty$, it follows that $\lim_\alpha T_\alpha = T$ in the strong operator topology.

Let f be any function in $L_p^t * C_{00}$. By hypothesis, the net $\{T_\alpha \circ W_f\}$ is $\|\cdot\|_p$ -Cauchy and so has some $\|\cdot\|_p$ -limit V in \mathfrak{B}_p . For each $g \in L_1 \cap L_p$, we have

$$V(g) = \lim_\alpha T_\alpha \circ W_f(g) = \lim_\alpha T_\alpha(g * f) = T(g * f) = T \circ W_f(g) .$$

Since $L_1 \cap L_p$ is dense in L_p , it follows that $V = T \circ W_f$. Thus, $\lim_\alpha \|(T_\alpha - T) \circ W_f\| \approx 0$. Since $\{W_f : f \in L_p^t * C_{00}\}$ spans a dense subset of \mathfrak{A}_p and since $\sup_\alpha \|T_\alpha\| < \infty$, it follows that $\lim_\alpha T_\alpha = T$ in $\mathfrak{K}(\mathfrak{B}_p, \mathfrak{A}_p)$.

THEOREM 7. *Let $\pi| \mathfrak{M}_p \rightarrow \mathfrak{B}_p^*$ be defined by, for each $T \in \mathfrak{M}_p$, letting the function $\pi_T| \mathfrak{A}_p \rightarrow \mathfrak{B}_p$ be given by $\pi_T(W) = T \circ W$ for all $W \in \mathfrak{A}_p$. Then π is an isometric algebra isomorphism \mathfrak{M}_p onto $m(\mathfrak{A}_p)$.*

Proof. We shall apply Theorem 6 for $B = \mathfrak{M}_p$ and $A = \mathfrak{A}_p$. That \mathfrak{A}_p has a minimal left approximate identity follows from Theorem 3. That condition (i) of Theorem 6 is satisfied follows from Lemma 1. That condition (iii) of Theorem 6 is satisfied follows from Lemma 2. To invoke Theorem 6 and so prove Theorem 7, it will suffice to show that $\|T\| = \sup\{\|T \circ W\| : W \in \mathfrak{A}_p, \|W\| = 1\}$ for each $T \in \mathfrak{M}_p$.

Let then T be any multiplier in \mathfrak{M}_p . That $\|T\| \geq \sup\{\|T \circ W\| : W \in \mathfrak{A}_p, \|W\| = 1\}$ is obvious. Let ε be any positive number. Choose $f \in L_p$ such that $\|f\|_p \leq 1$ and $\|T(f)\|_p > \|T\| - \varepsilon/2$. Let $\{W_\gamma\}$ be a minimal left approximate identity for \mathfrak{A}_p . Then $\lim_\gamma W_\gamma = I$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$ where I is the identity operator on L_p . By (4) we have $\lim_\gamma T \circ W_\gamma = T \circ I = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$. By Lemma 2 we know that $\lim_\gamma T \circ W_\gamma = T$ in the strong operator topology. In particular, there exists some index γ such that $\|T \circ W_\gamma(f) - T(f)\| < \varepsilon/2$. It follows that

$$\begin{aligned} \|T \circ W_\gamma(f)\|_p &\geq \|T(f)\|_p - \|T(f) - T \circ W_\gamma(f)\|_p \\ &\geq \|T\| - \varepsilon/2 - \varepsilon/2 = \|T\| - \varepsilon ; \end{aligned}$$

but $\|T \circ W_\gamma(f)\|_p \leq \|T \circ W_\gamma\| \cdot \|f\|_p \leq \|T \circ W_\gamma\|$, so that $\|T \circ W_\gamma\| \geq \|T\| - \varepsilon$. Since ε was arbitrary and $\|W_\gamma\| \leq 1$, we have shown that

$$\|T\| = \sup\{\|T \circ W\|: W \in A, \|W\| \leq 1\}.$$

We shall identify L_p^t and \mathfrak{A}_p for several particular cases.

Case I. $p = 1$. Since L_1 is a Banach algebra with 2-sided minimal approximate identity, it follows that $L_1^t = L_1$ and $\|W_f\| = \|f\|_1$ for all $f \in L_1$. Because $L_1 * C_{00}$ is dense in L_1 , it follows that \mathfrak{A}_p is isomorphic to L_1 as a Banach algebra. Thus, in this case, Theorem 7 is the well-known fact that a bounded linear operator on L_1 commutes with all left translation operators if and only if it commutes with all left multiplication by elements of L_1 .

Case II. G is Abelian and $p = 2$. Let X be the character group of G and θ the Haar measure on X such that $\|\hat{f}\|_2 = \|f\|_2$ for all $f \in L_2$. In this case there is an isometric isomorphism $\widehat{\cdot} : M_2 \rightarrow L_\infty(X)$ which is onto $L_\infty(X)$ and such that $\widehat{T(f)} = \widehat{T} \cdot \widehat{f}$ for all $g \in L_2$. Evidently, L_2^t is just $\{f \in L_2: \widehat{f} \in L_\infty(X)\}$. It is known that there is a net $\{g_\alpha\}$ in the set $\{\widehat{f}: f \in C_{00}(G)\}$ such that $\|g_\alpha\|_\infty = 1$ for each index α and $\lim g_\alpha(\chi) = 1$ uniformly on compact subsets of X . Consequently, the set $\{\widehat{h * f}: h \in L_2^t, f \in C_{00}\}$ is dense in the set $\{g \in L_2(X) \cap L_\infty(X): g \text{ vanishes at } \infty\}$. It follows that \mathfrak{A}_2 is isomorphic in this case to $\{f \in L_\infty(X): f \text{ vanishes at } \infty\}$.

Case III. G is compact and $p \neq 1$. In this case L_p is a convolution algebra ([2] 28.64). Thus, $L_p^t = L_p$ and W may be viewed as a non norm-increasing linear operator from L_p into \mathfrak{A}_p . Since $C_{00} \subset L_p \cap L_1$, it is not difficult to show that W is an isomorphism into \mathfrak{A}_p .

Let $f \in L_p$ and choose a minimal approximate identity $\{f_\alpha\}$ for L_1 out of C_{00} . Then $\{f * f_\alpha\}$ converges to f in L_p . Consequently, $\{W_{f * f_\alpha}\}$ converges to W_f in \mathfrak{A}_p . All this shows that, in this case, \mathfrak{A}_p is the closure in \mathfrak{B}_p of the set $\{W_f: f \in L_p\}$.

Suppose now that G is also infinite. Then L_p has no minimal 1-sided identity (see [2] 34.40. b); since \mathfrak{A}_p does have one, it follows that W is not a homeomorphism. Since W is a continuous isomorphism, the open mapping theorem implies that $W|_{L_p} \rightarrow \mathfrak{A}_p$ is not onto \mathfrak{A}_p .

Case IV. G is compact and $p = 2$. Let Σ be the dual object of G as in [2]. For the spaces $\mathfrak{C}_0(\Sigma)$, $\mathfrak{C}_\infty(\Sigma)$, and $\mathfrak{C}_2(\Sigma)$ and the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ on these spaces, see [2] 28.34. It is an easy consequence of [2] D. 54 that

$$(5) \quad \|E\|_\infty = \sup\{\|A \circ E\|_2: A \in \mathfrak{C}_2(\Sigma), \|A\|_2 \leq 1\}$$

for all $E \in \mathfrak{G}_\infty(\Sigma)$. For the definition of the Fourier-Stieltjes transform \hat{f} of a function $f \in L_2$, see [2] 28.34. By [2] 28.43, the mapping $\hat{\cdot} : L_2 \rightarrow \mathfrak{G}_2(\Sigma)$ is a surjective linear isometry and, by [2] 28.40, $\widehat{f * g} = \hat{f} \circ \hat{g}$ for all $f, g \in L_2$. Consequently, by (5),

$$(6) \quad \|W_f\| = \|\hat{f}\|_\infty \quad \text{for all } f \in L_2.$$

Since $C_0 \subset L_2$, it follows from [2] 28.39, 28.27, and 28.40 that the set $\{\hat{f} : f \in L_2\}$ is a dense subspace of $\mathfrak{G}_0(\Sigma)$. Since \mathfrak{A}_p is just the closure in \mathfrak{B}_p of the set $\{W_f : f \in L_2\}$, it follows from (6) that \mathfrak{A}_p is isomorphic to $\mathfrak{G}_0(\Sigma)$ as a Banach algebra.

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