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E. J. Vought has characterized hereditarily locally connected compact metric continua as those which are hereditarily aposyndetic, and (subsequently) as those which are aposyndetic and have only aposyndetic separating subcontinua. Also, Vought characterized hereditarily locally connected, cyclically connected compact metric continua as those having no cut point and separated only by aposyndetic subcontinua. In this paper it is shown that similar characterizations can be obtained when a larger class of subcontinua are allowed to separate, namely those which are semi-aposyndetic.

A *continuum* is a nondegenerate closed connected set. If x and y are points of the continuum M, we say that M is *aposyndetic at x with respect to y* if there exists a subcontinuum $H \subset M - \{y\}$ containing x in its interior. The continuum M is *aposyndetic at x* if M is aposyndetic at x with respect to each point of $M - \{x\}$. If M is aposyndetic at each point $x \in M$, then we say that M is *aposyndetic*. If x and y are points of a continuum M, then M is *semi-aposyndetic at $\{x, y\}$* if M is aposyndetic at one (at least) of x and y with respect to the other. If M is semi-aposyndetic at each 2-point subset, then we say that M is *semi-aposyndetic*. Thus every aposyndetic continuum must be semi-aposyndetic. But the converse does not hold, indeed, M may be aposyndetic at *none* of its points yet still be semi-aposyndetic, as shown in the example below. A set D *separates* M if $M - D$ is not connected, and a point z *cuts* M if there exist points $x, y \in M - \{z\}$ such that every subcontinuum of M containing both x and y also contains z. A continuum M is *cyclically connected* if each pair of points of M are contained in a simple closed curve in M. A property (e.g., locally connected, aposyndetic, or semi-aposyndetic) of a continuum M is *hereditary* if each subcontinuum of M has that property.

The notion of semi-aposyndesis has recently been shown to be useful in the study of n-mutual aposyndesis in the Cartesian products of continua [8]. Also, C. L. Hagopian has a number of results concerning semi-aposyndetic plane continua [2; 3; 4], the most interesting being that non-separating semi-aposyndetic plane continua are arcwise-connected [3]. That semi-aposyndesis is weaker than aposyndesis is evident: the cone over any regular Hausdorff space S is semi-aposyndetic [8, p. 240] but clearly not always aposyndetic.

EXAMPLE. *A compact planar semi-aposyndetic continuum which* *is aposyndetic at none of its points.* Let K be a cone over the Cantor set C (built in $[0, 1]$), i.e. $[0, 1] \times C$ with $\{0\} \times C$ identified. Let B denote the copy of $[0, 1] \times \{0\}$ in K. Assume that K is situated in the plane so that B coincides with the line segment $\{(x, \sqrt{3}/6) \mid -1/2 \leqq x \leqq 1/2\}$, with the order on B agreeing with that of L from $(-1/2, \sqrt{3}/6)$ to $(1/2, \sqrt{3}/6)$. Let f and g denote the rotation maps of 120° and 240° respectively. Finally, let $M = K \cup f(K) \cup g(K)$, with $B \cup f(B) \cup g(B)$ forming a triangle and the rest of M outside this triangle. It is clear that M has the required properties.

Vought [10, p. 96] showed that hereditary aposyndesis and hereditary local connectedness are equivalent. Since the cone over the Cantor set is hereditarily semi-aposyndetic, it is clear that his result does not hold when hereditary aposyndesis is replaced by hereditary semi-aposyndesis. However, in the event that the continuum is aposyndetic, such a substitution does work. It should be noted that the proofs of Theorems 2, 3, and 4 are patterned in general after those of Vought's in [9].

First we extract a result from [8, p. 242]:

LEMMA 1. *Let M be a compact metric semi-aposyndetic continuum. If M is irreducible between two points, then M is an arc.*

Another useful and well-known result is

LEMMA 2. *Let x be a point of a compact metric continuum M such that M is aposyndetic at each point of $M - \{x\}$ with respect to x. Then x cuts in M if and only if x separates in M.*

THEOREM 1. *Let M be a compact metric continuum. Then M is hereditarily locally connected if and only if M is aposyndetic and hereditarily semi-aposyndetic.*

Proof. Suppose that M is not hereditarily locally connected. Then [11, p. 18] there exist disjoint subcontinua $C_1, C_2 \cdots$ converging to a subcontinuum C disjoint from each C_i. Let x and y be distinct points of C. Let $x_i, y_i \in C_i$ (for each i) such that $x = \lim x_i$ and $y = \lim y_i$. For each i, let A_i be an irreducible subcontinuum of C_i from x_i to y_i. Then by Lemma 1, each A_i is an arc. Let $z \in \lim A_i - \{x, y\}$ [taking a subsequence, if necessary]. By the aposyndesis of M, there exist subcontinua H and K in $M - \{z\}$ such that $x \in H^\circ$ and $y \in K^\circ$ (for any set S, S° denotes the interior of S). We may assume that each A_i meets $H \cup K$ and that no A_i is contained in $H \cup K$. Select $z_i \in A_i - (H \cup K)$ [for each i] such that $z = \lim z_i$. Let A_i' be the subarc of A_i

which is the closure of the z_i-component of $A_i - (H \cup K)$. Let $A' = \lim A'_i$ [taking a subsequence, if necessary]. Let $w \in A' - (H \cup K \cup \{z\})$, and let $w_i \in A'_i$ (for each i) such that $w = \lim w_i$. Let p_i and q_i denote the endpoints of A'_i. We may assume that w_i precedes z_i in the order that A'_i has from p_i to q_i. For each i, let D_i be the subarc of A'_i defined by $D_i = [p_i, z_i]$ for odd i, and $D_i = [w_i, q_i]$ for even i. Finally, let B denote the continuum $C \cup H \cup K \cup (\cup D_i)$. By hypothesis, B must be semi-aposyndetic. However, it is easily seen that B is aposyndetic at neither of w and z with respect to the other. This contradiction concludes the proof of the theorem.

Bing [1, p. 499] showed that for compact metric continua in which no subcontinuum separates, aposyndesis at a point implied local connectedness at that point. Vought [9, p. 258] allowed aposyndetic subcontinua to separate and obtained the same conclusion. When semi-aposyndetic subcontinua are allowed to separate, we show that if M is both aposyndetic and semi-locally-connected at x, then M is connected im kleinen at x, but not necessarily locally connected at x. Whether the "semi-locally-connected at x" is actually necessary is unknown to the author. (Clearly semi-locally-connected at x without aposyndetic at x is not sufficient, because of the cone over the Cantor set.) First we prove a useful lemma.

LEMMA 3. *Suppose B is a subcontinuum of the compact metric continuum M, x is a point of $M - B$, and A is a subcontinuum of M irreducible from x to B. If $A \cup B$ is semi-aposyndetic, then A is an arc.*

Proof. By Lemma 1, we need only show that A is semi-aposyndetic. Suppose there exist distinct points $w, z \in A \cap B$. Since $A \cup B$ is semi-aposyndetic, there exists a subcontinuum H of $A \cup B$ such that, say, $w \in H^\circ$ and $z \notin H$. It $x \in H$ then any subcontinuum of H irreducible from x to B would contradict the irreducibility of A. Thus $x \notin H$. If $A - H$ is connected, then $\mathrm{Cl}\,(A - H)$ is a continuum missing w but containing x and z. This contradiction implies that $A - H = E \cup F$, separated, with $x \in E$. The continuum $H \cup E$ contains both x and w. Thus any subcontinuum of $H \cup E$ irreducible from x to B would contradict the irreducibility of A. Thus $A \cap B$ consists of only a single point w.

Suppose that $y, z \in A$ such that A is not semi-aposyndetic at $\{y, z\}$. By the semi-aposyndesis of $A \cup B$, there is a subcontinuum H of $A \cup B$ such that, say, $y \in H^\circ$ (relative to $A \cup B$) and $z \notin H$. By the choice of y and z, it follows that $H \not\subset A$. Then $H - \{w\} = E \cup F$, separated, with $y \in E$. Hence $E \cup \{w\}$ is a subcontinuum of A containing y in its interior (relative to A) and missing z. This contradiction completes the proof.

THEOREM 2. *Let M be a compact metric continuum in which only semi-aposyndetic subcontinua separate. If M is both aposyndetic at x and semi-locally-connected at x, then M is connected im kleinen at x.*

Proof. Suppose M is not connected im kleinen at x. Then [11, p. 18] there exists an open set U containing x, and a sequence $C_1, C_2, \cdots$ of closures of distinct components of U such that $x \in C = \lim C_i$, and $C \cap C_i = \phi$ (for each i).

We may assume that x is a non-separating point of M, since if K is a component of $M - \{x\}$, then x is a non-separating point of of $K \cup \{x\}$, and we would need only show that each $K \cup \{x\}$ is connected im kleinen at x in order to complete the proof.

Since M is semi-locally-connected at x, M is aposyndetic at each point of $M - \{x\}$ with respect to x. Hence $M - U$ can be covered by a collection of subcontinua missing x, and by compactness, a finite number of these cover $M - U$. Then since x does not separate, by Lemma 2 we have that x does not cut. Hence the union of this finite collection of subcontinua is contained in a subcontinuum missing x. Thus we may assume that $M - U$ is connected.

We first note that if B is any subcontinuum of C_i irreducible from x_i to Bd U [Bd denotes boundary], then $B \cup (M - U)$ is a separating subcontinuum of M and hence is semi-aposyndetic. Thus by Lemma 3, each such continuum B is an arc. Now for each i, let $p_i, q_i \in C_i - U$ [p_i and q_i possibly the same point] such that there are arcs T_i and S_i in $(C_i \cap U) \cup \{p_i\}$ and $(C_i \cap U) \cup \{q_i\}$ respectively irreducible from x_i to p_i and q_i respectively. Let $p = \lim p_i$ and $q = \lim q_i$ (taking a subsequence of $\{C_i\}_{i=1}^{\infty}$ if necessary). If $p = q$ for each possible choice of sequences $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$, then M would not be aposyndetic at x with respect to p. Hence there are sequences $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$ such that $p \neq q$. For each i, let A_i be an arc from p_i to q_i contained in $T_i \cup S_i$; hence $A_i - U = \{p_i, q_i\}$. Let $A = \lim A_i$ (taking a subsequence, if necessary), let w and z be distinct points of A, and let $w_i, z_i \in A_i - \{p_i, q_i\}$ (for each i) such that $w = \lim w_i$ and $z = \lim z_i$. We may assume that for each i, w_i precedes z_i in the order that A_i has from p_i to q_i. For each i, let D_i be the subarc of A_i defined by $D_i = [p_i, z_i]$ for odd i, and $D_i = [w_i, q_i]$ for even i. Finally, let B denote the continuum $(M - U) \cup A \cup (\cup D_i)$. Then B is not semi-aposyndetic at $\{w, z\}$ but it does separate M. This contradiction establishes the theorem.

A well-known example (see Figure 3–9 of [5, p. 113]) of a continuum which is connected im kleinen at x but not locally connected at x satisfies the hypotheses of Theorem 2 and hence shows that the conclusion cannot be improved to "locally connected" as in the cases of Bing's and Vought's results.

THEOREM 3. *A compact metric continuum M is hereditarily locally connected if and only if M is aposyndetic and each separating subcontinuum is semi-aposyndetic.*

Proof. Using Theorems 1 and 2 (and the fact that a continuum is locally connected if it is connected im kleinen at each point), the proof of Theorem 3 is essentially the same as Vought's proof [9, p. 259].

The final result is a "semi-aposyndetic version" of Vought's Theorem 3 of [9, p. 260], which generalizes Bing's result [1, p. 504] that a compact metric continuum in which no point cuts and no subcontinuum separates must be a simple closed curve.

We first prove two lemmas.

LEMMA 4. *Suppose that no point cuts in the compact metric continuum M, x is a point of the open set $U \subset M$,* Bd U *is nondegenerate, and each subcontinuum of M irreducible from x to* Bd U *is an arc. Then for each $\varepsilon > 0$, there exists an arc A in* Cl U *with end points in* Bd U *such that the distance from x to A is less than ε.*

Proof. We shall assume that each arc S irreducible from a point p of U to Bd U is ordered from p to Bd U. Furthermore, for $a, b \in S$, $S[a, b]$ denotes the closed interval of S from a to b; open and half-open interval notation denote analogous subsets of S.

Let T be an arc irreducible from x to Bd U, and let b be the point of $T \cap \text{Bd } U$. Let Q be the set of all points $y \in T$ such that there exists an arc S containing y and irreducible between two points of Bd U. Since no point cuts, there exists an arc S' containing x and intersecting Bd $U - \{b\}$ but missing b. Then in $T \cup S'$ there is an arc which contains a point of $T - \{b\}$ and is irreducible between b and some other point of Bd U. Hence $Q \neq \varnothing$. Let $q = \text{glb } Q$. We need only show that $q = x$.

Assume that $q \neq x$. Since q does not cut x from Bd U, there exists an arc D from x to Bd U missing q. Since $q = \text{glb } Q$, $D \cap T(q, b] \neq \varnothing$. Let y be the first point (with respect to the order on D) of $D \cap T(q, b]$. Let z be the last point (w.r.t. D) of $D[x, y] \cap T[x, q]$. We may assume that $D = T[x, z] \cup D[z, y] \cup T[y, b]$.

Since q is either in Q or a limit point of Q, there exists a point $w \in T(z, y) \cap Q$ (possibly $w = q$). Thus there are arcs A and B each from w to Bd U such that $A \cap B = \{w\}$. We may assume that w

precedes all other points of $(A \cup B) \cap T$ [w.r.t. T]. If $D \cap B = \emptyset$, then $z \in Q$ because of the arc $B \cup T[z, w] \cup D[z, b]$. But since this contradicts the fact that $q = \operatorname{glb} Q$, we have that $D \cap B \neq \emptyset$. Let v denote the first point (w.r.t. D) of $D \cap B$. If $A \cap D[z, v] = \emptyset$, then $z \in Q$ because of the continuum $A \cup T[z, w] \cup D[z, v] \cup B[v, b']$ where b' is the point of $B \cap \operatorname{Bd} U$. This contradiction implies that $A \cap D[z, v] \neq \emptyset$. Let p be the first point (w.r.t. D) of $A \cap D[z, v]$ and let a be the point of $A \cap \operatorname{Bd} U$. Then $A[p, a] \cup D[z, p] \cup T[z, w] \cup B$ shows that $z \in Q$. This contradiction implies that $q = x$ and the proof is complete.

LEMMA 5. *Suppose that M is a compact metric continuum in which no point cuts and only semi-aposyndetic subcontinua separate. If M is semi-aposyndetic at $\{x, y\}$, then M is aposyndetic at x with respect to y.*

Proof. Assume that M is not aposyndetic at x with respect to y. By semi-aposyndesis, there exists a subcontinuum $B \subset M - \{x\}$ such that $y \in B^\circ$. Let $C, C_1, C_2, \cdots$ be the closures of distinct components of $M - B$ such that $x \in \lim C_i \subset C$. Using Lemmas 3 and 4, we can construct (for each i) points p_i and q_i in $B \cap C_i$ and an arc A_i irreducible from p_i to q_i in C_i such that $A_i \cap B = \{p_i, q\}$ and $\lim A_i$ is non-degenerate [taking a subsequence, if necessary]. Let $A = \lim A_i$ and select distinct points $w, z \in A$. Let $w_i, z_i \in A_i - \{p_i, q_i\}$ (for each i) such that $w = \lim w_i$ and $z = \lim z_i$. We may assume that w_i precedes z_i in the order that A_i has from p_i to q_i. Let D_i be the subarc of A_i defined by $D_i = [p_i, z_i]$ for odd i, and $D_i = [w_i, q_i]$ for even i. Then $(\cup D_i) \cup A \cup B$ is a subcontinuum which separates M but which is not semi-aposyndetic at $\{w, z\}$. This contradiction concludes the proof of the lemma.

THEOREM 4. *A compact metric continuum M is hereditarily locally connected and cyclically connected if and only if no point cuts in M and only semi-aposyndetic subcontinua separate M.*

Proof. Since the necessity is obvious, we consider the sufficiency. Using Theorem 3, Lemma 2, and [7, p. 138], it is clear that we need only show that M is aposyndetic.

Suppose that x and u are points of M such that M is not aposyndetic at x with respect to u. Since no point cuts in M, M is both aposyndetic and semi-locally-connected on a dense G_δ-subset Z of M [6, p. 412]. By Theorem 2, M is connected im kleinen at each point of Z. Let $y, z \in Z - \{x, u\}$, and let H and K be disjoint subcontinua in $M - \{x, u\}$ such that $y \in H^\circ$ and $z \in K^\circ$.

Suppose that $M - (H \cup K)$ is connected. Then the continuum $\mathrm{Cl}\,[M - (H \cup K)]$ is semi-aposyndetic since it separates y from z. Hence M is semi-aposyndetic at $\{x, u\}$. By Lemma 5, M is aposyndetic at x with respect to u. This contradiction implies that $M - (H \cup K)$ is not connected.

Thus $M - (H \cup K) = D \cup E$, separated. One of $H \cup D \cup K$ and $H \cup E \cup K$ must be a continuum. We shall show that the other is also. Let $H \cup D \cup K$ be a continuum and suppose that $H \cup E \cup K = P \cup Q$, separated subcontinua, with $H \subset P$ and $K \subset Q$.

The continuum $H \cup D \cup K$ is not irreducible about $H \cup K$, or else points in D will cut P from Q. Let W be a proper subcontinuum of $H \cup D \cup K$ containing $H \cup K$. Suppose $P \neq H$ and $Q \neq K$. Then the three continua $H \cup D \cup K$, $P \cup W$, and $Q \cup W$ each separate M and hence are semi-aposyndetic. Also each of x and u is in the interior of one of them. Thus their union, namely M, is semi-aposyndetic at $\{x, u\}$. Then by Lemma 5, M is aposyndetic at x with respect to u. Thus it cannot be the case that $P \neq H$ and $Q \neq K$. We assume, without loss of generality, that $P = H$. Then $Q = K \cup E$.

In order to show that $x \in D$, we suppose that this is not the case, i.e., that $x \in E$. The continuum Q is not irreducible about $K \cup \{x\}$, or else x will be cut (in M) from K by any point of $E - \{x\}$. Let T be a proper subcontinuum of Q containing both x and K. In order to show that $Q - T$ is connected, we suppose that $Q - T = T_1 \cup T_2$, separated. Then $T \cup T_1$ and $T \cup T_2$ are separating, hence semi-aposyndetic, subcontinua. Assume that $u \notin T$, so that $u \in T_1$, say. Then $T \cup T_1$ is aposyndetic at either (1) u with respect to x, or (2) x with respect to u. In the first case, it would follow immediately that M is aposyndetic at u with respect to x, and by Lemma 5 we would have a contradiction. In the second case, M would be aposyndetic at x with respect to u because of the continuum which is the union of T_2, T, and the subcontinuum of $T \cup T_1$ missing u and containing x in its interior (relative to $T \cup T_1$). This contradiction implies that $u \in T$. Each of $T \cup T_1$ and $T \cup T_2$ are semi-aposyndetic at $\{x, u\}$. Without loss of generality, we may assume that there is a subcontinuum S_1 of $T \cup T_1$ such that $x \in S_1^\circ$ (relative to $T \cup T_1$) and $u \notin S_1$. Now $T \cup T_2$ cannot be aposyndetic at x with respect to u since it would follow that M also is aposyndetic at x with respect to u. Thus there is a subcontinuum S_2 of $T \cup T_2$ such that $u \in S_2^\circ$ (relative to $T \cup T_2$) and $x \notin S_2$. The continuum $T \cup S_1$ separates $T \cup T_1$ into sets A_1 and B_1 (otherwise $S_2 \cup \mathrm{Cl}\,(T_1 - S_1)$ would be a continuum with u in its interior and missing x, and by Lemma 5 we would arrive at a contradiction). Similarly $T \cup S_2$ separates $T \cup T_2$ into sets A_2 and B_2. Then $T \cup S_1 \cup S_2 \cup A_1 \cup A_2$ is a continuum. Since it separates M, it must be semi-aposyndetic. Thus it contains a subcontinuum S_3 which, say, misses

x and contains u in its relative interior. In a similar manner, $T \cup S_1 \cup S_2 \cup B_1 \cup B_2$ is a semi-aposyndetic subcontinuum of M. If it contains a continuum missing x and containing u in its relative interior, then the union of that continuum with S_3 will miss x and contain u in its interior (relative to M) and by Lemma 5, we would arrive at a contradiction. So there must be a subcontinuum S_4 missing u and containing x in its interior (relative to $T \cup S_1 \cup S_2 \cup B_1 \cup B_2$). Again in a similar manner, $T \cup S_1 \cup S_2 \cup B_1 \cup A_2$ is a continuum which separates M and hence is semi-aposyndetic. In case this continuum is aposyndetic at x with respect to u, then it follows that M is also. Thus there is a subcontinuum S_5 which misses x and contains u in its relative interior. Then $S_3 \cup S_5$ is a continuum missing x and containing u in its interior (relative to M) and by Lemma 5, M is aposyndetic at x with respect to u. This contradiction implies that $Q - T$ is connected. The dense G_δ-set Z intersects $Q - T$, so the continuum $\mathrm{Cl}\,(Q - T)$ is decomposable and hence can be written as the union of two proper subcontinua X and Y. Suppose X does not intersect T. Then x is in the interior of the continuum $Y \cup T$ that separates M. It follows that M is semi-aposyndetic at $\{x, u\}$. Then by Lemma 5, we arrive at a contradiction. Thus both X and Y must intersect T. Each of the continua $X \cup T$ and $Y \cup T$ separate M and hence are semi-aposyndetic. Using an argument similar to the one above (which involved $T \cup T_1$ and $T \cup T_2$), we arrive at a contradiction.

Since the assumption that $x \in E$ has led to a contradiction, it must be that $x \in D$. The set D cannot be connected, or else, $\mathrm{Cl}\, D$ is semi-aposyndetic since it separates M, and by Lemma 5 we would have a contradiction. Thus $D = D_1 \cup D_2$, separated, with $x \in D_1$. Let A denote the x-component of $D_1 \cup H \cup K$. Since $D_1 \cup H \cup K$ has at most two components, $x \in A^\circ$. If $K \subset A$, then A is a continuum which separates D_2 from E, and hence is semi-aposyndetic. Then by Lemma 5, we would arrive at a contradiction. Thus we suppose that $K \cap A = \phi$. Then A meets H, and $\mathrm{Cl}\, D_2$ meets both H and K. Let $D' = D_2 \cup E$ and $E' = D_1$. Then $H \cup K \cup D'$ is connected while $H \cup K \cup E'$ is not. However, earlier (the portion of the proof which preceded this paragraph) we showed that x could not lie in such a part of a separation of $M - (H \cup K)$. This contradiction implies that the original supposition that $H \cup E \cup K$ is not connected is false. Hence both $H \cup D \cup K$ and $H \cup E \cup K$ are continua.

Suppose both $H \cup D \cup K$ and $H \cup E \cup K$ are irreducible about $H \cup K$. Since M has no cut points, no point of D cuts any other point of D from $H \cap K$ in $H \cup D \cup K$. Assume that H cuts a point d of D from K in $H \cup D \cup K$. Since no point cuts in M and $H \cap \mathrm{Cl}\, D$ cuts the point d from K in M, then $H \cap \mathrm{Cl}\, D$ must contain more than one point. If $H \cap \mathrm{Cl}\, D \cap \mathrm{Cl}\, E \neq \phi$, then $\mathrm{Cl}\, D \cup \mathrm{Cl}\, E$ is a separating, hence semi-aposyndetic, subcontinuum, and by Lemma 5 we have a contradiction. Thus $H \cap \text{Cl}\, D \cap \text{Cl}\, E = \phi$. Consequently, $\text{Cl}\, H^\circ \cap \text{Cl}\, D \neq \phi$, or else the continuum $H \cup D \cup K$ would be the union of two separated sets $\text{Cl}\, H^\circ \cup (H \cap \text{Cl}\, E)$ and $K \cup \text{Cl}\, D$. Next, using Lemma 5 and the fact that the continuum $\text{Cl}\, D \cup K \cup \text{Cl}\, E$ is the complement of H°, it follows that H° is connected. Similarly, K° is connected. Suppose $\text{Cl}\, H^\circ$ contains a proper subcontinuum R which intersects both $H \cap \text{Cl}\, D$ and $H \cap \text{Cl}\, E$. Then the continuum $\text{Cl}\, D \cup R \cup \text{Cl}\, E$ is semi-aposyndetic since it separates $H^\circ - R$ from K°, and by Lemma 5 we reach a contradiction. Thus $\text{Cl}\, H^\circ$ is irreducible from $H \cap \text{Cl}\, D$ to $H \cap \text{Cl}\, E$. Similarly $\text{Cl}\, K^\circ$ is irreducible from $K \cap \text{Cl}\, D$ to $K \cap \text{Cl}\, E$. It follows that $\text{Cl}\, K^\circ \cup \text{Cl}\, D$ is irreducible from $H \cap \text{Cl}\, D$ to $K \cap \text{Cl}\, E$. Note that $\text{Cl}\, H^\circ$ and $\text{Cl}\, K^\circ \cup \text{Cl}\, D$ are the only two subcontinua of M irreducible from $H \cap \text{Cl}\, D$ to $\text{Cl}\, E$. Let $a \in \text{Cl}\, H^\circ \cap \text{Cl}\, D$ and let $b \in H \cap \text{Cl}\, D - \{a\}$. Since no point cuts in M, there exists a continuum R which contains b, intersects $\text{Cl}\, E$, and misses the point a. Then R must contain one of the two continua $\text{Cl}\, H^\circ$ and $\text{Cl}\, K^\circ \cup \text{Cl}\, D$, each of which contains the point a. Since $a \notin R$, we have a contradiction.

Using a similar argument for the case of K cutting b in D from H in $H \cup D \cup K$, we have that neither H nor K cuts the other from any point of D in $H \cup D \cup K$. Thus the upper semi-continuous decomposition whose elements are points of D together with the two sets H and K is an arc [1, p. 501]. Similarly, $H \cup E \cup K$ can be decomposed into an arc. Then M is aposyndetic at each point of $D \cup E$, hence at x. This contradiction implies that one of $H \cup D \cup K$ and $H \cup E \cup K$ is not irreducible about $H \cup K$.

Let N be a proper subcontinuum of $H \cup D \cup K$ irreducible about $H \cup K$. Since the G_δ-set Z is dense, there exist points p and q in $D - (N \cup \{x, u\})$ and $E - \{x, u\}$ respectively at which M is connected im kleinen. Thus there exist subcontinua P and Q such that $P \in P^\circ \subset P \subset D - (N \cup \{x, u\})$ and $q \in Q^\circ \subset Q \subset E - \{x, u\}$. As was shown above (with $M - (H \cup K)$), we have that $M - (P \cup Q) = S \cup T$, separated, such that $P \cup S \cup Q$ and $P \cup T \cup Q$ are continua. We may assume that $N \subset S$. Thus the continuum $P \cup T \cup Q$ misses N (hence $H \cup K$) and therefore is contained in $D \cup E$. But since $p \in D$ and $q \in E$, the continuum $P \cup T \cup Q$ intersects both parts of the separation $D \cup E$. This impossibility implies, contrary to our initial assumption, that M is aposyndetic at x. Thus the proof is complete.

Just as in [9, p. 262], an easy application of Theorem 4 yields the following result due to Bing [1, p. 504]:

COROLLARY. *Every compact metric continuum in which no point cuts and no subcontinuum separates is a simple closed curve.*

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