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It will be shown in this paper that the Banach algebra of all continuous multipliers on $L_p(G)$ (G a locally compact group, $p \in [0, \infty[$) may be viewed as the set of all multipliers on a natural Banach algebra with minimal approximate left identity.

Let G be an arbitrary locally compact group, λ its left Haar measure, and p a number in $[1, \infty[$. Write $\mathfrak{B}_p$ for the Banach algebra of all bounded linear operators on L_p and write $\mathfrak{M}_p$ for the subset of $\mathfrak{B}_p$ consisting of those operators which commute with all left translation operators; elements of $\mathfrak{M}_p$ are called *multipliers of type* (p, p). If A is a Banach algebra, then a bounded linear operator T on A such that $T(a \cdot b) = T(a) \cdot b$ for all $a, b \in A$ is called a *multiplier on* A; write $\mathfrak{m}(A)$ for the set of all such. By C_{00} will be meant the set of all continuous complex-valued functions on G which have compact support. A function f in L_p such that for each g in L_p, the function $g*f(x) = \int g(t)f(t^{-1}x)d\lambda(t)$ exists λ-almost everywhere, $g*f$ is in L_p, and $\| g*f \|_p \leqq \| g \|_p \cdot k$ where k is a positive number independent of g, is said to be *p-tempered*; write L_p^t for the set of all such. Evidently L_p^t is closed under convolution and C_{00} is a subset of L_p^t. Thus, for each f in L_p^t and h in C_{00}, there is precisely one operator W in $\mathfrak{B}_p$ such that $W(g) = g*f*h$ for all g in L_p; write $\mathfrak{A}_p$ for the norm closure in $\mathfrak{B}_p$ of the linear span of all such W. The principal result of this paper is that $\mathfrak{A}_p$ is a Banach algebra with minimal approximate left identity and that $\mathfrak{m}(\mathfrak{A}_p)$ and $\mathfrak{M}_p$ are isomorphic isometric Banach algebras.

THEOREM 1. *Let f be a function in L_p and k a positive number such that $\| g*f \|_p \leqq \| g \|_p \cdot k$ for all g in C_{00}. Then f is in L_p^t.*

Proof. First of all, suppose that h is a function in $L_1 \cap L_p$. As is well known, $h*f$ is in L_p and $\| h*f \|_p \leqq \| h \|_1 \cdot \| f \|_p$. Let $\{h_n\}$ be a sequence in C_{00} which converges to h in the L_p and L_1 norms both. It follows from the above that $\{h_n*f\}$ converges to $h*f$ in L_p. This fact and the hypothesis for f imply

$$\| h*f \|_p = \lim_n \| h_n*f \|_p \leqq \overline{\lim_n} \| h_n \|_p \cdot k = \| h \|_p \cdot k \, .$$

Let h be now an arbitrary function from L_p. We may assume that h vanishes off some σ-finite set A. Let $\{A_n\}$ be an increasing nest of λ-finite and λ-measurable subsets of G such that their union is A. Let for each $n \in N$, h_n be the product of h with the characteristic function of A_n. Let π_j ($j = 0, 1, 2, 3$) be the minimal non-negative functions on the complex field K such that $z = \sum_{j=0}^{3} i^j \pi_j(z)$ for each $z \in K$.

Fix j in $\{0, 1, 2, 3\}$. For each $x \in G$, define the measurable function w^x in $[0, \infty]^G$ by letting $w^x(t) = \pi_j[h(t) \cdot f(t^{-1}x)]$ for all $t \in G$. For each $x \in G$ and $n \in N$, define the measurable function w_n^x in $[0, \infty]^G$ by letting $w_n^x(t) = \pi_j[h_n(t) \cdot f(t^{-1}x)]$ for all $t \in G$. Since the sequence $\{w_n^x\}$ converges upwards to w^x for each $x \in G$, it follows from the monotone convergence theorem that $\lim_n \int w_n^x d\lambda = \int w^x d\lambda$. Define the function F in $[0, \infty]^G$ by letting $F(x) = \int w^x d\lambda$ for all $x \in G$. For each $n \in N$, define the function F_n in $[0, \infty]^G$ by letting $F_n(x) = \int w_n^x d\lambda$ for all $x \in G$. Thus, $\{F_n\}$ converges upwards to F at each point $x \in G$.

For each $n \in N$, h_n is in $L_1 \cap L_p$; it follows that $\pi_j[h_n * f]$ is in L_p, and so equals F_n almost everywhere. Hence, each F_n is measurable whence F is measurable. Further, by the monotone convergence theorem and the inequality which concludes the initial paragraph of this proof,

$$\begin{aligned} \| F \|_p &= \lim_n \| F_n \|_p \\ &= \lim_n \| \pi_j[h_n * f] \|_p \leqq \overline{\lim_n} \| h_n * f \|_p \leqq \overline{\lim_n} \| h_n \|_p \cdot k = \| h \|_p \cdot k \,. \end{aligned}$$

Recalling that $F(x) = \int \pi_j[h(t) \cdot f(t^{-1}x)] dt$ almost everywhere and j was arbitrary, we see that $h * f$ exists almost everywhere, is in L_p and $\| h * f \|_p \leqq \| h \|_p \cdot 4k$. This proves that f is p-tempered.

The condition given in Theorem 1 for a function in L_p to be in L_p^t is clearly necessary as well as sufficient. Another such condition was proved in [4], Theorem 1.3:

THEOREM 2. *Let f be a function in L_p such that $g * f$ is defined and in L_p for all g in L_p. Then f is in L_p^t.*

For each $f \in L_p^t$, there is precisely one operator $W_f \in \mathfrak{B}_p$ such that

$$W_f(g) = g * f \tag{1}$$

for all $g \in L_p$. For $f \in C_{00}$, we have as well (see [1] 20.13)

$$\| W_f \| \leqq \int \Delta^{-(p-1)/p} | f | d\lambda \,. \tag{2}$$

It is easy to check that

$$W_{f*h} = W_h \circ W_f \tag{3}$$

for all f and h in L_p^t.

THEOREM 3. *The set* $\mathfrak{A}_p$ *is a complete subalgebra of* $\mathfrak{M}_p$ *and it possesses a minimal left approximate identity (i.e., a net* $\{T_\alpha\}$ *such that* $\overline{\lim}_\alpha \| T_\alpha \| \leqq 1$ *and* $\lim \| T_\alpha \circ T - T \| = 0$ *for all* $T \in \mathfrak{A}_p$*).*

Proof. A simple calculation shows that, when f is in L_p^t, then W_f is in $\mathfrak{M}_p$. Evidently, $\mathfrak{M}_p$ is a Banach algebra; hence, $\mathfrak{A}_p$ is a subset of $\mathfrak{M}_p$. That $\mathfrak{A}_p$ is a Banach space is an elementary consequence of its definition. That $\mathfrak{A}_p$ is a Banach algebra is a consequence of the fact that $L_p^t * C_{00}$ is closed under convolution.

For each compact neighborhood E of the identity of G, let f_E be a nonnegative function in C_∞ which vanishes outside E and such that $\int f_E d\lambda = 1$. Directing the family of compact neighborhoods of the identity by letting $E > F$ when $E \subset F$, we obtain a net $\{f_E\}$ which is a minimal approximate identity for L_1. If $\{h_\gamma\}$ denotes the product net of $\{f_E\}$ with itself, then $\{h_\gamma\}$ is again a minimal approximate identity for L_1 and the net $\{W_{h_\gamma}\}$ is in $\mathfrak{A}_p$. Since Δ is unity and continuous at the identity of G, we have by (2),

$$\overline{\lim_\gamma} \| W_{h_\gamma} \| \leqq \overline{\lim_\gamma} \int \Delta^{-(p-1)/p} h_\gamma d\lambda \leqq 1 .$$

For $f \in L_p^t$ and $g \in C_{00}$, (3) and (2) imply

$$\begin{aligned} \overline{\lim_\gamma} \| W_{h_\gamma} \circ W_{f*g} - W_{f*g} \| &= \overline{\lim_\gamma} \| (W_{g*h_\gamma} - W_g) \circ W_f \| \\ &\leqq \overline{\lim_\gamma} \| W_{g*h_\gamma} - W_g \| \cdot \| W_f \| \leqq \left(\overline{\lim_\gamma} \int | g*h_\gamma - g | \cdot \Delta^{-(p-1)/p} d\lambda \right) \cdot \| W_f \| \\ &\leqq \overline{\lim_\gamma} \| g*h_\gamma - g \|_1 \cdot \sup \{ \Delta^{-(p-1)/p}(x) : g*h_\gamma(x) \neq g(x) \} \cdot \| W_f \| = 0 \end{aligned}$$

since $\overline{\lim}_\gamma \|g*h_\gamma - g\|_1 = 0$ and since the net of sets $\{x \in G: g*h_\gamma(x) \neq g(x)\}$ is eventually contained in some fixed compact set. Since $L_p^t * C_{00}$ generates a dense subset of $\mathfrak{A}_p$, we have $\lim \| W_{h_\gamma} \circ T - T \| = 0$ for all $T \in \mathfrak{A}_p$. Thus, $\{W_{h_\gamma}\}$ is a minimal left approximate identity for $\mathfrak{A}_p$.

We now turn to $\mathfrak{M}_p$. We shall need a theorem proved in [3] 4.2.

THEOREM 4. *Let* μ *and the elements of a net* $\{\mu_\alpha\}$ *be bounded, complex, regular Borel measures on G such that*

$$\lim_\alpha \| \mu_\alpha \| = \| \mu \| \tag{a}$$

and

(b) $$\lim_\alpha \int f d\mu_\alpha = \int f d\mu \quad \textit{for each} \quad f \in C_{00} .$$

Then, for each $g \in L_p$ $(p \in [1, \infty[)$, $\lim_\alpha \| \mu_\alpha * g - \mu * g \|_p = 0$.

COROLLARY. *For each multiplier* T *in* $\mathfrak{M}_p$ *and each bounded, complex, regular Borel measure* μ, *we have*

(i) $T(\mu * g) = \mu * T(g)$

for all $g \in L_p$. *In particular, for* $f \in L_1$, *we have*

(ii) $T(f * g) = f * T(g)$.

Proof. Since T commutes with left translation operators, it is evident that (i) holds when μ is a linear combination of Dirac measures. Now let μ be arbitrary. Since the extreme points of the unit ball of the conjugate space C_{00}^* (where C_{00} bears the uniform or supremum norm) are Dirac measures, and since Alaoglu's Theorem implies that the unit ball of C_{00}^* is $\sigma(C_{00}^*, C_{00})$-compact, it follows by the Krein-Milman Theorem that there exists a net $\{\mu_\alpha\}$ consisting of linear combinations of Dirac measures such that the hypotheses (a) and (b) of Theorem 4 are satisfied. By Theorem 4, we have $\lim_\alpha \| \mu_\alpha * g - \mu * g \|_p = 0$ for all $g \in L_p$. This implies that $\lim_\alpha \| T(\mu_\alpha * g) - T(\mu * g) \|_p = 0$ for all $g \in L_p$. Consequently,

$$\| T(\mu * g) - \mu * T(g) \|_p \leqq \overline{\lim_\alpha} \| T(\mu * g) - T(\mu_\alpha * g) \|_p$$
$$+ \overline{\lim_\alpha} \| T(\mu_\alpha * g) - \mu * T(g) \|_p = 0 + \overline{\lim_\alpha} \| \mu_\alpha * T(g) - \mu * T(g) \|_p = 0 .$$

This proves part (i). Part (ii) is a special case of (i).

THEOREM 5. *For each multiplier* T *in* $\mathfrak{M}_p$ *and each function* f *in* C_{00}, *the function* $T(f)$ *is in* L_p^t *and* $W_{T(f)} = T \circ W_f$.

Proof. Because f is in L_p, it follows from the corollary to Theorem 4 and (1) that $g * T(f) = T(g * f) = T \circ W_f(g)$ for all $g \in C_{00}$. This implies that $\| g * T(f) \|_p \leqq \| T \| \cdot \| W_f \| \cdot \| g \|_p$ for all $g \in C_{00}$. Thus, by Theorem 1, $T(f)$ is in L_p^t. Since C_{00} is dense in L_p, we have that $W_{T(f)} = T \circ W_f$.

We purpose to identify the multipliers on $\mathfrak{A}_p$. To accomplish this, we shall set down a general multiplier identification theorem.

Let B be a normed algebra with identity and let A be any subalgebra of B which is $\| \ \|_B$-complete and which has a minimal left approximate identity. Define $\mathfrak{R}(B, A)$ to be the coarsest topology with respect to which each of the seminorms ${}^a\| \ \|$ $(a \in A)$ is continuous where ${}^a\| b \| = \| b \cdot a \|_B$ for all $b \in B$. It is known (see [3] 1.4. (ii)) that

(4) the map $(a, b) \longrightarrow a \cdot b$ is $\mathfrak{K}(B, A)$-continuous

when a and b run through any $\| \|_B$-bounded subset of B.

THEOREM 6. *Let A and B be as above and suppose that the following hold:*

(i) *the unit ball A_1 of A is $\mathfrak{K}(B, A)$-dense in the unit ball B_1 of B;*

(ii) $\| b \|_B = \sup\{\| b \cdot a \|_B : a \in A_1\}$ *for each* $b \in B_1$;

(iii) *B_1 is $\mathfrak{K}(B, A)$-complete.*

Then $\mathfrak{m}(A)$ is isomorphic to B.

Proof. By [3] 1.8. (iv), A is a left ideal in B. Define the map $T | \to m(A)$ by letting $T_b(a) = b \cdot a$ for all $b \in B$ and $a \in A$. That T is an algebra homomorphism of B into $m(A)$ is easy to check. That T is an isometry follows from (ii). That T is onto is a consequence of [3] 1.12.

LEMMA 1. *The unit ball of $\mathfrak{A}_p$ is $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$-dense in the unit ball of $\mathfrak{M}_p$.*

Proof. Let T be any operator in the unit ball of $\mathfrak{M}_p$. Let $\{W_{h_\gamma}\}$ be the minimal left approximate identity for $\mathfrak{A}_p$ chosen in Theorem 3. For each index γ, we know from Theorem 5 and (3) that $T(h_\gamma)$ is in L_p^t and $W_{h_\gamma} \circ T \circ W_{h_\gamma} = W_{h_\gamma} \circ W_{T(h_\gamma)} = W_{T(h_\gamma) * h_\gamma}$. From (4), we see that $\{W_{h_\gamma} \circ T \circ W_{h_\gamma}\}$ converges to $I \circ T \circ I = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$: in other words, $\lim W_{T(h_\gamma) * h_\gamma} = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$.

Thus, we must have $\underline{\lim}_\gamma \| W_{T(h_\gamma) * h_\gamma} \| \geqq \| T \|$, as is easily seen. But $\overline{\lim}_\gamma \| W_{T(h_\gamma) * h_\gamma} \| = \overline{\lim}_\gamma \| W_{h_\gamma} \circ T \circ W_{h_\gamma} \| \leqq \overline{\lim}_\gamma \| W_{h_\gamma} \|^2 \cdot \| T \| \leqq \| T \|$. Thus, we have $\lim_\gamma \| W_{T(h_\gamma) * h_\gamma} \| = \| T \|$. It follows that $\lim_\gamma \| W_{T(h_\gamma) * h_\gamma} \|^{-1} \cdot W_{T(h_\gamma) * h_\gamma} = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$. We have shown that T is the $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$-limit of operators in the unit ball of $\mathfrak{A}_p$.

LEMMA 2. *Let $\{T_\alpha\}$ be any $\mathfrak{K}(\mathfrak{B}_p, \mathfrak{A}_p)$-Cauchy net in $\mathfrak{B}_p$ such that* $\sup_\alpha \| T_\alpha \| < \infty$. *Then there is an operator T in $\mathfrak{B}_p$ such that* $\lim_\alpha T_\alpha = T$ *in both the strong operator topology and the topology $\mathfrak{K}(\mathfrak{B}_p, \mathfrak{A}_p)$.*

Proof. Let S be the subspace of L_p spanned by the set $L_p * L_p^t * C_{00}$. If g is in L_p and $\{h_\gamma\}$ is the net in $L_p^t * C_\infty$ constructed in the proof of Theorem 3, then $\lim_\gamma \| g * h_\gamma - g \|_p = 0$ (see [1] 20.15. ii). It follows that S is dense in L_p.

Let $\sum_{j=1}^m f_j * h_j * g_j$ be a typical element of S where $f_j \in L_p$, $h_j \in L_p^t$, and $g_j \in C_{00}$ $(j = 1, 2, \cdots, m)$. Then $W_{h_j * g_j}$ is in $\mathfrak{A}_p$ $(j = 1, 2, \cdots, m)$ so that, by hypothesis, the net $\{T_\alpha \circ W_{h_j * g_j}\}$ is $\| \|$-Cauchy in $\mathfrak{B}_p$. Since

$T_\alpha(f_j*h_j*g_j) = T_\alpha \circ W_{h_j*g_j}(f_j)$ for each $j = 1, 2, \cdots, m$ and each index α, it follows that the net $\{T_\alpha(f_j*h_j*g_j)\}$ is $\| \|_p$-Cauchy for each $j = 1, 2, \cdots, m$. Thus, $\{T_\alpha(\sum_{j=1}^m f_j*h_j*g_j)\}$ is $\| \|_p$-Cauchy and so has some limit in L_p which we shall write as $T_0(\sum_{j=1}^m f_j*h_j*g_j)$. The operator $T_0 \mid S \to L_p$ thus defined is clearly linear and, by the hypothesis $\sup_\alpha \| T_\alpha \| < \infty$, is also bounded. Since S is dense in L_p, T_0 is the restriction to S of a unique operator T in $\mathfrak{B}_p$. Since the net $\{T_\alpha\}$ converges to T on the dense subspace S of L_p, and since $\sup_\alpha \| T_\alpha \| < \infty$, it follows that $\lim_\alpha T_\alpha = T$ in the strong operator topology.

Let f be any function in $L_p^t * C_{00}$. By hypothesis, the net $\{T_\alpha \circ W_f\}$ is $\| \|$-Cauchy and so has some $\| \|$-limit V in $\mathfrak{B}_p$. For each $g \in L_1 \cap L_p$, we have

$$V(g) = \lim_\alpha T_\alpha \circ W_f(g) = \lim_\alpha T_\alpha(g*f) = T(g*f) = T \circ W_f(g) .$$

Since $L_1 \cap L_p$ is dense in L_p, it follows that $V = T \circ W_f$. Thus, $\lim_\alpha \| (T_\alpha - T) \circ W_f \| = 0$. Since $\{W_f : f \in L_p^t * C_{00}\}$ spans a dense subset of $\mathfrak{A}_p$ and since $\sup_\alpha \| T_\alpha \| < \infty$, it follows that $\lim_\alpha T_\alpha = T$ in $\mathfrak{R}(\mathfrak{B}_p, \mathfrak{A}_p)$.

THEOREM 7. *Let $\pi \mid \mathfrak{M}_p \to \mathfrak{B}_p^{\mathfrak{A}_p}$ be defined by, for each $T \in \mathfrak{M}_p$, letting the function $\pi_T \mid \mathfrak{A}_p \to \mathfrak{B}_p$ be given by $\pi_T(W) = T \circ W$ for all $W \in \mathfrak{A}_p$. Then π is an isometric algebra isomorphism $\mathfrak{M}_p$ onto $\mathfrak{m}(\mathfrak{A}_p)$.*

Proof. We shall apply Theorem 6 for $B = \mathfrak{M}_p$ and $A = \mathfrak{A}_p$. That $\mathfrak{A}_p$ has a minimal left approximate identity follows from Theorem 3. That condition (i) of Theorem 6 is satisfied follows from Lemma 1. That condition (iii) of Theorem 6 is satisfied follows from Lemma 2. To invoke Theorem 6 and so prove Theorem 7, it will suffice to show that $\| T \| = \sup \{\| T \circ W \| : W \in \mathfrak{A}_p, \| W \| = 1\}$ for each $T \in \mathfrak{M}_p$.

Let then T be any multiplier in $\mathfrak{M}_p$. That $\| T \| \geqq \sup \{\| T \circ W \| : W \in \mathfrak{A}_p, \| W \| = 1\}$ is obvious. Let ε be any positive number. Choose $f \in L_p$ such that $\| f \|_p \leqq 1$ and $\| T(f) \|_p > \| T \| - \varepsilon/2$. Let $\{W_\gamma\}$ be a minimal left approximate identity for $\mathfrak{A}_p$. Then $\lim_\gamma W_\gamma = I$ in $\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)$ where I is the identity operator on L_p. By (4) we have $\lim_\gamma T \circ W_\gamma = T \circ I = T$ in $\mathfrak{R}(\mathfrak{M}_p, \mathfrak{A}_p)$. By Lemma 2 we know that $\lim_\gamma T \circ W_\gamma = T$ in the strong operator topology. In particular, there exists some index γ such that $\| T \circ W_\gamma(f) - T(f) \| < \varepsilon/2$. It follows that

$$\begin{aligned} \| T \circ W_\gamma(f) \|_p &\geqq \| T(f) \|_p - \| T(f) - T \circ W_\gamma(f) \|_p \\ &\geqq \| T \| - \varepsilon/2 - \varepsilon/2 = \| T \| - \varepsilon ; \end{aligned}$$

but $\| T \circ W_\gamma(f) \|_p \leqq \| T \circ W_\gamma \| \cdot \| f \|_p \leqq \| T \circ W_\gamma \|$, so that $\| T \circ W_\gamma \| \geqq \| T \| - \varepsilon$. Since ε was arbitrary and $\| W_\gamma \| \leqq 1$, we have shown that

$$\| T \| = \sup\{\| T \circ W \| : W \in A, \| W \| \leqq 1\}.$$

We shall identify L_p^t and $\mathfrak{A}_p$ for several particular cases.

Case I. $p = 1$. Since L_1 is a Banach algebra with 2-sided minimal approximate identity, it follows that $L_1^t = L_1$ and $\| W_f \| = \| f \|_1$ for all $f \in L_1$. Because $L_1 * C_{00}$ is dense in L_1, it follows that $\mathfrak{A}_p$ is isomorphic to L_1 as a Banach algebra. Thus, in this case, Theorem 7 is the well-known fact that a bounded linear operator on L_1 commutes with all left translation operators if and only if it commutes with all left multiplication by elements of L_1.

Case II. *G is Abelian and $p = 2$.* Let X be the character group of G and θ the Haar measure on X such that $\| \hat{f} \|_2 = \| f \|_2$ for all $f \in L_2$. In this case there is an isometric isomorphism $\hat{}\mid M_2 \rightarrow L_\infty(X)$ which is onto $L_\infty(X)$ and such that $\widehat{T(f)} = \hat{T} \cdot \hat{f}$ for all $g \in L_2$. Evidently, L_2^t is just $\{f \in L_2 : \hat{f} \in L_\infty(X)\}$. It is known that there is a net $\{g_\alpha\}$ in the set $\{\hat{f} : f \in C_{00}(G)\}$ such that $\| g_\alpha \|_\infty = 1$ for each index α and $\lim g_\alpha(\chi) = 1$ uniformly on compact subsets of X. Consequently, the set $\{\widehat{h * f} : h \in L_2^t, f \in C_{00}\}$ is dense in the set $\{g \in L_2(X) \cap L_\infty(X) : g$ vanishes at $\infty\}$. It follows that $\mathfrak{A}_2$ is isomorphic in this case to $\{f \in L_\infty(x) : f$ vanishes at $\infty\}$.

Case III. *G is compact and $p \neq 1$.* In this case L_p is a convolution algebra ([2] 28.64). Thus, $L_p^t = L_p$ and W may be viewed as a non norm-increasing linear operator from L_p into $\mathfrak{A}_p$. Since $C_{00} \subset L_p \cap L_1$, it is not difficult to show that W is an isomorphism into $\mathfrak{A}_p$.

Let $f \in L_p$ and choose a minimal approximate identity $\{f_\alpha\}$ for L_1 out of C_{00}. Then $\{f * f_\alpha\}$ converges to f in L_p. Consequently, $\{W_{f * f_\alpha}\}$ converges to W_f in $\mathfrak{A}_p$. All this shows that, in this case, $\mathfrak{A}_p$ is the closure in $\mathfrak{B}_p$ of the set $\{W_f : f \in L_p\}$.

Suppose now that G is also infinite. Then L_p has no minimal 1-sided identity (see [2] 34.40. b); since $\mathfrak{A}_p$ does have one, it follows that W is not a homeomorphism. Since W is a continuous isomorphism, the open mapping theorem implies that $W \mid L_p \rightarrow \mathfrak{A}_p$ is not onto $\mathfrak{A}_p$.

Case IV. *G is compact and $p = 2$.* Let Σ be the dual object of G as in [2]. For the spaces $\mathfrak{E}_0(\Sigma)$, $\mathfrak{E}_\infty(\Sigma)$, and $\mathfrak{E}_2(\Sigma)$ and the norms $\| \ \|_\infty$ and $\| \ \|_2$ on these spaces, see [2] 28.34. It is an easy consequence of [2] D. 54 that

$$(5) \qquad \| E \|_\infty = \sup\{\| A \circ E \|_2 : A \in \mathfrak{E}_2(\Sigma), \| A \|_2 \leqq 1\}$$

for all $E \in \mathfrak{C}_\infty(\Sigma)$. For the definition of the Fourier-Stieltjes transform $\hat{f}$ of a function $f \in L_2$, see [2] 28.34. By [2] 28.43, the mapping $\hat{\ } | L_2 \rightarrow \mathfrak{C}_2(\Sigma)$ is a surjective linear isometry and, by [2] 28.40, $\widehat{f*g} = \hat{f} \circ \hat{g}$ for all $f, g \in L_2$. Consequently, by (5),

$$\| W_f \| = \| \hat{f} \|_\infty \quad \text{for all} \quad f \in L_2 \ . \tag{6}$$

Since $C_{00} \subset L_2$, it follows from [2] 28.39, 28.27, and 28,40 that the set $\{\hat{f}: f \in L_2\}$ is a dense subspace of $\mathfrak{C}_0(\Sigma)$. Since $\mathfrak{A}_p$ is just the closure in $\mathfrak{B}_p$ of the set $\{W_f: f \in L_2\}$, it follows from (6) that $\mathfrak{A}_p$ is isomorphic to $\mathfrak{C}_0(\Sigma)$ as a Banach algebra.

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