

# Pacific Journal of Mathematics

**SEQUENCES OF QUASI-SUBORDINATE FUNCTIONS**

JAMES MILLER

## SEQUENCES OF QUASI-SUBORDINATE FUNCTIONS

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**In this paper a theorem is proved which connects bounded analytic functions in the unit disk and sequences of quasi-subordinate functions. As an application a necessary and sufficient condition for certain sequences of quasi-subordinate functions to converge is found.**

Let  $f$  and  $F$  be analytic functions in  $|z| < R$ . If there exist two functions  $\phi$  and  $\omega$  which are analytic in  $|z| < R$  and satisfy  $\omega(0) = 0$ ,  $|\phi(z)| \leq 1$ ,  $|\omega(z)| < R$ , and  $f(z) = \phi(z)F(\omega(z))$  for  $|z| < R$ , then we say that  $f$  is quasi-subordinate to  $F$  in  $|z| < R$  and write  $f <_q F$ . Without loss of generality we may assume that  $R = 1$ . This class was introduced by Robertson [2, 3].

We note that there are two special cases of quasi-subordination which are of interest: If  $\phi$  is the constant function one, then  $f$  is subordinate to  $F$ , and on the other hand, if  $\omega$  is the identity function, then  $f$  is majorized by  $F$ .

Let  $B$  denote the class of functions  $\theta$  which are analytic in  $|z| < 1$  and satisfy  $|\theta(z)| \leq 1$  for  $|z| < 1$ . Then the functions  $\phi$  and  $\omega$  which are defined above are elements of  $B$ . In this paper we prove a theorem which connects functions in  $B$  and sequences of quasi-subordinate functions. As an application we find necessary and sufficient conditions for certain sequences of quasi-subordinate functions to converge. This is a generalization of Pommerenke's results [1] on sequences of subordinate functions.

Let  $\{f_n\}$ ,  $n = 1, 2, \dots$ , be a sequence of functions which are analytic in  $|z| < 1$  such that  $f_n <_q f_{n+1}$  for each  $n$  or  $f_{n+1} <_q f_n$  for each  $n$ . When considering the convergence of such sequences we need to require that either the sequence  $\{f_n(0)\}$  converges or the functions agree at a single point. In this paper we shall assume that the functions agree at a single point. Further we may assume that the point is  $z = 0$  for if the functions  $f_n$  agree at the point  $a \neq 0$  then we could consider the functions  $g_n(z) = f_n((z-a)/(1-az))$ . We will use  $f_n(0) = 0$  for all  $n$ , otherwise the function  $\phi$  would be identically one. The proof for the case where  $\{f_n(0)\}$  is convergent is similar.

**THEOREM 1.** *Let  $\{f_n\}$  be a sequence of functions which are analytic in  $|z| < 1$  and satisfy  $f_n(0) = 0$ ,  $\alpha_n = f'_n(0) \neq 0$ , and  $f_n(z) <_q f_{n+1}$ , and let  $\phi_{n+1}, \omega_{n+1} \in B$  and  $\omega_{n+1}(0) = 0$  be such that*

$$f_n(z) = \phi_{n+1}(z)f_{n+1}(\omega_{n+1}(z))$$

for  $|z| < 1$ . If  $\sum_{n=2}^{\infty} \arg \phi_n(0)$  converges and  $\lim_{n \rightarrow \infty} \alpha_n = \alpha, |\alpha| < \infty$ , then  $\prod_{n=2}^{\infty} \phi_n(0)$  converges.

*Proof.* We observe that if  $m < n$ , then we have  $f_m <_q f_n$ . Thus for  $m < n$  there are functions  $\phi_{mn}, \omega_{mn} \in B$  where  $\omega_{mn}(0) = 0$  such that

$$f_m(z) = \phi_{mn}(z)f_n(\omega_{mn}(z))$$

for  $|z| < 1$ . Let  $\phi_{n+1}(z) = \phi_{n+1}(z)$ . We now observe that

$$f'_m(0) = \phi_{mn}(0)\omega'_{mn}(0)f'_n(0)$$

or

$$(1) \quad \alpha_m = \phi_{mn}(0)\omega'_{mn}(0)\alpha_n.$$

Since  $0 < |\alpha_m| \leq |\alpha_n|$  for  $m < n$  and  $\alpha_n \rightarrow \alpha$ , there exists an integer  $K$  such that if  $n > m > K$ , then

$$(2) \quad \left| \frac{\alpha_m}{\alpha_n} - 1 \right| < \varepsilon.$$

From (1) and (2) we see that

$$1 - \varepsilon < \left| \frac{\alpha_m}{\alpha_n} \right| = |\phi_{mn}(0)\omega'_{mn}(0)| \leq |\phi_{mn}(0)| \leq 1.$$

We now observe that

$$\phi_{mn}(0) = \prod_{k=m+1}^n \phi_k(0).$$

Thus we have

$$1 - \varepsilon < \left| \prod_{k=m+1}^n \phi_k(0) \right| \leq 1$$

for  $n > m > K$ . Since  $\sum_{n=2}^{\infty} \arg \phi_n(0)$  converges this says that  $\prod_{k=2}^{\infty} \phi_k(0)$  converges. Further we have that  $\omega'_n(0) \rightarrow 1$  and  $\omega'_{mn}(0) = 1$ .

In applying Theorem 1 to sequences of quasi-subordinate functions we will also need two lemmas for functions in  $B$ . The proofs of the lemmas are essentially in [1].

**LEMMA 1.** *Let  $\phi \in B, \phi(0) = 0$ , and satisfy  $|\phi(0)| \geq \sigma > 0$ . Then the mapping  $w = \phi(z)$  maps the disk*

$$|z| < \rho = \frac{\sigma}{1 + \sqrt{1 - \sigma^2}}$$

univalently onto a region that contains  $|w| < \rho^2$ .

LEMMA 2. For  $\varepsilon > 0$  and  $0 < r < 1$ , there exists an  $\eta > 0$  ( $\eta(\varepsilon, r)$ ) such that if  $\phi \in B$  satisfies  $\phi(z) = \sum_{n=0}^{\infty} \beta_n z^n$  and  $|\beta_k - 1| \leq \eta$ , then

$$|\phi(z) - z^k| < \varepsilon, \quad \text{for } |z| < r.$$

THEOREM 2. Let  $\{f_n\}$  be a sequence of analytic functions in  $|z| < 1$  such that  $f_n(0) = 0$ ,  $f_n \prec_q f_{n+1}$ , and  $\alpha_n = f'_n(0) \neq 0$ , and let  $\phi_{n+1}, \omega_{n+2} \in B$  and  $\omega_{n+1}(0) = 0$  be such that  $f'_n(z) = \phi_{n+1}(z)f'_{n+1}(\omega_{n+1}(z))$  for  $|z| < 1$  and  $\sum_{n=2}^{\infty} \arg \phi_n(0)$  converges. Then the sequence  $\{f_n\}$  converges uniformly in  $|z| < r$  for every  $0 \leq r < 1$  if and only if

$$\lim_{n \rightarrow \infty} \alpha_n = \alpha, \quad |\alpha| < \infty.$$

PROOF. If  $\{f_n\}$  converges uniformly in  $|z| \leq r$  for every  $0 < r < 1$  then  $\alpha_n = f'_n(0)$  converges. Further since  $|\alpha_n| \leq |\alpha_{n+1}|$ ,  $f_n(0) = 0$ , and  $\alpha_n \neq 0$  we see that  $\lim_{n \rightarrow \infty} \alpha_n = \alpha \neq 0$  and  $|\alpha| < \infty$ .

Let  $\omega_{n+1}, \phi_{n+1} \in B$ , and  $\omega_{n+1}(0) = 0$  be as defined in Theorem 2. Further for  $m < n$ , let  $\phi_{mn}, \omega_{mn} \in B$  with  $\omega_{mn}(0) = 0$  be such that

$$(3) \quad f_m(z) = \phi_{mn}(z)f'_n(\omega_{mn}(z)).$$

Suppose that  $\alpha_n \rightarrow \alpha, |\alpha| < \infty$ . Then by Theorem 1 the product  $\prod_{k=2}^{\infty} \phi_k(0)$  converges. We will first show that  $\{f_n\}$  is a normal family in  $|z| < 1$ .

Let  $r, 0 < r < 1$ , be fixed and  $\sigma$  determined by

$$\sqrt{r} = \frac{\sigma}{1 + \sqrt{1 - \sigma^2}}.$$

Since  $\sigma < 1$  and  $\alpha_n \rightarrow \alpha \neq 0$ , there exists an integer  $N_1$  such that

$$\left| \frac{\alpha_m}{\alpha_n} \right| > \sigma, \quad \text{for } n > m > N_1.$$

Further, since  $|\phi_{mn}(z)| \leq 1$ , we have  $|\phi_{mn}(0)|^{-1} \geq 1$ . For  $n > m > N_1$  we have  $\omega'_{mn}(0) = \alpha_m/(\alpha_n \phi_{mn}(0))$  or

$$(4) \quad |\omega'_{mn}(0)| = \left| \frac{1}{\phi_{mn}(0)} \frac{\alpha_m}{\alpha_n} \right| > \sigma.$$

Thus by Lemma 1 the mapping  $\zeta = \omega_{mn}(z)$  for  $n < m < N_1$  maps  $|z| < \sqrt{r}$  univalently onto a domain that contains  $|\zeta| < r$ . Let  $\psi_{mn}$  be the inverse of  $\zeta = \omega_{mn}(z)$  in  $|\zeta| < r$ , then

$$|\psi_{mn}(\zeta)| \leq \sqrt{r}.$$

From (3) we may write

$$f_n(\zeta) = \frac{1}{\phi_{mn}(\psi_{mn}(\zeta))} f_m(\psi_{mn}(\zeta)), \quad \text{for } |\zeta| < r.$$

For  $|\zeta| \leq r$  we have

$$|f_n(\zeta)| \leq \max_{|z| \leq \sqrt{r}} \left| \frac{f_m(z)}{\phi_{mn}(z)} \right| \leq \frac{1}{\min_{|z| \leq \sqrt{r}} |\phi_{mn}(z)|} \max_{|z| \leq \sqrt{r}} |f_m(z)|.$$

From Lemma 2 with  $k = 0$ , given  $\varepsilon > 0$ , there exists an  $\eta$  such that if  $|\beta_0 - 1| < \eta$  then  $|\phi(z) - 1| < \varepsilon$  for  $|z| < r$ . Since  $\prod_{k=2}^{\infty} \phi_k(0)$  converges by Theorem 1 and  $\phi_{mn}(0) = \prod_{k=m+1}^n \phi_k(0)$ , there exists an integer  $N_2$  such that if  $n > m > N_2$  then  $|\phi_{mn}(0) - 1| < \eta$ . Let  $N = \max(N_1, N_2)$ . Thus, by Lemma 2 we have that  $|\phi_{mn}(z) - 1| < \varepsilon$  for  $|z| \leq r$  and  $n > m > N$  or

$$\min_{|z| \leq r} |\phi_{mn}(z)| \geq 1 - \varepsilon.$$

Hence, for  $n > N$  and  $|\zeta| \leq r$  we have

$$|f_n(\zeta)| \leq \frac{1}{1 - \varepsilon} \max_{|z| \leq \sqrt{r}} |f_{N+1}(z)|.$$

Thus there exists  $M(r)$  such that

$$(5) \quad |f_n(z)| \leq M(r)$$

for all  $n$ , that is,  $\{f_n\}$  is locally uniformly bounded. Therefore  $\{f_n\}$  is normal.

Let  $\{f_{n_\nu}\}$  be a subsequence of  $\{f_n\}$  which is uniformly convergent in  $|z| \leq r_0$ , for every  $r_0 < 1$ . Let  $f$  be the limit function of  $\{f_{n_\nu}\}$ . Let  $\varepsilon > 0$  and  $r < 1$ . Then choose  $\nu_0$  such that

$$|f_{n_\nu}(z) - f(z)| < \varepsilon/3$$

for  $\nu \geq \nu_0$  and  $|z| \leq r$ . From inequality (5) we have that the sequence  $\{f_n\}$  is bounded in  $|z| \leq r$  and thus equicontinuous in  $|z| \leq r$ . Therefore there exists a  $\delta > 0$  such that

$$|f_n(z_1) - f_n(z_2)| < \varepsilon/3$$

for  $|z_1 - z_2| < \delta$ ,  $|z_1| \leq r + \delta$ ,  $|z_2| \leq r + \delta$ , and for all  $n$ .

Using (4), the convergence of  $\sum_{n=2}^{\infty} \arg \phi_n(0)$ , and applying Lemma 2 we have that there exists an integer  $M_1$  such that if  $n \geq m \geq M_1$ , then

$$|\omega_{mn}(z) - z| < \delta, \quad \text{for } |z| \leq r$$

where  $M_1$  is chosen so that  $|\omega'_{mn}(0) - 1| < \eta$  for a suitable  $\eta$ . Again making use of Lemma 2 we have that there exists an integer  $M_2$  such that if  $n > m > M_2$  then

$$|\phi_{mn}(z) - 1| < \varepsilon/3M(r), \quad \text{for } |z| < r.$$

Let  $M = \max\{M_1, M_2, n_{\nu_0}\}$ . If  $M \leq k < n_{\nu}$  and  $|z| < r$  then

$$\begin{aligned} |f_k(z) - f(z)| &\leq |f_k(z) - f_{n_{\nu}}(z)| + |f_{n_{\nu}}(z) - f(z)| \\ &< \varepsilon/3 + |f_{n_{\nu}}(z) - \phi_{kn_{\nu}}(z)f_{n_{\nu}}(\omega_{kn_{\nu}}(z))| \\ &\leq \varepsilon/3 + |f_{n_{\nu}}(z) - f_{n_{\nu}}(\omega_{kn_{\nu}}(z))| \\ &\quad + |f_{n_{\nu}}(\omega_{kn_{\nu}}(z)) [1 - \phi_{kn_{\nu}}(z)]| \\ &< \varepsilon/3 + \varepsilon/3 + M(r) \varepsilon/3M(r) = \varepsilon \end{aligned}$$

for  $|z| \leq r$  and  $k > M$ . This completes the proof of Theorem 2.

**THEOREM 3.** *Let  $\{f_n\}$  be a sequence of functions analytic in  $|z| < 1$  such that  $f_n(0) = 0$ ,  $\alpha_n = f'_n(0) \neq 0$ , and  $f_{n+1} \prec_q f_n$ , and let  $\phi_{n+1}, \omega_{n+1} \in B$  and  $\omega_{n+1}(0) = 0$  be such that*

$$f_{n+1}(z) = \phi_{n+1}(z)f_n(\omega_{n+1}(z))$$

for  $|z| < 1$  and  $\sum_{n=2}^{\infty} \arg \phi_n(0)$  converges. Then the sequence  $\{f_n\}$  converges uniformly in  $|z| \leq r$  for every  $r < 1$  if the sequence  $\{\alpha_n\}$  converges. The limit function is constant if and only if

$$\lim_{n \rightarrow \infty} \alpha_n = 0.$$

The proof of this theorem is similar to that of Theorem 2 and Pommerenke's Theorem 2 [1].

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