MULTIPLIERS OF TYPE \((p, p)\)

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It will be shown in this paper that the Banach algebra of all continuous multipliers on \(L_p(G)\) (\(G\) a locally compact group, \(p \in [0, \infty]\)) may be viewed as the set of all multipliers on a natural Banach algebra with minimal approximate left identity.

Let \(G\) be an arbitrary locally compact group, \(\lambda\) its left Haar measure, and \(p\) a number in \([1, \infty]\). Write \(\mathfrak{B}_p\) for the Banach algebra of all bounded linear operators on \(L_p\) and write \(\mathfrak{W}_p\) for the subset of \(\mathfrak{B}_p\) consisting of those operators which commute with all left translation operators; elements of \(\mathfrak{W}_p\) are called multipliers of type \((p, p)\).

If \(A\) is a Banach algebra, then a bounded linear operator \(T\) on \(A\) such that \(T(ab) = T(a)b\) for all \(a, b \in A\) is called a multiplier on \(A\); write \(m(A)\) for the set of all such. By \(C_0\) will be meant the set of all continuous complex-valued functions on \(G\) which have compact support.

A function \(f\) in \(L_p\) such that for each \(g\) in \(C_0\), the function \(g * f (x) = \int g(t)f(x-t)d\lambda(t)\) exists \(\lambda\)-almost everywhere, \(g * f\) is in \(L_p\), and \(\| g * f \|_p \leq g \|_p \cdot k\) where \(k\) is a positive number independent of \(g\), is said to be \(p\)-tempered; write \(L'_p\) for the set of all such. Evidently \(L'_p\) is closed under convolution and \(C_0\) is a subset of \(L'_p\). Thus, for each \(f\) in \(L'_p\) and \(h\) in \(C_0\), there is precisely one operator \(W\) in \(\mathfrak{B}_p\) such that \(W(g) = g * f * h\) for all \(g\) in \(L_p\); write \(\mathfrak{U}_p\) for the norm closure in \(\mathfrak{B}_p\) of the linear span of all such \(W\). The principal result of this paper is that \(\mathfrak{U}_p\) is a Banach algebra with minimal approximate left identity and that \(m(\mathfrak{U}_p)\) and \(\mathfrak{W}_p\) are isomorphic isometric Banach algebras.

**Theorem 1.** Let \(f\) be a function in \(L_p\) and \(k\) a positive number such that \(\| g * f \|_p \leq \| g \|_p \cdot k\) for all \(g\) in \(C_0\). Then \(f\) is in \(L'_p\).

**Proof.** First of all, suppose that \(h\) is a function in \(L_1 \cap L_p\). As is well known, \(h * f\) is in \(L_p\) and \(\| h * f \|_p \leq \| h \|_1 \cdot \| f \|_p\). Let \(\{h_n\}\) be a sequence in \(C_0\) which converges to \(h\) in the \(L_p\) and \(L_1\) norms both. It follows from the above that \(\{h_n * f\}\) converges to \(h * f\) in \(L_p\). This fact and the hypothesis for \(f\) imply

\[
\| h * f \|_p = \lim_n \| h_n * f \|_p \leq \lim_n \| h_n \|_p \cdot k = \| h \|_p \cdot k .
\]

Let \(h\) be now an arbitrary function from \(L_p\). We may assume that \(h\) vanishes off some \(\sigma\)-finite set \(A\). Let \(\{A_n\}\) be an increasing nest of \(\lambda\)-finite and \(\lambda\)-measurable subsets of \(G\) such that their union...
is \( A \). Let for each \( n \in N \), \( h_n \) be the product of \( h \) with the characteristic function of \( A_\cdot \). Let \( \pi_j \ (j = 0, 1, 2, 3) \) be the minimal non-negative functions on the complex field \( K \) such that 
\[
z = \sum_{j=0}^3 \hat{\pi}_j(z)
\]
for each \( z \in K \).

Fix \( j \) in \( \{0, 1, 2, 3\} \). For each \( x \in G \), define the measurable function \( w^z \) in \([0, \infty]^G\) by letting 
\[
w^z(t) = \pi_j[h(t) \cdot f(t^{-1}x)] \quad \text{for all } t \in G.
\]
For each \( x \in G \) and \( n \in N \), define the measurable function \( w^z_n \) in \([0, \infty]^G\) by letting 
\[
w^z_n(t) = \pi_j[h_n(t) \cdot f(t^{-1}x)] \quad \text{for all } t \in G.
\]
Since the sequence \( \{w^z_n\} \) converges upwards to \( w^z \) for each \( x \in G \), it follows from the monotone convergence theorem that 
\[
\lim_n \int w^z_n d\lambda = \int w^z d\lambda.
\]
Define the function \( F \) in \([0, \infty]^G\) by letting 
\[
F(x) = \int w^z d\lambda \quad \text{for all } x \in G.
\]
For each \( n \in N \), \( h_n \) is in \( L_1 \cap L_p \); it follows that \( \pi_j[h_n * f] \) is in \( L_p \), and so equals \( F_n \) almost everywhere. Hence, each \( F_n \) is measurable whence \( F \) is measurable. Further, by the monotone convergence theorem and the inequality which concludes the initial paragraph of this proof,

\[
|| F ||_p = \lim_n || F_n ||_p
= \lim_n || \pi_j[h_n * f] ||_p \leq \lim_n || h_n * f ||_p \leq \lim_n || h_n ||_p \cdot k = || h ||_p \cdot k.
\]

Recalling that \( F(x) = \int \pi_j[h(t) \cdot f(t^{-1}x)] dt \) almost everywhere and \( j \) was arbitrary, we see that \( h * f \) exists almost everywhere, is in \( L_p \) and 
\[
|| h * f ||_p \leq || h ||_p \cdot 4k.
\]
This proves that \( f \) is \( p \)-tempered.

The condition given in Theorem 1 for a function in \( L_p \) to be in \( L_\cdot_p \) is clearly necessary as well as sufficient. Another such condition was proved in [4], Theorem 1.3:

**Theorem 2.** Let \( f \) be a function in \( L_p \) such that \( g * f \) is defined and in \( L_p \) for all \( g \) in \( L_p \). Then \( f \) is in \( L_p \).

For each \( f \in L_p \), there is precisely one operator \( W_f \in \mathcal{B}_p \) such that

\[
(1) \quad W_f(g) = g * f
\]

for all \( g \in L_p \). For \( f \in C_0^0 \), we have as well (see [1] 20.13)

\[
(2) \quad || W_f || \leq \int A^{-(p-1)/p} | f | d\lambda.
\]

It is easy to check that
for all $f$ and $h$ in $L_p^\mu$.

**Theorem 3.** The set $\mathcal{A}_p$ is a complete subalgebra of $\mathcal{M}_p$ and it possesses a minimal left approximate identity (i.e., a net $\{T_\alpha\}$ such that $\varliminf_\alpha ||T_\alpha|| \leq 1$ and $\lim ||T_\alpha \circ T - T|| = 0$ for all $T \in \mathcal{A}_p$).

**Proof.** A simple calculation shows that, when $f$ is in $L_p^\mu$, then $W_f$ is in $\mathcal{M}_p$. Evidently, $\mathcal{M}_p$ is a Banach algebra; hence, $\mathcal{A}_p$ is a subset of $\mathcal{M}_p$. That $\mathcal{A}_p$ is a Banach space is an elementary consequence of its definition. That $\mathcal{A}_p$ is a Banach algebra is a consequence of the fact that $L_p^\mu \ast C_0$ is closed under convolution.

For each compact neighborhood $E$ of the identity of $G$, let $f_E$ be a nonnegative function in $C_\infty$ which vanishes outside $E$ and such that $\int f_E d\lambda = 1$. Directing the family of compact neighborhoods of the identity by letting $E > F$ when $E \subset F$, we obtain a net $\{f_\eta\}$ which is a minimal approximate identity for $L_\eta$. If $\{h_\eta\}$ denotes the product net of $\{f_\eta\}$ with itself, then $\{h_\eta\}$ is again a minimal approximate identity for $L_\eta$ and the net $\{W_{h_\eta}\}$ is in $\mathcal{A}_p$. Since $\Delta$ is unity and continuous at the identity of $G$, we have by (2),

$$\varlimsup_{\eta} ||W_{h_\eta}|| \leq \varlimsup_{\eta} \int \Delta^{-\frac{p-1}{p}} h_\eta d\lambda \leq 1. $$

For $f \in L_p^\mu$ and $g \in C_\infty$, (3) and (2) imply

$$\varlimsup_{\eta} \left| \left| W_{h_\eta} \circ W_{f,g} - W_{f,g} \right| \right| \leq \varlimsup_{\eta} \left| \left| (W_{g,h_\eta} - W_g) \circ W_f \right| \right|$$

$$\leq \varlimsup_{\eta} \left| \left| W_{g,h_\eta} - W_g \right| \right| \cdot \left| \left| W_f \right| \right| \leq \left( \varlimsup_{\eta} \int |g \ast h_\eta - g \ast \Delta^{-\frac{p-1}{p}} d\lambda \right) \cdot \left| \left| W_f \right| \right|$$

$$\leq \varlimsup_{\eta} \left| \left| g \ast h_\eta - g \right| \right| \cdot \sup \{ \Delta^{-\frac{p-1}{p}}(x) : g \ast h_\eta(x) \neq g(x) \} \cdot \left| \left| W_f \right| \right| = 0$$

since $\varlimsup_{\eta} \left| \left| g \ast h_\eta - g \right| \right| = 0$ and since the net of sets $\{x \in G : g \ast h_\eta(x) \neq g(x)\}$ is eventually contained in some fixed compact set. Since $L_p^\mu \ast C_0$ generates a dense subset of $\mathcal{A}_p$, we have $\lim ||W_{h_\eta} \circ T - T|| = 0$ for all $T \in \mathcal{A}_p$. Thus, $\{W_{h_\eta}\}$ is a minimal left approximate identity for $\mathcal{A}_p$.

We now turn to $\mathcal{M}_p$. We shall need a theorem proved in [3] 4.2.

**Theorem 4.** Let $\mu$ and the elements of a net $\{\mu_\alpha\}$ be bounded, complex, regular Borel measures on $G$ such that

(a) $\lim_\alpha ||\mu_\alpha|| = ||\mu||$
and

\[ \lim_{\alpha} \int f \, d\mu_\alpha = \int f \, d\mu \quad \text{for each } f \in C_{\infty}. \]

Then, for each \( g \in L_p \) (\( p \in [1, \infty[ \)), \( \lim_{\alpha} \| \mu_\alpha * g - \mu^* g \|_p = 0. \)

**Corollary.** For each multiplier \( T \) in \( M_p \) and each bounded, complex, regular Borel measure \( \mu \), we have

- \( (i) \quad T(\mu * g) = \mu * T(g) \)
- \( (ii) \quad T(f * g) = f * T(g) \).

**Proof.** Since \( T \) commutes with left translation operators, it is evident that (i) holds when \( \mu \) is a linear combination of Dirac measures. Now let \( \mu \) be arbitrary. Since the extreme points of the unit ball of the conjugate space \( C_{\infty}^* \) (where \( C_{\infty} \) bears the uniform or supremum norm) are Dirac measures, and since Alaoglu's Theorem implies that the unit ball of \( C_{\infty}^* \) is \( \sigma(C_{\infty}^*, C_{\infty}) \)-compact, it follows by the Krein-Milman Theorem that there exists a net \( \{\mu_\alpha\} \) consisting of linear combinations of Dirac measures such that the hypotheses (a) and (b) of Theorem 4 are satisfied. By Theorem 4, we have \( \lim_{\alpha} \| \mu_\alpha * g - \mu^* g \|_p = 0 \) for all \( g \in L_p \). This implies that \( \lim_{\alpha} \| T(\mu_\alpha * g) - T(\mu^* g) \|_p = 0 \) for all \( g \in L_p \). Consequently,

\[ \| T(\mu^* g) - \mu^* T(g) \|_p \leq \lim_{\alpha} \| T(\mu_\alpha * g) - T(\mu_\alpha^* g) \|_p, \]

\[ + \lim_{\alpha} \| T(\mu_\alpha^* g) - \mu^* T(g) \|_p = 0 \quad \text{for all } g \in L_p. \]

This proves part (i). Part (ii) is a special case of (i).

**Theorem 5.** For each multiplier \( T \) in \( M_p \) and each function \( f \) in \( C_{\infty} \), the function \( T(f) \) is in \( L_p^t \) and \( W_{T(f)} = T \circ W_f \).

**Proof.** Because \( f \) is in \( L_p \), it follows from the corollary to Theorem 4 and (1) that \( g^* T(f) = T(g^* f) = T \circ W_f(g) \) for all \( g \in C_{\infty} \). This implies that \( \| g^* T(f) \|_p \leq \| T \|_p \| W_f \|_p \| g \|_p \) for all \( g \in C_{\infty} \). Thus, by Theorem 1, \( T(f) \) is in \( L_p^t \). Since \( C_{\infty} \) is dense in \( L_p \), we have that \( W_{T(f)} = T \circ W_f \).

We purpose to identify the multipliers on \( A_p \). To accomplish this, we shall set down a general multiplier identification theorem.

Let \( B \) be a normed algebra with identity and let \( A \) be any subalgebra of \( B \) which is \( \| \|_B \)-complete and which has a minimal left approximate identity. Define \( \mathfrak{A}(B, A) \) to be the coarsest topology with respect to which each of the seminorms \( a || (a \in A) \) is continuous where \( a || b || = || b \cdot a ||_B \) for all \( b \in B \). It is known (see [3] 1.4. (ii)) that
the map \((a, b) \rightarrow a \cdot b\) is \(\mathfrak{R}(B, A)\)-continuous when \(a\) and \(b\) run through any \(||\cdot||_B\)-bounded subset of \(B\).

**Theorem 6.** Let \(A\) and \(B\) be as above and suppose that the following hold:

(i) the unit ball \(A_i\) of \(A\) is \(\mathfrak{R}(B, A)\)-dense in the unit ball \(B_i\) of \(B\);

(ii) \(\|b\|_B = \sup \{\|b \cdot a\|_B : a \in A_i\}\) for each \(b \in B_i\);

(iii) \(B_i\) is \(\mathfrak{R}(B, A)\)-complete.

Then \(m(A)\) is isomorphic to \(B\).

**Proof.** By [3] 1.8. (iv), \(A\) is a left ideal in \(B\). Define the map \(\Gamma|\rightarrow m(A)\) by letting \(T_b(a) = \delta_\alpha\) for all \(b \in B\) and \(a \in A\). That \(T\) is an algebra homomorphism of \(B\) into \(m(A)\) is easy to check. That \(T\) is an isometry follows from (ii). That \(T\) is onto is a consequence of [3] 1.12.

**Lemma 1.** The unit ball of \(\mathfrak{A}_p\) is \(\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)\)-dense in the unit ball of \(\mathfrak{M}_p\).

**Proof.** Let \(T\) be any operator in the unit ball of \(\mathfrak{M}_p\). Let \(\{W_{h_\gamma}\}\) be the minimal left approximate identity for \(\mathfrak{A}_p\) chosen in Theorem 3. For each index \(\gamma\), we know from Theorem 5 and (3) that \(T(h_\gamma)\) is in \(L_p\) and \(W_{h_\gamma} \circ T \circ W_{h_\gamma} = W_{h_\gamma} \circ W_{T(h_\gamma)} = W_{T(h_\gamma) + h_\gamma}\). From (4), we see that \(\{W_{h_\gamma} \circ T \circ W_{h_\gamma}\}\) converges to \(I \circ T \circ I = T\) in \(\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)\); in other words, \(\lim W_{T(h_\gamma) + h_\gamma} = T\) in \(\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)\).

Thus, we must have \(\lim_T ||W_{T(h_\gamma) + h_\gamma}|| \geq ||T||\), as is easily seen. But \(\lim_T ||W_{T(h_\gamma) + h_\gamma}|| = \lim_T ||W_{h_\gamma} \circ T \circ W_{h_\gamma}|| \leq \lim_T ||W_{h_\gamma}|| \cdot ||T|| \leq ||T||\).

Thus, we have \(\lim_T ||W_{T(h_\gamma) + h_\gamma}|| = ||T||\). It follows that \(\lim_T ||W_{T(h_\gamma) + h_\gamma}||^{-1} - W_{T(h_\gamma) + h_\gamma} = T\) in \(\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)\). We have shown that \(T\) is the \(\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)\)-limit of operators in the unit ball of \(\mathfrak{A}_p\).

**Lemma 2.** Let \(\{T_\alpha\}\) be any \(\mathfrak{R}(\mathfrak{B}_p, \mathfrak{A}_p)\)-Cauchy net in \(\mathfrak{B}_p\) such that \(\sup_\alpha ||T_\alpha|| < \infty\). Then there is an operator \(T\) in \(\mathfrak{B}_p\) such that \(\lim_\alpha T_\alpha = T\) in both the strong operator topology and the topology \(\mathfrak{R}(\mathfrak{B}_p, \mathfrak{A}_p)\).

**Proof.** Let \(S\) be the subspace of \(L_p\) spanned by the set \(L_p \ast L_b \ast C_{00}\). If \(g\) is in \(L_p\) and \(\{h_\gamma\}\) is the net in \(L_{p^\\infty} \ast C_{00}\) constructed in the proof of Theorem 3, then \(\lim_\gamma ||g * h_\gamma - g||_p = 0\) (see [1] 20.15. ii). It follows that \(S\) is dense in \(L_p\).

Let \(\sum_{j=1}^m f_j * h_j * g_j\) be a typical element of \(S\) where \(f_j \in L_p\), \(h_j \in L_{p^\\infty}\), and \(g_j \in C_{00}\) \((j = 1, 2, \cdots, m)\). Then \(W_{h_j * g_j}\) is in \(\mathfrak{A}_p\) \((j = 1, 2, \cdots, m)\) so that, by hypothesis, the net \(\{T_\alpha \circ W_{h_j * g_j}\}\) is \(\|\|\)-Cauchy in \(\mathfrak{B}_p\). Since
\[ T_a(f_j h_j g_j) = T_0 \circ W_{h_j g_j}(f_j) \text{ for each } j = 1, 2, \ldots, m \text{ and each index } \alpha, \text{ it follows that the net } \{ T_a(f_j h_j g_j) \} \text{ is } \|\cdot\|_p\text{-Cauchy for each } j = 1, 2, \ldots, m. \text{ Thus, } \{ T_a(\sum_{j=1}^m f_j h_j g_j) \} \text{ is } \|\cdot\|_p\text{-Cauchy and so has some limit in } L_p \text{ which we shall write as } T_0(\sum_{j=1}^m f_j h_j g_j). \text{ The operator } T_0 | S \to L_\alpha \text{ thus defined is clearly linear and, by the hypothesis } \sup_{\alpha} \| T_a \| < \infty, \text{ is also bounded. Since } S \text{ is dense in } L_p, \text{ } T_0 \text{ is the restriction to } S \text{ of a unique operator } T \text{ in } \mathcal{B}_p. \text{ Since the net } \{ T_a \} \text{ converges to } T \text{ on the dense subspace } S \text{ of } L_p, \text{ and since } \sup_{\alpha} \| T_a \| < \infty, \text{ it follows that } \lim_{\alpha} T_a = T \text{ in the strong operator topology.}

Let } f \text{ be any function in } L_p \ast C_\infty. \text{ By hypothesis, the net } \{ T_0 \circ W_f \} \text{ is } \|\cdot\|\text{-Cauchy and so has some } \|\cdot\|\text{-limit } V \text{ in } \mathcal{B}_p. \text{ For each } g \in L_1 \cap L_p, \text{ we have}

\[ V(g) = \lim_{\alpha} T_0 \circ W_f(g) = \lim_{\alpha} T_0(g \ast f) = T(g \ast f) = T_0 W_f(g). \]

Since } L_1 \cap L_p \text{ is dense in } L_p, \text{ it follows that } V = T_0 W_f. \text{ Thus, } \lim_{\alpha} \| (T_0 - T) \circ W_f \| = 0. \text{ Since } \{ W_f : f \in L_p \ast C_\infty \} \text{ spans a dense subset of } \mathcal{A}_p \text{ and since } \sup_{\alpha} \| T_a \| < \infty, \text{ it follows that } \lim_{\alpha} T_a = T \text{ in } \mathcal{A}(\mathcal{B}_p, \mathcal{A}_p).

**Theorem 7.** Let } \pi | \mathcal{M}_p \to \mathcal{B}_p^a \text{ be defined by, for each } T \in \mathcal{M}_p, \text{ letting the function } \pi_T | \mathcal{A}_p \to \mathcal{B}_p \text{ be given by } \pi_T(W) = T_0 W \text{ for all } W \in \mathcal{A}_p. \text{ Then } \pi \text{ is an isometric algebra isomorphism } \mathcal{M}_p \text{ onto } \mathcal{M}(\mathcal{A}_p).

**Proof.** We shall apply Theorem 6 for } B = \mathcal{M}_p \text{ and } A = \mathcal{A}_p. \text{ That } \mathcal{A}_p \text{ has a minimal left approximate identity follows from Theorem 3. That condition (i) of Theorem 6 is satisfied follows from Lemma 1. That condition (iii) of Theorem 6 is satisfied follows from Lemma 2. To invoke Theorem 6 and so prove Theorem 7, it will suffice to show that } \| T \| = \sup \{ \| T_0 W \| : W \in \mathcal{A}_p, \| W \| = 1 \} \text{ for each } T \in \mathcal{M}_p. \text{ Let then } T \text{ be any multiplier in } \mathcal{M}_p. \text{ That } \| T \| \geq \sup \{ \| T_0 W \| : W \in \mathcal{A}_p, \| W \| = 1 \} \text{ is obvious. Let } \varepsilon \text{ be any positive number. Choose } f \in L_p \text{ such that } \| f \|_p \leq 1 \text{ and } \| T(f) \|_p > \| T \| - \varepsilon/2. \text{ Let } \{ W_{f,\gamma} \} \text{ be a minimal left approximate identity for } \mathcal{A}_p. \text{ Then } \lim_{\gamma} W_{f,\gamma} = I \text{ in } \mathcal{A}(\mathcal{M}_p, \mathcal{A}_p) \text{ where } I \text{ is the identity operator on } L_n. \text{ By (4) we have } \lim_{\gamma} T_0 W_{f,\gamma} = T_0 I = T \text{ in } \mathcal{A}(\mathcal{M}_p, \mathcal{A}_p). \text{ By Lemma 2 we know that } \lim_{\gamma} T_0 W_{f,\gamma} = T \text{ in the strong operator topology. In particular, there exists some index } \gamma \text{ such that } \| T_0 W_{f,\gamma}(f) - T(f) \| < \varepsilon/2. \text{ It follows that}

\[
\| T_0 W_{f,\gamma}(f) \|_p \geq \| T(f) \|_p - \| T(f) - T_0 W_{f,\gamma}(f) \|_p \\
\geq \| T \| - \varepsilon/2 - \varepsilon/2 = \| T \| - \varepsilon;
\]

but } \| T_0 W_{f,\gamma}(f) \|_p \leq \| T_0 W_{f,\gamma} \| \cdot \| f \|_p \leq \| T_0 W_{f,\gamma} \|, \text{ so that } \| T_0 W_{f,\gamma} \| \geq \| T \| - \varepsilon. \text{ Since } \varepsilon \text{ was arbitrary and } \| W_{f,\gamma} \| \leq 1, \text{ we have shown that}
We shall identify $L_\rho'$ and $\mathcal{A}_p$ for several particular cases.

Case I. $p = 1$. Since $L_1$ is a Banach algebra with 2-sided minimal approximate identity, it follows that $L_1' = L_1$ and $\| W_f \| = \| f \|$ for all $f \in L_1$. Because $L_1^* C_{00}$ is dense in $L_1$, it follows that $\mathcal{A}_p$ is isomorphic to $L_1$ as a Banach algebra. Thus, in this case, Theorem 7 is the well-known fact that a bounded linear operator on $L_1$ commutes with all left translation operators if and only if it commutes with all left multiplication by elements of $L_1$.

Case II. $G$ is Abelian and $p = 2$. Let $X$ be the character group of $G$ and $\theta$ the Haar measure on $X$ such that $\| \hat{f} \|_2 = \| f \|_2$ for all $f \in L_2$. In this case there is an isometric isomorphism $\hat{\cdot} : M_2 \to L_{\omega}(X)$ which is onto $L_{\omega}(X)$ and such that $\hat{T}(f) = \hat{T} \cdot \hat{f}$ for all $g \in L_2$. Evidently, $L_2^* \text{ is just } \{ f \in L_2 : \hat{f} \in L_{\omega}(X) \}$. It is known that there is a net $\{ g_\alpha \}$ in the set $\{ \hat{f} : f \in C_{00}(G) \}$ such that $\| g_\alpha \|_{\infty} = 1$ for each index $\alpha$ and $\lim g_\alpha(\chi) = 1$ uniformly on compact subsets of $X$. Consequently, the set $\{ \hat{h} \hat{\cdot} \hat{f} : h \in L_2^*, f \in C_{00} \}$ is dense in the set $\{ g \in L_2(X) \cap L_{\omega}(X) : g \text{ vanishes at } \infty \}$. It follows that $\mathcal{A}_2$ is isomorphic in this case to $\{ f \in L_{\omega}(x) : f \text{ vanishes at } \infty \}$.

Case III. $G$ is compact and $p \neq 1$. In this case $L_p$ is a convolution algebra ([2] 28.64). Thus, $L_p' = L_p$ and $W$ may be viewed as a non norm-increasing linear operator from $L_p$ into $\mathcal{A}_p$. Since $C_{00} \subset L_p \cap L_1$, it is not difficult to show that $W$ is an isomorphism into $\mathcal{A}_p$.

Let $f_3 L_p$ and choose a minimal approximate identity $\{ f_\alpha \}$ for $L_1$ out of $C_{00}$. Then $\{ f_\alpha f_3 \}$ converges to $f$ in $L_p$. Consequently, $\{ W f_\alpha f_3 \}$ converges to $W f_3$ in $\mathcal{A}_p$. All this shows that, in this case, $\mathcal{A}_p$ is the closure in $\mathcal{B}_p$ of the set $\{ W f : f \in L_p \}$.

Suppose now that $G$ is also infinite. Then $L_p$ has no minimal 1-sided identity (see [2] 34.40. b); since $\mathcal{A}_p$ does have one, it follows that $W$ is not a homeomorphism. Since $W$ is a continuous isomorphism, the open mapping theorem implies that $W | L_p \to \mathcal{A}_p$ is not onto $\mathcal{A}_p$.

Case IV. $G$ is compact and $p = 2$. Let $\Sigma$ be the dual object of $G$ as in [2]. For the spaces $C_0(\Sigma), C_s(\Sigma)$, and $C_\omega(\Sigma)$ and the norms $\| \|_\infty$ and $\| \|_2$ on these spaces, see [2] 28.34. It is an easy consequence of [2] D. 54 that

$$\| A \|_\infty = \sup \{ \| A \circ E \|_2 : A \in C_s(\Sigma), \| A \|_2 \leq 1 \}$$
for all \( E \in \mathcal{E}_c(\Sigma) \). For the definition of the Fourier-Stieltjes transform \( \hat{f} \) of a function \( f \in L_2 \), see [2] 28.34. By [2] 28.43, the mapping \( \hat{\cdot} : L_2 \to \mathcal{E}_c(\Sigma) \) is a surjective linear isometry and, by [2] 28.40, \( \hat{f} \hat{g} = \hat{f \circ g} \) for all \( f, g \in L_2 \). Consequently, by (5),

\[
\| W_f \| = \| \hat{f} \|_\infty \quad \text{for all } f \in L_2.
\]

Since \( C_0 \subset L_2 \), it follows from [2] 28.39, 28.27, and 28.40 that the set \( \{ \hat{f} : f \in L_2 \} \) is a dense subspace of \( \mathcal{E}_c(\Sigma) \). Since \( \mathcal{A}_p \) is just the closure in \( \mathcal{B}_p \) of the set \( \{ W_f : f \in L_2 \} \), it follows from (6) that \( \mathcal{A}_p \) is isomorphic to \( \mathcal{E}_c(\Sigma) \) as a Banach algebra.

**References**


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