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SEQUENCES OF QUASI-SUBORDINATE FUNCTIONS

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In this paper a theorem is proved which connects bounded analytic functions in the unit disk and sequences of quasi-subordinate functions. As an application a necessary and sufficient condition for certain sequences of quasi-subordinate functions to converge is found.

Let f and F be analytic functions in $|z| < R$. If there exist two functions ϕ and ω which are analytic in $|z| < R$ and satisfy $\omega(0) = 0$, $|\phi(z)| \leq 1$, $|\omega(z)| < R$, and $f(z) = \phi(z)F(\omega(z))$ for $|z| < R$, then we say that f is quasi-subordinate to F in $|z| < R$ and write $f <_q F$. Without loss of generality we may assume that $R = 1$. This class was introduced by Robertson [2, 3].

We note that there are two special cases of quasi-subordination which are of interest: If ϕ is the constant function one, then f is subordinate to F , and on the other hand, if ω is the identity function, then f is majorized by F .

Let B denote the class of functions θ which are analytic in $|z| < 1$ and satisfy $|\theta(z)| \leq 1$ for $|z| < 1$. Then the functions ϕ and ω which are defined above are elements of B . In this paper we prove a theorem which connects functions in B and sequences of quasi-subordinate functions. As an application we find necessary and sufficient conditions for certain sequences of quasi-subordinate functions to converge. This is a generalization of Pommerenke's results [1] on sequences of subordinate functions.

Let $\{f_n\}$, $n = 1, 2, \dots$, be a sequence of functions which are analytic in $|z| < 1$ such that $f_n <_q f_{n+1}$ for each n or $f_{n+1} <_q f_n$ for each n . When considering the convergence of such sequences we need to require that either the sequence $\{f_n(0)\}$ converges or the functions agree at a single point. In this paper we shall assume that the functions agree at a single point. Further we may assume that the point is $z = 0$ for if the functions f_n agree at the point $a \neq 0$ then we could consider the functions $g_n(z) = f_n((z-a)/(1-az))$. We will use $f_n(0) = 0$ for all n , otherwise the function ϕ would be identically one. The proof for the case where $\{f_n(0)\}$ is convergent is similar.

THEOREM 1. *Let $\{f_n\}$ be a sequence of functions which are analytic in $|z| < 1$ and satisfy $f_n(0) = 0$, $\alpha_n = f'_n(0) \neq 0$, and $f_n(z) <_q f_{n+1}$, and let $\phi_{n+1}, \omega_{n+1} \in B$ and $\omega_{n+1}(0) = 0$ be such that*

$$f_n(z) = \phi_{n+1}(z)f_{n+1}(\omega_{n+1}(z))$$

for $|z| < 1$. If $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges and $\lim_{n \rightarrow \infty} \alpha_n = \alpha, |\alpha| < \infty$, then $\prod_{n=2}^{\infty} \phi_n(0)$ converges.

Proof. We observe that if $m < n$, then we have $f_m \prec_q f_n$. Thus for $m < n$ there are functions $\phi_{mn}, \omega_{mn} \in B$ where $\omega_{mn}(0) = 0$ such that

$$f_m(z) = \phi_{mn}(z)f_n(\omega_{mn}(z))$$

for $|z| < 1$. Let $\phi_{n+1}(z) = \phi_{n+1}(z)$. We now observe that

$$f'_m(0) = \phi_{mn}(0)\omega'_{mn}(0)f'_n(0)$$

or

$$(1) \quad \alpha_m = \phi_{mn}(0)\omega'_{mn}(0)\alpha_n.$$

Since $0 < |\alpha_m| \leq |\alpha_n|$ for $m < n$ and $\alpha_n \rightarrow \alpha$, there exists an integer K such that if $n > m > K$, then

$$(2) \quad \left| \frac{\alpha_m}{\alpha_n} - 1 \right| < \varepsilon.$$

From (1) and (2) we see that

$$1 - \varepsilon < \left| \frac{\alpha_m}{\alpha_n} \right| = |\phi_{mn}(0)\omega'_{mn}(0)| \leq |\phi_{mn}(0)| \leq 1.$$

We now observe that

$$\phi_{mn}(0) = \prod_{k=m+1}^n \phi_k(0).$$

Thus we have

$$1 - \varepsilon < \left| \prod_{k=m+1}^n \phi_k(0) \right| \leq 1$$

for $n > m > K$. Since $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges this says that $\prod_{k=2}^{\infty} \phi_k(0)$ converges. Further we have that $\omega'_n(0) \rightarrow 1$ and $\omega'_{mn}(0) = 1$.

In applying Theorem 1 to sequences of quasi-subordinate functions we will also need two lemmas for functions in B . The proofs of the lemmas are essentially in [1].

LEMMA 1. *Let $\phi \in B, \phi(0) = 0$, and satisfy $|\phi(0)| \geq \sigma > 0$. Then the mapping $w = \phi(z)$ maps the disk*

$$|z| < \rho = \frac{\sigma}{1 + \sqrt{1 - \sigma^2}}$$

univalently onto a region that contains $|w| < \rho^2$.

LEMMA 2. For $\varepsilon > 0$ and $0 < r < 1$, there exists an $\eta > 0$ ($\eta(\varepsilon, r)$) such that if $\phi \in B$ satisfies $\phi(z) = \sum_{n=0}^{\infty} \beta_n z^n$ and $|\beta_k - 1| \leq \eta$, then

$$|\phi(z) - z^k| < \varepsilon, \quad \text{for } |z| < r.$$

THEOREM 2. Let $\{f_n\}$ be a sequence of analytic functions in $|z| < 1$ such that $f_n(0) = 0$, $f_n \prec_q f_{n+1}$, and $\alpha_n = f'_n(0) \neq 0$, and let $\phi_{n+1}, \omega_{n+2} \in B$ and $\omega_{n+1}(0) = 0$ be such that $f_n(z) = \phi_{n+1}(z)f_{n+1}(\omega_{n+1}(z))$ for $|z| < 1$ and $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges. Then the sequence $\{f_n\}$ converges uniformly in $|z| < r$ for every $0 \leq r < 1$ if and only if

$$\lim_{n \rightarrow \infty} \alpha_n = \alpha, \quad |\alpha| < \infty.$$

PROOF. If $\{f_n\}$ converges uniformly in $|z| \leq r$ for every $0 < r < 1$ then $\alpha_n = f'_n(0)$ converges. Further since $|\alpha_n| \leq |\alpha_{n+1}|$, $f_n(0) = 0$, and $\alpha_n \neq 0$ we see that $\lim_{n \rightarrow \infty} \alpha_n = \alpha \neq 0$ and $|\alpha| < \infty$.

Let $\omega_{n+1}, \phi_{n+1} \in B$, and $\omega_{n+1}(0) = 0$ be as defined in Theorem 2. Further for $m < n$, let $\phi_{mn}, \omega_{mn} \in B$ with $\omega_{mn}(0) = 0$ be such that

$$(3) \quad f_m(z) = \phi_{mn}(z)f_n(\omega_{mn}(z)).$$

Suppose that $\alpha_n \rightarrow \alpha, |\alpha| < \infty$. Then by Theorem 1 the product $\prod_{k=2}^{\infty} \phi_k(0)$ converges. We will first show that $\{f_n\}$ is a normal family in $|z| < 1$.

Let $r, 0 < r < 1$, be fixed and σ determined by

$$\sqrt{r} = \frac{\sigma}{1 + \sqrt{1 - \sigma^2}}.$$

Since $\sigma < 1$ and $\alpha_n \rightarrow \alpha \neq 0$, there exists an integer N_1 such that

$$\left| \frac{\alpha_m}{\alpha_n} \right| > \sigma, \quad \text{for } n > m > N_1.$$

Further, since $|\phi_{mn}(z)| \leq 1$, we have $|\phi_{mn}(0)|^{-1} \geq 1$. For $n > m > N_1$ we have $\omega'_{mn}(0) = \alpha_m / (\alpha_n \phi_{mn}(0))$ or

$$(4) \quad |\omega'_{mn}(0)| = \left| \frac{1}{\phi_{mn}(0)} \frac{\alpha_m}{\alpha_n} \right| > \sigma.$$

Thus by Lemma 1 the mapping $\zeta = \omega_{mn}(z)$ for $n < m < N_1$ maps $|z| < \sqrt{r}$ univalently onto a domain that contains $|\zeta| < r$. Let ψ_{mn} be the inverse of $\zeta = \omega_{mn}(z)$ in $|\zeta| < r$, then

$$|\psi_{mn}(\zeta)| \leq \sqrt{r}.$$

From (3) we may write

$$f_n(\zeta) = \frac{1}{\phi_{mn}(\psi_{mn}(\zeta))} f_m(\psi_{mn}(\zeta)), \quad \text{for } |\zeta| < r.$$

For $|\zeta| \leq r$ we have

$$|f_n(\zeta)| \leq \max_{|z| \leq \sqrt{r}} \left| \frac{f_m(z)}{\phi_{mn}(z)} \right| \leq \frac{1}{\min_{|z| \leq \sqrt{r}} |\phi_{mn}(z)|} \max_{|z| \leq \sqrt{r}} |f_m(z)|.$$

From Lemma 2 with $k = 0$, given $\varepsilon > 0$, there exists an η such that if $|\beta_0 - 1| < \eta$ then $|\phi(z) - 1| < \varepsilon$ for $|z| < r$. Since $\prod_{k=2}^{\infty} \phi_k(0)$ converges by Theorem 1 and $\phi_{mn}(0) = \prod_{k=m+1}^n \phi_k(0)$, there exists an integer N_2 such that if $n > m > N_2$ then $|\phi_{mn}(0) - 1| < \eta$. Let $N = \max(N_1, N_2)$. Thus, by Lemma 2 we have that $|\phi_{mn}(z) - 1| < \varepsilon$ for $|z| \leq r$ and $n > m > N$ or

$$\min_{|z| \leq r} |\phi_{mn}(z)| \geq 1 - \varepsilon.$$

Hence, for $n > N$ and $|\zeta| \leq r$ we have

$$|f_n(\zeta)| \leq \frac{1}{1 - \varepsilon} \max_{|z| \leq \sqrt{r}} |f_{N+1}(z)|.$$

Thus there exists $M(r)$ such that

$$(5) \quad |f_n(z)| \leq M(r)$$

for all n , that is, $\{f_n\}$ is locally uniformly bounded. Therefore $\{f_n\}$ is normal.

Let $\{f_{n_\nu}\}$ be a subsequence of $\{f_n\}$ which is uniformly convergent in $|z| \leq r_0$, for every $r_0 < 1$. Let f be the limit function of $\{f_{n_\nu}\}$. Let $\varepsilon > 0$ and $r < 1$. Then choose ν_0 such that

$$|f_{n_\nu}(z) - f(z)| < \varepsilon/3$$

for $\nu \geq \nu_0$ and $|z| \leq r$. From inequality (5) we have that the sequence $\{f_n\}$ is bounded in $|z| \leq r$ and thus equicontinuous in $|z| \leq r$. Therefore there exists a $\delta > 0$ such that

$$|f_n(z_1) - f_n(z_2)| < \varepsilon/3$$

for $|z_1 - z_2| < \delta$, $|z_1| \leq r + \delta$, $|z_2| \leq r + \delta$, and for all n .

Using (4), the convergence of $\sum_{n=2}^{\infty} \arg \phi_n(0)$, and applying Lemma 2 we have that there exists an integer M_1 such that if $n \geq m \geq M_1$, then

$$|\omega_{mn}(z) - z| < \delta, \quad \text{for } |z| \leq r$$

where M_1 is chosen so that $|\omega'_{mn}(0) - 1| < \eta$ for a suitable η . Again making use of Lemma 2 we have that there exists an integer M_2 such that if $n > m > M_2$ then

$$|\phi_{mn}(z) - 1| < \varepsilon/3M(r), \quad \text{for } |z| < r.$$

Let $M = \max\{M_1, M_2, n_0\}$. If $M \leq k < n_\nu$ and $|z| < r$ then

$$\begin{aligned} |f_k(z) - f(z)| &\leq |f_k(z) - f_{n_\nu}(z)| + |f_{n_\nu}(z) - f(z)| \\ &< \varepsilon/3 + |f_{n_\nu}(z) - \phi_{kn_\nu}(z)f_{n_\nu}(\omega_{kn_\nu}(z))| \\ &\leq \varepsilon/3 + |f_{n_\nu}(z) - f_{n_\nu}(\omega_{kn_\nu}(z))| \\ &\quad + |f_{n_\nu}(\omega_{kn_\nu}(z)) [1 - \phi_{kn_\nu}(z)]| \\ &< \varepsilon/3 + \varepsilon/3 + M(r) \varepsilon/3M(r) = \varepsilon \end{aligned}$$

for $|z| \leq r$ and $k > M$. This completes the proof of Theorem 2.

THEOREM 3. *Let $\{f_n\}$ be a sequence of functions analytic in $|z| < 1$ such that $f_n(0) = 0$, $\alpha_n = f'_n(0) \neq 0$, and $f_{n+1} \prec_q f_n$, and let ϕ_{n+1} , $\omega_{n+1} \in B$ and $\omega_{n+1}(0) = 0$ be such that*

$$f_{n+1}(z) = \phi_{n+1}(z)f_n(\omega_{n+1}(z))$$

for $|z| < 1$ and $\sum_{n=2}^\infty \arg \phi_n(0)$ converges. Then the sequence $\{f_n\}$ converges uniformly in $|z| \leq r$ for every $r < 1$ if the sequence $\{\alpha_n\}$ converges. The limit function is constant if and only if

$$\lim_{n \rightarrow \infty} \alpha_n = 0.$$

The proof of this theorem is similar to that of Theorem 2 and Pommerenke's Theorem 2 [1].

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