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**THE EQUATION  $y'(t) = F(t, y(g(t)))$**

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**A functional differential equation, in general, is a relationship in which the rate of change of the state of the system at time  $t$  depends on the state of the system at values of time, perhaps other than the present.**

**In this paper, sufficient conditions are given for  $g$  so that the initial value problem  $y'(t) = F(t, y(g(t)))$ ,  $y(p) = q$ , may be solved uniquely; where  $F$  is both continuous into the Banach space  $B$ , and is Lipschitzean in the second position.**

1. DEFINITIONS. If  $p$  is a real number and  $I = \{I_1, I_2, \dots\}$  is a collection of intervals so that  $p \in I_1$  and  $I_n \subseteq I_{n+1}$  for each positive integer  $n$ , then  $I$  is said to be a nest of intervals about  $p$ . Let  $I_0 = \{p\}$  and  $a_0 = b_0 = p$ . Also, let  $[a_n, b_n] = I_n$  for each nonnegative integer  $n$ . Let  $I^*$  denote the union of all elements of  $I$ .

In general  $B$  denotes a Banach space; and if  $D$  is a real number set, let  $C[D, B]$  denote the set of continuous functions from  $D$  into  $B$ . Whenever  $D$  is an interval,  $C[D, B]$  is taken to be a Banach space with supremum norm  $|\cdot|$ .

If  $g$  is a continuous function from  $I^*$  into  $I^*$  so that  $g(I_n) \subseteq I_n$  for each positive integer  $n$ , then  $g$  is said to be an  $I$ -function. If  $g$  is an  $I$ -function then for each positive integer  $n$ , define the following:

$$\begin{aligned} A_n &= \{x \in [a_n, a_{n-1}]: g(x) \notin I_{n-1}\}, \\ B_n &= \{x \in [b_{n-1}, b_n]: g(x) \notin I_{n-1}\}, \text{ and} \\ E_n(s) &= [p, g(s)] \cap (A_n \cup B_n), \text{ for each } s \in I_n. \end{aligned}$$

Let  $\int_D h(s)ds$  denote the Lebesgue integral of  $h$  over the subset  $D$  of the domain of the Lebesgue integrable function  $h$ .

Let  $F$  denote a continuous function from  $I^* \times B$  into  $B$  so that  $\|F(x, y) - F(x, z)\| \leq M(x) \cdot \|y - z\|$  for all  $x \in I^*$  and  $y, z \in B$ , where  $M$  is Lebesgue integrable on each  $I_n$ . Furthermore, if  $f$  is a continuous nonnegative valued function from  $I^*$  to the reals, and  $m$  is a positive integer, let  $\int_p^x (M, f, g, m)$  denote

$$\left| \int_p^x M(s_1) \right| \left| \int_p^{g_1} M(s_2) \right| \cdots \left| \int_p^{g(s_{m-1})} M(s_m) f(s_m) ds_m \right| \cdots |ds_2| |ds_1|.$$

If  $D$  is either  $A_n$  or  $B_n$ , let  $\int (M, f, D, m)$  denote

$$\int_D M(s_1) \int_{E_n(s_1)} M(s_2) \cdots \int_{E_n(s_{m-1})} M(s_m) f(s_m) ds_m \cdots ds_2 ds_1.$$

If  $D$  is a subset of the domain of the function  $h$ , let  $h|_D$  denote

the restriction of  $h$  to  $D$ . Also, let  $f \circ g$  denote the composition of  $f$  with  $g$ , whenever applicable;  $f \circ g(x) = f(g(x))$ .

2. Main results.

**THEOREM A.** *Suppose  $I$  is a nest of intervals about  $p, q \in B, g$  is an  $I$ -function,  $k$  is a sequence of positive integers, and for each positive integer  $n, \alpha_n = \int (M, 1, A_n, k(n)) < 1$  and  $\beta_n = \int (M, 1, B_n, k(n)) < 1$ . Then there is a unique function  $y \in C[I^*, B]$  so that  $y'(t) = F(t, y(g(t)))$  and  $y(p) = q$ , for all  $t \in I^*$ . [We say then that the initial value problem (IVP) has unique solution.]*

*Proof.* Since,  $I_0 = \{p\}$ , then certainly  $y_0 = \{(p, q)\}$  is the unique function in  $C[I_0, B]$  so that for all  $t \in I_0, y_0(t) = q + \int_p^t F(s, y_0(g(s)))ds$ .

Next, suppose  $n$  is a nonnegative integer so that there is a unique function  $y_n \in C[I_n, B]$  so that, for each  $t \in I_n, y_n(t) = q + \int_p^t F(s, y_n(g(s)))ds$ . The following is the construction of  $y_{n+1}$ . Let  $D = \{f \in C[I_{n+1}, B]: f|_{I_n} = y_n\}$  and let  $m = k(n + 1)$ . Then, if  $f \in D$  and  $t \in I_{n+1}$ , let  $T$  be so that  $Tf(t) = q + \int_p^t F(s, f(g(s)))ds$ . Then, certainly  $T$  is from  $D$  into  $D$ .

**LEMMA 1.** *If  $f, h \in D$  and  $t \in I_{n+1}$ , then*

$$\|T^m f(t) - T^m h(t)\| \leq \int_p^t (M, \|f \circ g - h \circ g\|, g, m),$$
 for each positive integer  $m$ .

*Proof of Lemma 1.* (by induction on  $m$ ) If  $m = 1$ ,

$$\begin{aligned} \|Tf(t) - Th(t)\| &= \left\| \int_p^t [F(s, f(g(s))) - F(s, h(g(s)))]ds \right\| \\ &\leq \left| \int_p^t \|F(s, f(g(s))) - F(s, h(g(s)))\| ds \right| \\ &\leq \left| \int_p^t M(s) \cdot \|f(g(s)) - h(g(s))\| ds \right| = \int_p^t (M, \|f \circ g - h \circ g\|, g, 1) . \end{aligned}$$

Now, suppose the lemma holds for  $m = r$ . Then,

$$\begin{aligned} &\|T^{r+1}f(t) - T^{r+1}h(t)\| \\ &= \left\| \int_p^t [F(s, T^r f(g(s))) - F(s, T^r h(g(s)))]ds \right\| \\ &\leq \left| \int_p^t \|F(s, T^r f(g(s))) - F(s, T^r h(g(s)))\| ds \right| \\ &\leq \left| \int_p^t M(s) \cdot \|T^r f(g(s)) - T^r h(g(s))\| ds \right| \\ &\leq \left| \int_p^t M(s_1) \cdot \int_p^{g(s_1)} (M, \|f \circ g - h \circ g\|, g, r) ds_1 \right| , \end{aligned}$$

by the induction hypothesis, but this equals  $\int_p^t (M, \|f \circ g - h \circ g\|, g, r + 1)$ .

LEMMA 2. *If  $N$  is a bounded, measurable function from  $I_{n+1}$  to the reals so that  $N(s) = 0$  whenever  $s$  is in  $I_{n+1} \setminus (A_{n+1} \cup B_{n+1})$ , then*

$$\int_p^{a_{n+1}} (M, N, g, m) = \int (M, N, A_{n+1}, m),$$

and

$$\int_p^{b_{n+1}} (M, N, g, m) = \int (M, N, B_{n+1}, m).$$

*Proof of Lemma 2.* (by induction on  $m$ ) If  $m = 1$ ,  $\int_p^{a_{n+1}} (M, N, g, 1) = \left| \int_p^{a_{n+1}} M(s)N(s)ds \right| = \int_{A_{n+1}} M(s)N(s)ds = \int (M, N, A_{n+1}, 1)$ , because  $N$  is 0 at each point of  $[p, a_{n+1}] \setminus A_{n+1}$ . Suppose the lemma is true for  $m = r$ . Then,  $\int_p^{a_{n+1}} (M, N, g, r + 1) = \int_p^{a_{n+1}} (M, U, g, r)$ , where  $U(s) = \left| \int_p^{g(s)} M(t)N(t)dt \right|$ , for all  $s \in I_{n+1}$ . If  $s \in I_{n+1} \setminus (A_{n+1} \cup B_{n+1})$ ,  $g(s) \in I_n$ . Thus,  $N$  is 0 on  $[p, g(s)]$ , and so  $U(s) = 0$ . Whence,  $U$  satisfies the conditions for  $N$  in the lemma. So, by the induction hypothesis,  $\int_p^{a_{n+1}} (M, U, g, r) = \int (M, U, A_{n+1}, r) = \int (M, N, A_{n+1}, r + 1)$ , because  $U(s) = \int_{E_{n+1}(s)} M(t)N(t)dt$ . The proof of the second equality in the lemma is similar. Thus, Lemma 2 is proven.

Now, the two lemmas are applied. By Lemma 1,  $\|T^m f(t) - T^m h(t)\| \leq \int_p^t (M, \|f \circ g - h \circ g\|, g, m)$ , for all  $t \in I_m$ ,  $\leq \max \left\{ \int_p^{a_{n+1}} (M, \|f \circ g - h \circ g\|, g, m), \int_p^{b_{n+1}} (M, \|f \circ g - h \circ g\|, g, m) \right\}$  which by Lemma 2 is  $= \max \left\{ \int (M, \|f \circ g - h \circ g\|, A_{n+1}, m), \int (M, \|f \circ g - h \circ g\|, B_{n+1}, m) \right\}$ , because  $\|f(g(s)) - h(g(s))\| = 0$  for all  $s \in I_{n+1} \setminus (A_{n+1} \cup B_{n+1})$ . Thus,  $\|T^m f - T^m h\| \leq \max \left\{ \int (M, \|f \circ g - h \circ g\|, A_{n+1}, m), \int (M, \|f \circ g - h \circ g\|, B_{n+1}, m) \right\} \leq \max \left\{ \int (M, 1, A_{n+1}, m), \int (M, 1, B_{n+1}, m) \right\} \cdot |f - h|$ . Thus,  $T^m$  is a contraction map from the complete metric space  $D$  into  $D$ . Thus  $T^m$  has a unique fixed point  $y_{n+1}$ . It is a known result that this implies that  $y_{n+1}$  is the unique fixed point of  $T$ . [ $(Ty_{n+1}) = T(T^m(Ty_{n+1})) = T^m(Ty_{n+1})$ , but only  $y_{n+1}$  is so that  $y_{n+1} = T^m y_{n+1}$ . So  $Ty_{n+1} = y_{n+1}$ , and uniqueness is clear.]

Thus,  $y_{n+1}(t) = Ty_{n+1}(t) = q + \int_p^t F(s, y_{n+1}(g(s)))ds$ , for all  $t \in I_{n+1}$ , and is the unique such function. Hence, by inductive definition, for each positive integer  $i$ , there is a unique function  $y_i \in C[I_i, B]$  so that for all  $t \in I_i$ ,  $y_i(t) = q + \int_p^t F(s, y_i(g(s)))ds$ . Now, define  $y \in$

$C[I^*, B]$  so that  $y(t) = y_n(t)$ , whenever  $t \in I_n$ . Since  $m \leq n$  implies  $y_n|_{I_m} = y_m$ ,  $y$  is well-defined, and  $y(t) = q + \int_p^t F(s, y(g(s)))ds$ , for all  $t \in I^*$ . Now, suppose  $z(t) = q + \int_p^t F(s, z(g(s)))ds$ , for all  $t \in I^*$ , and  $z \in C[I^*, B]$ . Then, if  $n$  is a positive integer, and  $t \in I_n$ ,  $z|_{I_n}(t) = q + \int_p^t F(s, z|_{I_n}(g(s)))ds$ . So,  $z|_{I_n} = y_n = y|_{I_n}$  for each positive integer  $n$ . Thus,  $z = y$ .

**COROLLARY 1.** *Let  $M$  be the constant 1 function, and let  $k(n) = 2$ , for all  $n$ . Suppose for each  $n$ ,  $\int_{A_n} \min\{|g(x) - a_{n-1}|, |g(x) - b_{n-1}|\}dx < 1$ , and  $\int_{B_n} \min\{|g(x) - a_{n-1}|, |g(x) - b_{n-1}|\}dx < 1$ . Then, the IVP has a unique solution. [See Figure 1. All the shaded area between each pair of vertical dashed lines is less than one.]*

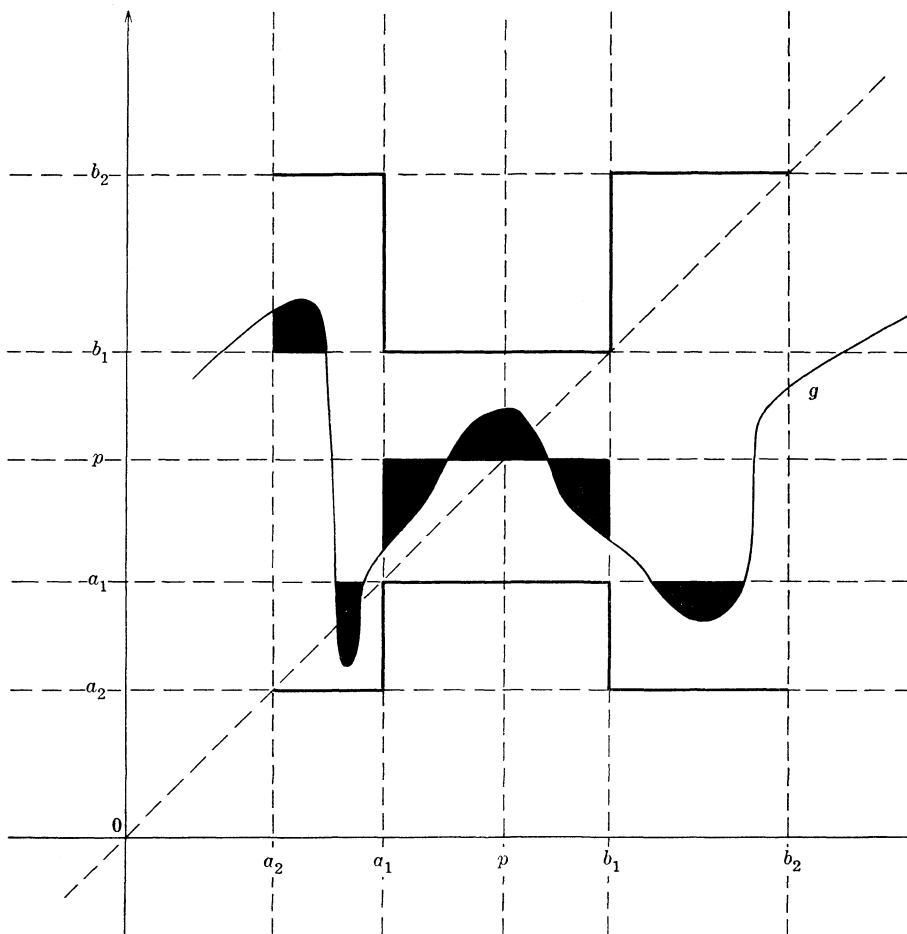


FIGURE 1

*Proof.*  $\alpha_n = \int_{A_n} M(s_1) \int_{E_n(s_1)} M(s_2) ds_2 ds_1 = \int_{A_n} \int_{E_n(s_1)} ds_2 ds_1$ . Now,  $s_1 \in A_n$  implies

$$E_n(s_1) = \begin{cases} A_n \cap [p, g(s_1)], & \text{if } g(s_1) \in A_n, \text{ and} \\ B_n \cap [p, g(s_1)], & \text{if } g(s_1) \in B_n. \end{cases}$$

Thus,  $E_n(s_1) \subseteq [g(s_1), a_{n-1}]$  if  $g(s_1) \in A_n$ , and in this case,  $|g(s_1) - a_{n-1}| \leq |g(s_1) - b_{n-1}|$ . Also,  $E_n(s_1) \subseteq [b_{n-1}, g(s_1)]$  if  $g(s_1) \in B_n$ , and in this case,  $|g(s_1) - b_{n-1}| \leq |g(s_1) - a_{n-1}|$ . Thus,  $E_n(s_1)$ , which is certainly measurable, must have measure  $\leq \min\{|g(s_1) - a_{n-1}|, |g(s_1) - b_{n-1}|\}$ . Hence,  $\int_{A_n} \int_{E_n(s_1)} ds_2 ds_1 \leq \int_{A_n} \min\{|g(s_1) - a_{n-1}|, |g(s_1) - b_{n-1}|\} ds_1$ , because  $\int_{E_n(s_1)} ds_2$  is the measure of  $E_n(s_1)$ . Thus,  $\alpha_n < 1$ , and similarly  $\beta_n < 1$ , for each positive integer  $n$ . Apply Theorem A.

**COROLLARY 2.** *Suppose  $k(n) = 1$  for each  $n$ . Then, if  $\int_{A_n} M < 1$  and  $\int_{B_n} M < 1$ , for each  $n$ , the IVP has unique solution.*

*Proof.* Immediate.

**COROLLARY 3.** *Suppose  $M$  is the constant 1 function and  $k(n) = 1$  for each  $n$ . Then if  $\max\{b_n - b_{n-1}, a_{n-1} - a_n\} < 1$ , for each  $n$ , the IVP has unique solution.*

*Proof.*  $A_n \subseteq [a_{n-1}, a_n]$  and  $B_n \subseteq [b_{n-1}, b_n]$  implies  $\int_{A_n} 1 \leq \int_{a_n}^{a_{n+1}} 1 = a_{n-1} - a_n$  and  $\int_{B_n} 1 \leq \int_{b_{n-1}}^{b_n} 1 = b_n - b_{n-1}$ . Apply Corollary 2.

The following example illustrates the advantage of allowing  $k(n)$  to assume integral values other than 1.

**EXAMPLE.** Let  $F$  be so that  $M = 1$  in the IVP— $y(p) = q, y'(t) = F(t, y(g(t)))$ , where

$$g(x) = \begin{cases} 2x & , \text{ if } x \in [0, p], \text{ and} \\ 4p - 2x, & \text{ if } x \in [p, 2p]. \end{cases}$$

then it is straightforward to show that if  $J$  is a subinterval of  $[0, 2p]$  and  $g(J) \subseteq J$ , then  $J = [0, 2p]$ . Thus, if  $I$  is a nest of intervals about any point of  $[0, 2p]$  and  $I^* = [0, 2p]$ , then  $I_n = [0, 2p]$  for each positive integer  $n$ , if  $g$  is to be an  $I$ -function. Thus, in order to apply Corollary 3, it seems necessary to require  $p < 1$ , in order to solve

the IVP. However, if Theorem A is applied with  $k(n) = m$  for all  $n$ , then Theorem B, which follows, shows that the condition  $p < 2^{(m-1)/m}$  gives the best apparent bound for the size of  $p$  in order to solve the IVP. Now, since  $m$  is arbitrary, clearly,  $p$  may be any positive number less than 2.

**THEOREM B.** *If  $g$  is as in the above example, and for each positive integer  $n$ ,  $F_n(x) = \int_p^x (1, 1, g, n + 1)$ , then*

(1)  $F_n$  is symmetric about  $p$ . That is, for each  $n$ ,  $F_n(x) = F_n(2p - x)$ , for all  $x \in [0, p]$ ; and

(2)  $F_n(x) + F_n(p - x) = p^{n+1}/2^n$ , for each  $n$ , and for all  $x \in [0, p/2]$ .

*Proof.* (induction on  $n$ ) Suppose  $n = 1$ . Then, if  $x \in [0, 2p]$ ,  $F_1(x) = \left| \int_p^x |g(s) - p| ds \right|$ , which is

$$F_1(x) = \begin{cases} p^2/2 - px + x^2, & \text{if } x \in [0, p/2], \\ px - x^2, & \text{if } x \in [p/2, p], \\ -2p^2 + 3px - x^2, & \text{if } x \in [p, 3p/2], \text{ and} \\ 5p^2/2 - 3px + x^2, & \text{if } x \in [3p/2, 2p]. \end{cases}$$

It is straightforward to show that  $F_1$  satisfies the conditions (1) and (2) of the theorem. Now, suppose the theorem is true for the positive integer  $k$ . Then, for each  $x \in [0, 2p]$ ,  $F_{k+1}(x) = \left| \int_p^x F_k(g(s)) ds \right|$ . If  $x \in [0, p]$ ,  $F_{k+1}(2p - x) = \left| \int_p^{2p-x} F_k(g(s)) ds \right|$ . Thus, if  $x \leq s \leq p$ ,  $g(s) = 2s = 4p - 2(2p - s) = g(2p - s)$ . So,  $F_{k+1}(x) = \int_p^x F_k(g(s)) ds = \int_{2p-x}^p F_k(g(2p - s))(-1) ds$ , by change of variable, but this is  $\int_p^{2p-x} F_k(g(2p - s)) ds = \int_p^{2p-x} F_k(g(s)) ds = F_{k+1}(2p - x)$ . Thus,  $F_{k+1}$  is symmetric about  $p$ .

Now, suppose  $x \in [0, p/2]$ . Then,

$$\begin{aligned} & F_{k+1}(x) + F_{k+1}(p - x) \\ &= \int_x^p F_k(g(s)) ds + \int_{p-x}^p F_k(g(s)) ds \\ &= \int_x^p F_k(2s) ds + \int_{p-x}^p F_k(2s) ds, \text{ because } g(s) = 2s \\ &= \int_x^p F_k(2s) ds + \int_{p-x}^p F_k(2p - 2s) ds, \text{ because } g(z) = g(2p - z) \\ &= \int_x^p F_k(2s) ds - \int_{2x}^0 (1/2) F_k(s) ds, \text{ by change of variable} \end{aligned}$$

$$\begin{aligned}
 &= \int_x^p F_k(2s)ds + (1/2) \int_0^{2x} F_k(s)ds \\
 &= \int_x^p F_k(2s)ds + \int_0^x F_k(2s)ds, \text{ by change of variable} \\
 &= \int_0^p F_k(2s)ds \\
 &= (1/2) \int_0^{2p} F_k(s)ds, \text{ by change of variable} \\
 &= \int_0^p F_k(s)ds, \text{ because } F_k \text{ is symmetric about } p \\
 &= \int_0^{p/2} F_k(s)ds + \int_{p/2}^p F_k(s)ds \\
 &= \int_0^{p/2} F_k(s)ds - \int_0^{p/2} F_k(p-s)(-1)ds, \text{ by change of variable} \\
 &= \int_0^{p/2} \{F_k(s) + F_k(p-s)\}ds \\
 &= \int_0^{p/2} \{p^{k+1}/2^k\}ds, \text{ by the induction hypothesis} \\
 &= p^{k+2}/2^{k+1}.
 \end{aligned}$$

By Theorem B,  $F_n(0) + F_n(p-0) = p^{n+1}/2^n$ . But,  $F_n(p) = 0$ , by definition of  $F_n$ , and thus  $F_n(0) = p^{n+1}/2^n$ . Also,  $F_n(2p) = F_n(2p-0) = F_n(0) = p^{n+1}/2^n$ . Thus, if  $p^{n+1}/2^n < 1$ , then  $\alpha_{n+1} \leq F_n(0) = p^{n+1}/2^n < 1$ , and  $\beta_{n+1} \leq F_n(2p) = p^{n+1}/2^n < 1$ . Apply Theorem A.

**3. Applications.** The following is a generalization of a theorem by Anderson [1].

Let  $F$  be a continuous real-valued function with domain  $D$  of the plane  $R \times R$  so that the partial derivative  $F_2$  is continuous on  $D$  and  $(0, b) \in D$ . Let  $h'$  and  $k$  be so that if  $|x| \leq h'$  and  $|y - b| \leq k$ , then  $(x, y) \in D$ . Let  $K = \sup\{|F(x, y)| : |x| \leq h' \text{ and } |y - b| \leq k\}$ ,  $h = \min\{h', k/K\}$ , and  $M = \sup\{|F_2(x, y)| : |x| \leq h \text{ and } |y - b| \leq k\}$ .

**THEOREM C.** *Suppose there are intervals  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_m = [-h, h]$  so that  $\max\{b_n - b_{n-1}, a_{n-1} - a_n\} \cdot M < 1$  for each integer in  $[1, m]$ , and so that  $0 \in I_1$ . Let  $I_j = I_m$  for each  $j \geq m$ . Then, if  $g$  is an  $I$ -function, there is a unique function  $y$  so that  $y(0) = b$  and  $y'(t) = F(t, y(g(t)))$ , for all  $t \in [-h, h]$ .*

*Proof.* Let  $E = \{(x, y) : |x| \leq h, |y - b| \leq k\}$ , and let  $G$  be an extension of  $F|_E$  so that

$$G(x, y) = \begin{cases} F(x, b - k), & \text{if } y \leq b - k, \text{ and} \\ F(x, b + k), & \text{if } y \geq b + k. \end{cases}$$



By continuity of  $F_2$  and the mean value theorem, it follows that  $F$  is Lipschitzean in the second position with constant  $M$ . It follows, also, that  $G$  has the same Lipschitz constant  $M$ . Then, by Corollary 2, there is a unique function  $y \in C[I^*, B] = C[[-h, h], R]$  so that  $y'(t) = G(t, y(g(t)))$ ,  $y(0) = b$ , for all  $t \in [-h, h]$ . Equivalently,  $y(t) = b + \int_0^t G(s, y(g(s))) ds$ , for all  $|t| \leq h$ . Thus,  $|y(t) - b| = \left| \int_0^t G(s, y(g(s))) ds \right| \leq h \cdot \sup \{|G(s, y(g(s)))|: |s| \leq h\}$ , and since the range of  $G$  is a subset of the range of  $F|_E$ , we have that this is  $\leq h \cdot \sup \{|F(x, v)|: |x| \leq h, |v - b| \leq k\} = h \cdot K \leq k$ , by definition of  $h$ . Thus,  $G(x, y(g(x))) = F(x, y(g(x)))$ , for all  $|x| \leq h$ . So,  $y'(t) = F(t, y(g(t)))$ ,  $y(0) = b$ , for all  $t \in [-h, h]$ .

The following is a generalization of a theorem by Kuller [3].

**THEOREM D.** *Suppose only that  $g$  is a continuous function with connected, real domain  $E$  so that  $g$  is not the identity, but  $g \circ g$  is the identity. Then, if  $M = 1$  and  $q \in B$ , there is a segment  $Q$  about the unique fixed point  $p'$  of  $g$  so that if  $p \in Q \cap E$ , the IVP has unique solution.*

*Proof.* Kuller proves that  $g$  has a unique fixed point  $p'$  and that  $g$  is strictly decreasing. Let  $0 < k < 1/2$ . Let  $\beta_0 = p$  and let  $\beta$  be a nondecreasing sequence of reals so that  $\beta_i - \beta_{i-1} < k$ , for each positive integer  $i$ , and so that  $\beta$  converges to the right boundary of  $E$ , which may be  $+\infty$ . Then, for each positive integer  $i$ , let  $\{\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in_i}\}$  be so that  $g(\beta_i) = \alpha_{in_i} \geq \dots \geq \alpha_{i2} \geq \alpha_{i1} = g(\beta_{i-1})$  and also so that  $\alpha_{ij} - \alpha_{i,j+1} < k$ , for all  $j$ . Then,  $\{[\alpha_{ij}, g(\alpha_{ij})]: i \geq 1 \text{ and } 1 \leq j \leq n_i\}$  is a monotonic collection of intervals, each containing  $p$ . Let  $I_1 = [\alpha_{11}, g(\alpha_{11})]$ . Suppose  $I_m$  has been defined to be  $[\alpha_{ij}, g(\alpha_{ij})]$ . Then, let  $g(\alpha_{ij})$

$$I_{m+1} = \begin{cases} [\alpha_{i,j+1}, g(\alpha_{i,j+1})], & \text{if } j < n_i, \text{ and} \\ [\alpha_{i+1,2}, g(\alpha_{i+1,2})], & \text{if } j = n_i. \end{cases}$$

Relabel  $I_n$  to be  $[a_n, b_n]$ . Then,  $\max \{a_{n-1} - a_n, b_n - b_{n-1}\} < 1$ , for each positive integer  $n$ . Let  $Q = (a_1, b_1)$ . Then apply Corollary 3.

Kuller required differentiability of  $g$  in order to solve  $y' = y \circ g$ ,  $y(p') = q$ , where  $p'$  is the unique fixed point of  $g$ .

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