THE HASSE-WITT-MATRIX OF SPECIAL PROJECTIVE VARIETIES

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The Hasse-Witt-matrix of a projective hypersurface defined over a perfect field \( k \) of characteristic \( p \) is studied using an explicit description of the Cartier-operator. We get the following applications. If \( L \) is a linear variety of dimension \( n + 1 \) and \( X \) a generic hypersurface of degree \( d \), which divides \( p - 1 \), then the Frobenius-operator \( \mathcal{F} \) on \( H^i(X \cdot L; O_{X \cdot L}) \) is invertible.

As another application we prove the invertibility of the Hasse-Witt-matrix for the generic curve of genus two. We don't study the Frobenius \( \mathcal{F} \) directly, but the Cartier-operator [1]. It is well-known, that for curves Frobenius and Cartier-operator are dual to each other under the duality of the Riemann-Roch theorem. A similar fact is true for higher dimension via Serre duality. We have therefore to extend to the whole “De Rham” ring the description of the Cartier-operator given in [4] for 1-forms. We give this extension in §1. Diagonal hypersurfaces are studied in §2 and the invertibility of the Hasse-Witt-matrix is proved, if the degree divides \( p - 1 \). The same theorem for the generic hypersurface follows then from the semicontinuity of the matrix rank. The §3 is devoted to hyperelliptic curves and is intended as a preparation for a detailed study of curves of genus two.

1. The Cartier-operator of a projective hypersurface. We extend the explicit construction of the Cartier-operator given in [4] to the whole “De Rham” ring, but restrict ourself to projective hypersurfaces.

As an application we show: Let \( V \) be a projective hypersurface of dimension \( n - 1 \), defined by a diagonal equation \( F(X) = \sum_{i=0}^r a_i X_i \), \( a_i \in k \) a perfect field of char \( k = p > 0, a_i \neq 0 \). Let \( X \) be a linear variety of dimension \( t + 1 \). If \( r \) divides \( p - 1 \), then

\[
\mathcal{F} : H^i(X \cdot V, O_{X \cdot V}) \to H^i(X \cdot V, O_{X \cdot V})
\]

is invertible, \( \mathcal{F} \) being the induced Frobenius endomorphism. We have to rely on a technical proposition, which is a collection of some lemmas in [4]. We give first the proposition.

**Proposition 1.** Let

\[
\psi : k[T] \to k[T] \quad (T = (T_1, \cdots, T_n))
\]
be \( k \ p^{-1} \)-linear and

\[
\psi(T^\mu) = \begin{cases} 
T^\mu & \text{if } \mu = p \cdot \nu \\
0 & \text{else} 
\end{cases}.
\]

Then the following holds:

1. \( \psi(T_{\mu_1} \cdots T_{\mu_h}) = T_{\mu_1} \cdots T_{\mu_h}, \) for some \( \mu \in k[T] \)
2. Let \( D_{\mu} = T_{\mu} \frac{\partial}{\partial T_{\mu}} \) and \( D_{\mu}g = 0 \) for a given \( 1 \leq \mu \leq n, \) then
   \( \psi(D_{\mu}h \cdot g) = 0 \)
3. Let \( D_{\mu}g = 0, \) then \( \psi(h^{p-1}D_{\mu}h \cdot g) = D_{\mu}h \psi(g). \)

Proof.

1. By the \( p^{-1} \)-linearity of \( \psi \) we may assume \( h \) to be a monomial. The statement follows then directly from the definition of \( \psi. \)

2. \( \psi \) is \( p^{-1} \)-linear, so we may assume \( h \) to be a monomial
   \[ h = T_{r_1} \cdots T_{r_n}, \quad 0 \leq r_i \leq p - 1 \]
   (say \( \mu = n \)), then \( D_{\mu}h = r_n \cdot h. \) If \( r_n = 0 \) then (2) is trivially true. So \( r_n \neq 0. \) Again because of \( p^{-1} \)-linearity we may also assume \( g \) to be monomial.
   
   But \( D_{\mu}g = 0, \) so
   \[ g = T_{v_1} \cdots T_{v_n}^{p-1}, \quad 0 \leq v_i \leq p - 1. \]
   
   So the exponent of \( T_n \) in \( D_{\mu}h \cdot g \) is \( r_n \) and \( 0 < r_n \leq p - 1, \) therefore not divisible by \( p. \) The definition of \( \psi \) gives
   \[ \psi(D_{\mu}h \cdot g) = 0. \]

3. We may write
   \[ h = f_0 + f_1 \cdot T_\mu + \cdots + f_r \cdot T_\mu^r, \quad 0 \leq r \leq p - 1 \]
   and
   \[ D_{\mu}f_i = 0. \]
   
   We proceed by induction on \( T. \) \( r = 0 \) clear. Let \( r \geq 1, \) then \( h = f + T_\mu \bar{h} \) with \( D_{\mu}f = 0 \ deg f, \bar{h} < r. \) Now
   \[ T_{\mu}^{p-1} \bar{h}^{p-1} D_{\mu}(T_\mu \bar{h}) = (T_\mu \bar{h})^p \left( \frac{D_{\mu}T_\mu}{T_\mu} + \frac{D_{\mu}\bar{h}}{\bar{h}} \right). \]
   
   By \( p^{-1} \)-linearity of \( \psi \) and induction assumption for \( \bar{h} \) we get
   \[
   \psi(g \cdot T_{\mu}^{p-1} \bar{h}^{p-1} D_{\mu}(T_\mu \bar{h})) = T_\mu \bar{h} \psi(g) + T_\mu \psi(g \cdot \bar{h}^{p-1} D\bar{h}) 
   = \psi(g)(T_\mu \bar{h} + T_\mu D_{\mu} \bar{h}) 
   = D_{\mu}(T_\mu \bar{h}) \cdot \psi(g). \]
On the other hand

\[ T_{\mu}^p h^p = (h - f)^p = h^p + \frac{\partial P}{\partial h}, \]

where \( P \) is a polynomial in \( f \) and \( h \). We have

\[ D_{\mu}(T_{\mu}h) = D_{\mu}(h - f) = D_{\mu}h. \]

So

\[ T_{\mu}^p h^p D_{\mu}(T_{\mu}h) = h^p D_{\mu}h + D_{\mu}P. \]

Multiply by \( g \) and apply \( \psi \), then one gets

\[ D_{\mu}h \psi(g) = D_{\mu}(T_{\mu}h)\psi(g) = \psi(h^p D_{\mu}h \cdot g) + \psi(D_{\mu}P \cdot g). \]

But by (2)

\[ \psi(D_{\mu}P \cdot g) = 0. \]

Let \( F(X_1 \cdot \cdot \cdot X_n) \) define a absolutely irreducible hypersurface \( V/k \) in \( \mathcal{P}_{k,k} \) char \( k = p > 0 \). We denote by \( f(X_1 \cdot \cdot \cdot X_n) \) an affinization of \( F \). Let \( F_{\mu} = (\partial/\partial X_{\mu}) F \), similar \( 1 \leq \mu \leq n \). We assume \( f_n \) not to be the zero function on \( V \). Let \( K = K(V) \) be the function field of \( V \). We assume that \( K = K^p(x_1 \cdot \cdot \cdot x_j \cdot \cdot \cdot x_n) \) for any index \( j \). The \( x_i \) are the coordinate functions and \( \bar{x}_j \) means omit \( x_j \). As a consequence of these assumptions, we have that for a given index \( j \) any function \( z \in K \) can be represented modulo \( F \) by a rational function \( G(X_1 \cdot \cdot \cdot X_n) \), which is \( X_j \)-constant, i.e. such that \( \partial G/\partial X_j = 0 \). Write

\[ F_{i_1 \cdot \cdot \cdot i_r} = (X_{i_1} \cdot \cdot \cdot X_{i_r} \cdot X_n)^{-1} F. \]

**Definition 1.** Let

\[ \psi_{F_{i_1 \cdot \cdot \cdot i_r}} = F_{i_1 \cdot \cdot \cdot i_r} \cdot F_{i_1 \cdot \cdot \cdot i_r}. \]

Let \( \omega = \sum_{i_1 \cdot \cdot \cdot i_r} h_{i_1 \cdot \cdot \cdot i_r} \cdot dx_{i_1} \wedge \cdot \cdot \cdot \wedge dx_{i_r} \) be \( r \)-form on \( V \). Put

\[ \omega_{i_1 \cdot \cdot \cdot i_r} = \frac{dx_{i_1} \wedge \cdot \cdot \cdot \wedge dx_{i_r}}{f_n}. \]

Define

\[ C(\omega) = \sum_{i_1 \cdot \cdot \cdot i_r} \psi_{F_{i_1 \cdot \cdot \cdot i_r}}(h_{i_1 \cdot \cdot \cdot i_r} - f_n) \omega_{i_1 \cdot \cdot \cdot i_r}. \]

The definition is justified by the following theorem.

**Theorem 1.** (1) \( C \) is \( p^{-1} \)-linear

(2) If \( \omega = d\varphi \), then \( C(\omega) = 0 \)
If \( \omega = z_{i_1}^{r-1} \cdots z_{i_r}^{r-1} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \), then \( C(\omega) = dz_{i_1} \wedge \cdots \wedge dz_{i_r} \).

In other words, if one restricts \( C \) to \( Z_{r|k} \), the closed forms, then

\[
C: Z_{r|k} \rightarrow \Omega_{r|k}
\]

is the Cartier-operator of \( V \) [1].

**Proof of the theorem.**

1. The \( p \)-linearity follows from the \( p \)-linearity of \( \psi \).
2. Let \( \varphi = \sum_{i_1, \ldots, i_{r-1}} \varphi_{i_1, \ldots, i_{r-1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{r-1}} \) be a \((r-1)\)-form, then

\[
d\varphi = \sum_{j} \sum_{i_1, \ldots, i_{r-1}} \frac{\partial}{\partial x_j} (\varphi_{i_1, \ldots, i_{r-1}}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{r-1}}.
\]

To simplify the notation we put for the moment

\[
\varphi_{i_1, \ldots, i_{r-1}} = \tilde{\varphi}
\]

and

\[
F_{i_1, \ldots, i_{r-1}} = \tilde{F}.
\]

To compute \( C(d\varphi) \) we have to compute

\[
\varphi_{\tilde{T}} \left( \frac{\partial}{\partial x_j} \tilde{\varphi} \cdot f_{m} \right)
\]

for every system \((j, i_1, \ldots, i_{r-1})\).

Now remembering the definition of \( \varphi^{\tilde{T}} \) we have to show

\[
\varphi(F^{p-1}D_xFX_{i_1} \cdots X_{i_{r-1}}D_j\varphi) = 0
\]

in order to get \( C(d\varphi) = 0 \).

We have to use the above proposition. We apply first (3) and then (2) and get:

\[
\varphi(F^{p-1}D_xFX_{i_1} \cdots X_{i_{r-1}}D_j\varphi) = D_x\varphi(F^{p-1}X_{i_1} \cdots X_{i_{r-1}}D_j\varphi) = 0.
\]

Remark, that we assume \( j \neq (i_1, \ldots, i_{r-1}) \) otherwise

\[
dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{r-1}} = 0.
\]

That shows \( C(d\varphi) = 0 \)

3. Let \( \omega = z_{i_1}^{r-1} \cdots z_{i_r}^{r-1} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \).

We have
\[ dz_i \wedge \cdots \wedge dz_{i_r} = \sum_{i_1, \ldots, i_r} D_{i_1} z_{i_1} \cdots D_{i_r} z_{i_r} \frac{dx_{i_1} \wedge \cdots \wedge dx_{i_r}}{x_{i_1} \cdots x_{i_r}}, \]

\[ D_j = x_j \frac{\partial}{\partial x_j}. \]

To compute \( C(\omega) \), we have to work out

\[ U = \psi(F^{p^{-1}}D_n F \cdot Z_i^{\gamma_i} \cdot D_{i_1} Z_{i_1} \cdots Z_{i_r}^{\gamma_r} D_{i_r} Z_{i_r}) \mod F. \]

\[ Z_j \mod F = z_j. \]

We apply several times (3) of the proposition and get

\[ U \equiv D_n F D_{i_1} Z_{i_1} \cdots D_{i_r} Z_{i_r} \mod (F). \]

Therefore

\[ C(\omega) = \sum_{i_1, \ldots, i_r} D_n f D_{i_1} z_{i_1} \cdots D_{i_r} z_{i_r} \frac{dx_{i_1} \wedge \cdots \wedge dx_{i_r}}{x_{i_1} f_n x_{i_1} \cdots x_{i_r}}. \]

All forms of highest degree \( n - 1 \) are closed. We use the fact, that \( H^\omega(V, \Omega^{n-1}) \) has a basis of the following form

\[ \omega_n = x_i^{u_i} \cdots x_n^{u_n} \omega_0. \]

where

\[ \omega_0 = \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{x_1 \cdots x_n f_n}, \]

\[ \sum_{i=1}^{n} u_i \leq r; \ r = \deg V \quad \text{and} \quad 1 \leq u_i. \]

Recall \( x_i = X_i / X_0 \) are coordinate functions on \( V \) and of the affinization of \( F, f_n = \partial f / \partial x_n. \)

We get the important corollary to the theorem.

**Corollary 1.** Let \( A_{x,v} \) be the matrix of the Cartier-operator on \( H(V, \Omega^{n-1}) \) with respect to the above basis \( \omega_n \). Then

\[ A_{x,v} = \text{coefficient of } X^v \text{ in } \psi(F^{p^{-1}} \cdot X^v) \]

\[ X^v = X_0^{u_0} \cdots X_n^{u_n}, \quad \sum_{i=0}^{n} u_i = \sum_{i=0}^{n} v_i = r \]

\[ 1 \leq u_i \quad \text{for} \quad i = 1 \cdots n. \]
Proof. By definition
\[ C(\omega_u) = \psi_F(\prod_{i=0}^{n-1} x_i^{u_i-1} \cdots x_n^{u_n-1}) \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{f_n} \]
\[ = \psi(f^{p-1} \cdot x^u) \omega_v. \]

Now recall
\[ \psi(f^{p-1} \cdot x^u) = \psi \left( \frac{F^{p-1} X_0^{u_0} \cdots X_r^{u_r}}{X_0^{pr}} \right) \mod F \]
\[ \sum_{i=0}^{n} u_i = r, \quad 1 \leq u_i, \quad i = 1 \cdots n. \]

If \( A_{u,v} \) is the coefficient of \( X^v \) in \( \psi(F^{p-1} \cdot X^u) \).

Then
\[ C(\omega_u) = \sum_{\sum_{i=1}^{n} u_i = r} A_{u,v} x_1^{u_1} \cdots x_n^{u_n} \omega_v = \sum_{r} A_{u,v} \omega_v. \]

Notice
\[ \sum_{i=0}^{n} u_i = \sum_{i=0}^{n} v_i = r, \quad 1 \leq u_i, 1 \leq v_i, \quad i = 1 \cdots n. \]

REMARK. We have now an explicit description for the Cartier-operator on \( H^0(V, \Omega_{\psi}^1) \). We can use Serre duality \( H^0(V, \Omega_{\psi}^1) \cong H^{n-1}(V, \mathcal{O}_V) \). Under this duality \( \psi \) is the Frobenius \( \mathcal{F} \) on \( H^{n-1}(V, \mathcal{O}_V) \). We have therefore also an explicit description for \( \mathcal{F} \).

2. The Cartier-operator of a diagonal hypersurface. Let \( F(X) = \sum_{i=0}^{r} a_i X_i^r \) define a "generic" hypersurface. To compute the Cartier-operator, by the preceding discussion we have to analyse
\[ \psi(F^{p-1} X^u) \quad \left( \sum_{i=0}^{n} u_i = r, \quad u_i > 0 \right). \]

Let us adapt the following notation:
\[ \rho^i = \rho_0^i \cdots \rho_n^i, \quad \sigma^i = \prod_{i=0}^{n} a_i^{u_i}, \quad X^{u+1} = \prod_{i=0}^{n} X_i^{u+1}, \]
\[ |u| = \sum_{i=0}^{n} u_i, \quad u > 0 \iff u_i > 0 \quad (i = 0 \cdots n). \]

THEOREM 2. Let
\[ \text{char } k = p > 0, \quad F(X) = \sum_{i=0}^{n} a_i X_i^r, \quad \prod_{i=0}^{n} a_i \neq 0 \in k \]
\( V/k \) is defined by \( F \). Suppose \( r \) divides \( p - 1 \). Then the Cartier-operator
is invertible.

Proof.

\[ F^{p-1} = \sum_{m = p-1}^{\infty} \frac{(p-1)!}{m!} a_m X^{r_m}. \]

Using \( p^{-1} \)-linearity of \( \psi \) we get

\[ \psi(F^{p-1}X^u) = \sum_{m = p-1}^{\infty} \frac{-1}{m!} \alpha_m (X^{r_m+u}) = \sum_{m = p-1}^{\infty} \frac{-1}{m!} \alpha_m X^v. \]

We put \( \alpha = \alpha^{\frac{1}{sp}} \), and \( rm + u = pv \). Notice if \( u > 0 \) and \( |u| = r \), then also \( v > 0 \) and \( |v| = r \). If we write

\[ \psi(F^{p-1}X^u) = \sum_{|v| = r} A_{u,v}X^v, \]

then we have

\[
A_{u,v}^p = \begin{cases} 
- \frac{1}{m!} a_m & \text{if } rm = (p-1)v + v - u, |u| = |v| = r, u > 0, v > 0 \\
0 & \text{else .}
\end{cases}
\]

Let us now assume:

\[ p - 1 = rs. \]

If \( r \) divides \( v - u \) put \( v - u = r \cdot E(u, v) \) then

\[
A_{u,v}^s = \begin{cases} 
- \frac{1}{m!} a_m & \text{if } r|v - u \quad \text{and } m = sv + E(u, v) \\
0 & \text{else .}
\end{cases}
\]

We fix now a total ordering of \( u, v \). Let us order the \( n \)-tuples \((u_1, \ldots, u_n)\) resp \((v_1, \ldots, v_n)\) lexicographically and put

\[
u_0 = r - \sum_{i=1}^{n} u_i \text{ resp. } v_0 = r - \sum_{i=1}^{n} v_i.
\]

\( v < u \) means now, that either \( v_i < u_i \) or \( v_i = u_i \) for \( i = 1 \cdots j - 1 \) but \( v_j < u_j \). If any case, if \( v < u \), then \( v_j < u_j \) for some \( j \). We claim if \( v < u \), the \( A_{u,v} = 0 \).

**Case 1.** \( r \) does not divide \( u - v \), then \( A_{u,v} = 0 \).

**Case 2.** \( r \) divides \( u - v \). Now if \( v < u \) then for some \( j \) \( u_j - v_j > 0 \)
and $r$ divides $u_i - v_j$. But $r \geq u_j$ and $v_j \geq 1$, so $r - 1 \geq u_j - v_j$, therefore $r$ cannot divide $u_i - v_j$. This contradiction shows, if $v < u$, then $A_{u,v} = 0$. $A_{u,v}$ is therefore a triangle matrix.

What is the diagonal?

$$A^p_{u,v} = -\frac{1}{m!} a^m$$

with $m = s \cdot u$. Therefore

$$(\det A_{u,v})^p = \prod_u \left(-\frac{1}{(su)!}\right) a^s u^s \neq 0$$

**Corollary 2.** The assumptions are the same as in the theorem. Then

$$\mathcal{F}: H^{n-1}(V, \mathcal{O}_V) \rightarrow H^{n-1}(V, \mathcal{O}_V) \quad (\mathcal{F} \text{ is the Frobenius morphism})$$

is invertible.

**Proof.** Clear by Serre duality and the fact that $C = \mathcal{F}$.

The Cartier-operator of $W \cdot H$. The differential operator $C$ as given in Definition 1 on $\Omega^1$ is by $p^{-1}$-linearity completely determined on $\Omega^1$ by its value on $\omega = h \cdot dx$, where $x$ runs through a set of coordinate functions.

We have $C(\omega) = x^{-1} \psi(xh)dh$, that notation is only intrinsic, if $d\omega = 0$, because $\psi$ depends on the coordinate system. If we choose a different coordinate system, then we get in general a different operator; but for $\omega$ with $d\omega = 0$, we get the same, namely the Cartier-operator.

That fact can be exploited in the following way. Suppose

$$W = \{x_1 = x_2 = \cdots = x_t = 0\} \cap H.$$ 

We write now $C_H$ resp. $C_w$ for the the operators. The above definition shows $\bigoplus_{i=1}^t Kdx_i$ is stable under $C_H$. But by the property of $\psi$, $\psi(X_i H) = X_i \mathcal{H}$ for some $\mathcal{H}$, we have for

$$\omega = x_i hdx_i \quad i \neq j \quad i, j \text{ arbitrary}$$

$$C_H(\omega) = x_i \mathcal{H}dx_i.$$ 

Let $\mathcal{H} = \{x_i \cdots x_t\}$, then $\mathcal{H}\Omega^1_{H/k} \bigoplus \bigoplus_{i=1}^t \mathcal{O}_H dx_i$ is stable under $C_H$. By the exact sequence

$$0 \rightarrow \mathcal{H}\Omega^1_{H/k} + \bigoplus_{i=1}^t \mathcal{O}_H dx_i \rightarrow \Omega^1_{H/k} \rightarrow \Omega^1_{w/k} \rightarrow 0$$

$C_H$ induces an operator $C_w$ on $\Omega^1_{w/k}$. $C_w$ has again the properties
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(1) $C_w$ is $p^{-1}$-linear
(2) $C_w(dh) = 0$
(3) $C_w(h^{p-1}dh) = dh$.

If we restrict $C_w$ to the closed forms on $W$, then $C_w$ is the Cartier-operator.

Let now $L$ be an arbitrary linear variety. After a suitable coordinate change we may assume $L$ is the intersection of some coordinate hyperplanes. $W = L \cdot H$ has then the above shape.

Let us assume that the hypersurface $H$ has a diagonal defining equation of degree $d$ dividing $p - 1$, $p = \text{char } k$. Then the above Theorem 1 shows that $C_w$ is semisimple on $\mathcal{O}_{k, L}$. In the same way as before we can extend $C_w$ to any $\mathcal{O}_{k, L}$, in particular to $\mathcal{O}_{k, V}$, where $m = \dim W$. As result of this discussion we get:

**THEOREM 3.** If $L$ is a linear variety of dimension $m + 1$, then there exists a hypersurface $H$ of degree $d$, which divides $p - 1$, such that

$$\mathcal{F}: H^m(L \cdot H, \mathcal{O}_{L, H}) \to H^m(L \cdot H, \mathcal{O}_{L, H})$$

is invertible.

3. The Cartier-operator of plane curves. For curves the explicit description of the Cartier-operator is of special interest if one wants to study, how the Cartier-operator varies with the moduli of the curve. Unfortunately one is restricted to plane curves, because the above explicit form of the Cartier-operator is available only for hypersurfaces.

If one specializes the above results to plane curves, one has to assume, that the curve is singularity free.

The space $W = \{\text{homogenous forms of degree } d - 3\}$ is for non-singular curves $V$ of degree $d$ isomorphic to $H^0(V, \Omega^1_{k, V})$ under

$$W \simeq H^0(V, \mathcal{O}_{k, V})$$
$$P(X) \mapsto P(x)\omega_0$$

where the coordinate functions are given by

$$x = X_1/X_0, \quad y = X_2/X_0 \mod F,$$

$F$ being the defining equation for $V$ and $f(x, y)$ the affinization, $f_\gamma$ denotes $\partial f/\partial y$. With that notation $\omega_0 = dx/f_\gamma$.

But it is important to know, that one can give a similar description also for singular curves. Then $W$ is the space of $P(X)$, which define the “adjoint” curves to $V$. These are those curves, which cut out at least the “double point divisor”.

3* The Cartier-operator of plane curves* For curves the explicit description of the Cartier-operator is of special interest if one wants to study, how the Cartier-operator varies with the moduli of the curve. Unfortunately one is restricted to plane curves, because the above explicit form of the Cartier-operator is available only for hypersurfaces. If one specializes the above results to plane curves, one has to assume, that the curve is singularity free.

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To give an explicit basis depends on nature of the singularities.

Hyperelliptic curves: Let $p = \text{char } k > 2$.

For a detailed study of the Hasse-Witt-matrix of hyperelliptic curves one needs the explicit Cartier-operator with respect to various “normal forms”.

Let the hyperelliptic $V$ be given by $y^2 = f(x)$, $\deg f(x) = 2g + 1$ and such that $f(x)$ has no multiple roots. $V$ has a singularity at “infinity”. One could apply the above method and work out the adjoint curves in order to get a basis for $H^q(V, \Omega_{V/k})$. But we have already a basis, namely if $\omega = dx/y$ then $\{x^i\omega \mid i = 0 \cdots g - 1\}$ form a basis.

We specialize the results of §2 and get from Corollary 1 as matrix for the Cartier-operator with respect to the above basis (let us put $p - 1/2 = m$):

$$A_{u,v} = \text{coefficient of } x^{u+1} \text{ in } \psi(f(x)^m x^{u+1}) \quad 0 \leq u \leq g - 1.$$  

Legendre form: We assume now the defining equation in Legendre form.

$$f(x) = x(x - 1) \prod_{i=1}^{r} (x - \lambda_i) \quad r = 2g - 1 \quad \lambda_i \neq \lambda_j \neq 0, 1.$$  

Notation: Let

$$|\rho| = \rho_i + \cdots + \rho_r$$

$$\lambda^\rho = \lambda_1^{\rho_1} \cdots \lambda_r^{\rho_r}.$$  

The permutation group of $r$ elements $S_r$ operates on the monomials

$$\lambda^\rho \rightarrow \lambda^{\pi\rho}, \pi \in S_r.$$  

Let $G_\rho$ be the fix group of $\lambda^{m-\rho}$ and $G^{(\rho)} = S_r/G_\rho$. Let

$$H^{(\rho)}(\lambda) = \sum_{\pi \in G^{(\rho)}} \lambda^{m-\gamma^{(\rho)}}.$$  

Apparently

$$H^{(\rho)} = H^{(\bar{\rho})}, \text{ iff } \bar{\rho} = \bar{\pi}(\rho).$$  

We may therefore assume

$$0 \leq \rho_1 \leq \rho_2 \leq \rho_r \leq m.$$  

For given

$$0 \leq u \leq g - 1 \text{ let } \rho_v = |\rho| - vp + u.$$  

Put

$$a^{(p)}_{u,v} = (-1)^{v+m}(m) \cdots (m)$$

and

$$A^{p}_{u,v} = \sum_{\rho} a^{(p)}_{u,v}(\rho)\,0 \leq u \leq g - 1, \ r = 2g - 1$$

the summation condition being:

$$0 \leq \rho_{1} \leq \cdots \leq \rho_{r} \leq m \ , \ \rho_{0} = |\rho| - vp + u \ , \ 0 \leq \rho_{0} \leq m$$

$$vp - u + m \geq |\rho| \geq vp - u .$$

We state as a proposition

**PROPOSITION 2.** Let be $A_{u,v}, \ 0 \leq \frac{u}{v} \leq g - 1$, as defined above, and $\omega = dx/y$, then

$$C(x^{a}\omega) = \sum_{0 \leq u \leq g - 1} A^{p}_{u,v}x^{a}\omega$$

is the Cartier-operator.

**Applications:** We want to investigate, when the Cartier-operator is invertible. It seems that an answer to that question, without any restrictions is not available. It is therefore worthwhile to have various methods even in special cases. \(^1\)

We restrict ourself to genus 2, although the method could be applied to higher genus, but the calculations would be very easy. Let $p > 2$ and $g = 2$

i.e. $y^{2} = x(x - 1)(x - \lambda_{1})(x - \lambda_{2})(x - \lambda_{3}) \ , \ \lambda_{i} \neq \lambda_{j} \neq 0, 1 \ i \neq j .$

The notation is the same as above.

$H^{(p)}(\lambda)$ is homogeneous in the $\lambda$'s of degree $3m - |\rho|$, $m = (p - 1)/2$.

We have

$$A^{p}_{u,v} = \sum_{0 \leq \rho_{1} \leq \rho_{2} \leq \rho_{3} \leq m} a^{(p)}_{u,v}(\rho)\,0 \leq \frac{u}{v} \leq 1$$

$$\rho_{0} = |\rho| - vp + u \ \ vp - u \leq |\rho| \leq vp - u + m .$$

We want to know of $A^{p}_{u,v}$, what the forms of lowest homogeneous degree in the $\lambda$'s are. We have to give $|\rho|$ the maximal possible value.

We use the shorthands

\(^1\) Added in proof: We settled this question in the meantime, see \([6]\).
and \( D(u, v) = \text{degree of the lowest homogeneous term in } A_{x, v} \). In the list below is \( \rho_0 = \max |\rho| - vp + u \).

| \((u, v)\) | \(\max |\rho|\) | \(\rho_0\) | \(D(u, v)\) |
|-------|-------|-------|-------|
| \((0, 0)\) | \(m\) | \(m\) | \(p - 1\) |
| \((0, 1)\) | \(3m\) | \(m - 1\) | \(0\) |
| \((1, 0)\) | \(m - 1\) | \(m\) | \(p\) |
| \((1, 1)\) | \(3m\) | \(m\) | \(0\) |

We get therefore:

\[ A_{x, v}^p A_{x, v}^p = \text{terms of degree } p - 1 + \text{higher terms} \]

\[ A_{x, v}^p A_{x, v}^p = \text{terms of degree } p + \text{higher terms} . \]

The lowest degree term \( L \) in \( \det (A_{x, v})^p \) is given by

\[
L = m \sum_{\rho} \binom{m}{\rho} H^{(\rho)}(\lambda)
\]

\[
\rho_1 + \rho_2 + \rho_3 = m
\]

\[
0 \leq \rho_1 \leq \rho_2 \leq \rho_3 .
\]

Notice, if \( \rho \neq \bar{\rho} \), then \( H^{(\rho)} \) and \( H^{(\bar{\rho})} \) have no monomial in common. Therefore \( L \) is not the zero polynomial. We are able to specialize the variables \( (\lambda_0, \lambda_3, \lambda_3) \) in the algebraic closure of \( k \), such that \( \det (A_{x, v}) \neq 0 \). In other words, there exist curves of genus two with invertible Cartier-operator.

We do not know, what the smallest finite field is, over which such a curve exists.

REMARK. For large \( p \) we could push through a similar discussion for higher genus. We omit that, because there is a more elegant method for large \( p \) by Lubin (unpublished). Let \( y^2 = x^{p+1} + ax^{p+1} + x \). The claim is, that for large \( p \) (depending on \( g \)) and variable \( a \) the Hasse-Witt-matrix of that curve is a permutation matrix.

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