Pacific Journal of Mathematics

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Vol. 43, No. 3 May 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 3, 1972

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In this paper it is proved that a C^*-algebra A is strongly amenable iff A satisfies a certain fixed point property when acting on a compact convex set, or iff a certain Hahn-Banach type extension theorem is true for all Banach A-modules. It is proved that a C^*-algebra A is amenable iff A satisfies a weaker Hahn-Banach type extension theorem.

A topological group G is said to be amenable if there is a left invariant mean on the space of bounded continuous complex functions on G. A number of papers have been published which give equivalent definitions of amenability (for example, see the papers [4, 7, 11] or the book [3]). It has recently been proven that a locally compact group G is amenable iff for all two-sided $L^1(G)$-modules X and bounded derivations D of $L^1(G)$ into X^*, we have that D is the inner derivation induced by an element of X^* [5, Theorem 2.5]. This result motivates the definition of amenable and strongly amenable C^*-algebras [5, sections 5 and 7]. In § 2 of this paper we give some conditions on a C^*-algebra that are equivalent to amenability or strong amenability and are analogous to some of the known equivalent definitions of amenable groups. In § 3 we show that the generalized Stone-Weierstrass theorem for separable C^*-algebras is true when the C^*-subalgebra in question is strongly amenable.

1. Preliminaries. Let A be a C^*-algebra. Then a complex Banach space X is called a Banach A-module if it is a two-sided A-module and there exists a positive real number M such that for all $a \in A$ and $x \in X$ we have

$$||ax|| \leqq M||a|| \, ||x||$$

and

$$||xa|| \leqq M||x|| \, ||a|| \, .$$

If X is a Banach A-module, then the dual space X^* becomes a Banach A-module if we define for $a \in A,, f \in X^*$, and $x \in X$,

$$(af)(x) = f(xa)$$
$$(fa)(x) = f(ax) \, .$$

A derivation from A into X^* is a bounded linear map D from A into

X^* such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If $f \in X^*$, the function $\delta(f)$ from A into X^* given by

$$\delta(f)(a) = af - fa$$

is called the inner derivation induced by f.

DEFINITION 1. [5, § 5]. A C^*-algebra A is said to be *amenable* if every derivation from A into X^* is inner for all Banach A-modules X.

DEFINITION 2. [5, § 7]. A C^*-algebra A is said to be *strongly amenable* if, whenever X is a Banach A-module and D is a derivation of A into X^*, there is a $f \in \text{co}\,\{D(u)u^*: u \in U(A_e)\}$ with $D = -\delta(f)$, where A_e is the C^*-algebra obtained by adjoining the identity e to A, X is made into a unital A_e-module by defining $xe = ex = x$ for all $x \in K$, D is extended to A_e by defining $D(e) = 0$, $U(A_e)$ is the unitary group of A_e, and $\text{co}\,S$ denotes the w^*-closed convex hull of a set S contained in X^*.

A C^*-algebra A is strongly amenable iff A_e is strongly amenable, and a C^*-algebra A with identity is strongly amenable iff the definition is satisfied for all unital A-modules X with A_e replaced throughout by A [5, § 7]. The class of strongly amenable C^*-algebras includes all C^*-algebras which are GCR, uniformly hyperfinite, or the C^*-group algebra of a locally compact amenable group [5, § 7]. It is not known if there exist amenable C^*-algebras which are not strongly amenable.

For A a C^*-algebra, let $A \hat{\otimes} A$ be the completion of the algebraic tensor product $A \otimes A$ in the greatest cross-norm. Then we can identify $(A \hat{\otimes} A)^*$ with the space of bounded bilinear functionals on $A \times A$ [13, p. 438]. We see that $A \hat{\otimes} A$ becomes a Banach A-module if we define for $a, b, c \in A$,

$$a(b \otimes c) = ab \otimes c$$
$$(b \otimes c)a = b \otimes ca\ .$$

Hence $(A \hat{\otimes} A)^*$ becomes a Banach A-module under the dual action: If $f \in (A \hat{\otimes} A)$ and $a, b, c \in A$,

$$(af)(b \otimes c) = f(b \otimes ca)$$
$$(fa)(b \otimes c) = f(ab \otimes c)\ .$$

We can also make $A \hat{\otimes} A$ and $(A \hat{\otimes} A)^*$ into Banach A-modules by defining for $f \in (A \hat{\otimes} A)^*$ and $a, b, c \in A$:

$$a \circ (b \otimes c) = b \otimes ac$$
$$(b \otimes c) \circ a = ba \otimes c$$

$$(a\circ f)(b \otimes c) = f(ba \otimes c)$$
$$(f\circ a)(b \otimes c) = f(b \otimes ac) \; .$$

Note that the two operations on $A \hat{\otimes} A$ do not interact; that is, if $a, b, c, d \in A$,

$$\begin{aligned} a\circ(b(c \otimes d)) &= b(a\circ(c \otimes d)) \\ ((c \otimes d)b)\circ a &= ((c \otimes d)\circ a)b \\ a\circ((c \otimes d)b) &= (a\circ(c \otimes d))b \end{aligned}$$

and so forth.

2. Amenable and strongly amenable C^*-algebras.

PROPOSITION 1. *Let A be a C^*-algebra with unit e. Then the following seven statements are equivalent:*

(a) *A is strongly amenable.*

(b) *For all unital Banach A-modules X and $f \in X^*$, there exists $g \in \mathrm{co}\,\{ufu^*: u \in U(A)\}$ such that $ag = ga$ for all $a \in A$.*

(c) *For any $f \in (A \hat{\otimes} A)^*$ there exists $g \in \mathrm{co}\,\{ufu^*: u \in U(A)\}$ such that $ag = ga$ for all $a \in A$.*

(d) *There is a linear map T of $(A \hat{\otimes} A)^*$ into $C = \{g \in (A \hat{\otimes} A)^*: ag = ga$ all $a \in A\}$ such that $T(a\circ f) = a\circ T(f)$, $T(f\circ a) = T(f)\circ a$, and $T(f) \in \mathrm{co}\,\{ufu^*: u \in U(A)\}$ for all $a \in A, f \in (A \hat{\otimes} A)^*$.*

(e) *Let X be a Banach A-module, S a w^*-closed convex subset of X^* such that $usu^* \in S$ for all $s \in S, u \in U(A)$. Then there exists an element $s \in S$ such that $usu^* = s$ for all $u \in U(A)$.*

(f) *Let Y be a Banach A-module and X a subspace of Y such that $uxu^* \in X$ for all $x \in X, u \in U(A)$. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends f, there is an $h \in \mathrm{co}\,\{ugu^*: u \in U(A)\}$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y$ and $u \in U(A)$.*

(g) *Let Y be a Banach A-module and X a two-sided A-submodule of Y. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends f, there is an $h \in \mathrm{co}\,\{ugu^*: u \in U(A)\}$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y$ and $u \in U(A)$.*

Before proving the proposition, we make some remarks. The implications (a) implies (b) and (b) implies (d) were proven in [1]. The map T in (d) takes the place of the invariant mean that is present in amenable groups. The condition in (e) is a fixed point property; it is known that a locally compact group is amenable iff it has a certain fixed point property [7]. The condition (f) and (g) might be called the strong invariant extension property for subspaces, and the strong invariant extension property for submodules respectively. A locally compact group is amenable iff it has a certain Hahn-Banach type extension property similar to (f) and (g) [11].

Proof of Proposition 1. (*a*) *implies* (*b*): Let $f \in X^*$ and $\delta(f)$ be the inner derivation induced by f. Then there is a $a \in \mathrm{co}\,\{\delta(f)(u)u^*: u \in U(A)\}$ such that $\delta(f) = -\delta(g)$. But $\delta(f)(u)u^* = ufu^* - f$, hence $f + g \in \mathrm{co}\,\{ufu^*: u \in U(A)\}$. Also $\delta(f)(a) = -\delta(g)(a)$ for all $a \in A$, thus $(f + g)a = a(f + g)$ for all $a \in A$.

(*b*) *implies* (*c*): Clear.

(*c*) *implies* (*d*): The proof is an adaption to the present situation of a proof of J. Schwartz [10, Lemma 5]. Let Λ be the set of all linear mappings T of $(A \hat{\otimes} A)^*$ into $(A \hat{\otimes} A)^*$ such that $T(f) \in \mathrm{co}\,\{ufu^*: u \in U(A)\}$ and $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$ for all $f \in (A \hat{\otimes} A)^*$, $a \in A$. The set Λ is nonempty since the identity map is in Λ. We order Λ by defining $T_1 \geqq T_2$ if for all $f \in (A \hat{\otimes} A)^*$,

$$\mathrm{co}\,\{uT_1(f)u^*: u \in U(A)\} \subseteqq \mathrm{co}\,\{uT_2(f)u^*: u \in U(A)\}\ .$$

Then $\geqq$ defines a quasi-order on Λ. Suppose $\{T_\alpha: \alpha \in \Delta\}$ is a totally ordered subset of Λ. We have $\|T_\alpha(f)\| \leqq \|f\|$, thus for all $d \in A \hat{\otimes} A$, $|T_\alpha(f)(d)| \leqq \|f\|\,\|d\|$ and $\{T_\alpha(f)(d): \alpha \in \Delta\}$ is a bounded function on Δ. Let LIM be a Banach limit on the directed set Δ (see [10, p. 21] for information on Banach limits). Then set $T(f)(d) = \mathrm{LIM}\, T_\alpha(f)(d)$ for all $f \in (A \hat{\otimes} A)^*$ and $d \in A \hat{\otimes} A$. Then T is a bounded linear map from $(A \hat{\otimes} A)^*$ into $(A \hat{\otimes} A)^*$. An easy calculation shows that $T(a \circ f) = a \circ T(f)$ and $T(f \circ a) = T(f) \circ a$. We show that $T(f) \in \mathrm{co}\,\{ufu^*: u \in U(A)\}$ and $T \geqq T_\alpha$ for all $\alpha \in \Delta$. If $\beta \geqq \alpha$ and $f \in (A \hat{\otimes} A)^*$ then

$$T_\beta(f) \in \mathrm{co}\,\{uT_\alpha(f)u^*: uU(A)\} = K\ .$$

Suppose, for contradiction, that $T(f) \notin K$. Then by the strong separation theorem, there exists $d \in A \hat{\otimes} A$, λ real and $\varepsilon > 0$ such that for all $g \in K$,

$$\mathrm{Re}\, T(f)(d) \leqq \lambda < \lambda + \varepsilon \leqq \mathrm{Re}\, g(d)\ .$$

Hence $\mathrm{Re}\, T(f)(d) \leqq \lambda < \lambda + \varepsilon \leqq \mathrm{Re}\, T_\beta(f)(d)$ for all $\beta \geqq \alpha$. But applying the Banach limit to this equation we obtain $\mathrm{Re}\, T(f)(d) < \mathrm{Re}\, T(f)(d)$. Hence $T(f) \in K$. Thus

$$\mathrm{co}\,\{uT(f)u^*: u \in U(A)\} \subseteqq \mathrm{co}\,\{uT_\alpha(f)u^*: u \in U(A)\}$$

for all $\alpha \in \Delta$. Hence $T \in \Lambda$ and $T \geqq T_\alpha$. Hence Λ is inductive, so by Zorn's lemma Λ has a maximal element T. We show that $T(f) \in C$ for all $f \in (A \hat{\otimes} A)^*$. If $g \in (A \hat{\otimes} A)^*$ is such that $T(g) \notin C$, then $\mathrm{co}\,\{uT(g)u^*: u \in U(A)\}$ contains more than one element. Since we assumed (c), $C \cap \mathrm{co}\{uT(g)u^*: u \in U(A)\}$ is nonempty. Let $\sum \lambda uT(g)u^*$ be a net indexed by a directed set Δ which converge w^* to an element h of C (we suppress all indices in the sum). Define for $f \in (A \hat{\otimes} A)^*$ and $d \in (A \hat{\otimes} A)$,

$$T'(f)(d) = \mathrm{LIM} \sum \lambda uT(f)u^*(d) .$$

Then T' is a bounded linear map from $(A \hat{\otimes} A)^*$ to $(A \hat{\otimes} A)^*$ and another application of the strong separation theorem shows that $T'(f) \in \mathrm{co}\{uT(f)u^*: u \in U(A)\}$. If we show that $T'(a \circ f) = a \circ T'(f)$ and $T'(f \circ a) = T'(f) \circ a$, we will know that $T' \in \Lambda$ and $T' \geqq T$. But

$$\begin{aligned} T'(a \circ f)(b \otimes c) &= \mathrm{LIM} \sum \lambda uT(a \circ f)u^*(b \otimes c) \\ &= \mathrm{LIM} \sum \lambda(a \circ T(f))(u^*b \otimes cu) \\ &= \mathrm{LIM} \sum \lambda T(f)(u^*ba \otimes cu) \\ &= T'(f)(ba \otimes c) \\ &= (a \circ T'(f))(b \otimes c) . \end{aligned}$$

Hence $T'(a \circ f) = a \circ T'(f)$ and likewise $T'(f \circ a) = T'(f) \circ a$. But the net $\sum \lambda uT(g)u^*(d)$ has the actual limit $h(d)$. Thus $T'(g) = h$ and $\mathrm{co}\{uT'(g)'(g)u^*: u \in U(A)\} = \{h\}$, and $\{h\}$ is properly contained in $\mathrm{co}\{uT(g)u^*: u \in U(A)\}$, hence it is not true that $T \geqq T'$. But this contradicts the maximality of T, and we have that $T(f) \in C$ for all f. The completes the proof.

(*d*) *implies* (*e*): Let $x \in X$ and fix $s \in S$. Define a bounded bilinear function $F(x, s)$ on $A \times A$ by $F(x, s)(a, b) = s(axb)$ for all $a, b \in A$. We consider $F(x, s)$ as an element of $(A \hat{\otimes} A)^*$. Let $G(s)(x) = T(F(x, s))(e \otimes e)$. Then clearly $G(s) \in X^*$. Now if $u \in U(A)$, then

$$\begin{aligned} F(u^*xu, s)(a \otimes b) &= s(au^*xub) \\ &= F(x, s)(u \circ (a \otimes b) \circ u^*) \\ &= (u^* \circ F(x, s) \circ u)(a \otimes b) . \end{aligned}$$

Thus $F(u^*xu, s) = u^* \circ F(x, s) \circ u$. Hence for all $x \in X$ and $u \in U(A)$, by using the properties of the map T,

$$\begin{aligned} (uG(s)u^*)(x) &= G(s)(u^*xu) \\ &= T(F(u^*xu, s))(e \otimes e) \\ &= T(u^* \circ F(x, s) \circ u)(e \otimes e) \\ &= (uT(F(x, s))u^*)(e \otimes e) \\ &= T(F(x, s))(e \otimes e) \\ &= G(s)(x) . \end{aligned}$$

Thus $uG(s)u^* = G(s)$ for all $u \in U(A)$. We will be done when we show that $G(s) \in S$. If $G(s) \notin S$, then there exists $x \in X$, a real number

λ and $\varepsilon > 0$ such that for all $t \in S$ we have

$$\operatorname{Re} G(s)(x) \leqq \lambda < \lambda + \varepsilon \leqq \operatorname{Re} t(x) .$$

Now $T(F(x, s)) \in \operatorname{co}\{uF(x, s)u^*: u \in U(A)\}$, and $(uF(x, s)u^*)(e \otimes e) = (usu^*)(x)$. Since $usu^* \in S$ for all $u \in U(A)$, this implies that $\operatorname{Re} G(s)(x) \geqq \lambda + \varepsilon$. This contradiction proves that $G(s) \in S$.

(*e*) *implies* (*f*): Let X, Y and $f \in X^*$ be as in (f). Let $g \in Y^*$ be any extension of f and let $S = \operatorname{co}\{ugu^*: u \in U(A)\}$. Then S is w^*-closed convex subset of Y^* and if $s \in S$ then $usu^* \in S$ for all $u \in U(A)$. Hence by (e) there is an element $h \in S$ such that $uhu^* = h$ for all $u \in U(A)$. Since $uXu^* \subseteqq X$ for all $u \in U(A), f(uxu^*) = f(x)$ for all $x \in X$ and g extends f, it is easily seen that h extends f.

(*f*) *implies* (*g*): Clear.

(*g*) *implies* (*c*): Given $g \in (A \hat{\otimes} A)^*$, let $Y = A \hat{\otimes} A$, $X = \{0\}, f = 0$ and apply (*g*). Thus there is an $h \in \operatorname{co}\{ugu^*; u \in U(A)\}$ such that $h(uyu^*) = h(y)$ for all $y \in A \hat{\otimes} A$ and $u \in A$; that is, $ah = ha$ for all a in A.

(*d*) *implies* (*a*): Let $D: A \rightarrow X^*$ be a derivation. Let $x \in X$ and define a bounded bilinear functional $f(x)$ on $A \times A$ by $f(x)(b, c) = D(b)(xc)$. Then define an element $G \in X^*$ by $G(x) = T(f(x))(e \otimes e)$. We will show that $D = \delta(G)$ and $-G \in \operatorname{co}\{D(u)u^*: u \in U(A)\}$. For $x \in X$ and $a \in A$, define a bounded bilinear functional $g(x, a)$ on $A \times A$ by $g(x, a)(b, c) = D(a)(xcb)$. Then a computation shows that $a \circ f(x) = f(ax) + g(x, a)$ and $f(xa) = f(x) \circ a$. If $a \in A$, $x \in X$, then

$$\begin{aligned}
(\delta(G)(a))(x) &= (aG - Ga)(x) \\
&= G(xa - ax) \\
&= T(f(xa - ax))(e \otimes e) \\
&= T(f(x) \circ a - a \circ f(x) + g(x, a))(e \otimes e) \\
&= T(f(x))(e \otimes a) - T(f(x))(a \otimes e) + T(g(x, a))(e \otimes e) \\
&= T(g(x, a))(e \otimes e) .
\end{aligned}$$

The last equality is true because T maps into C. Now for $u \in U(A)$, $(ug(x, a)u^*)(e \otimes e) = g(x, a)(u^* \otimes u) = D(a)(x)$, and $T(g(x, a))$ is in $\operatorname{co}\{ug(x, a)u^*: u \in U(A)\}$, hence $(\delta(G)(a))(x) = D(a)(x)$ for all $x \in X$ and $a \in A$, thus $D = \delta(G)$. An application of the strong separation theorem shows that $-G \in \operatorname{co}\{D(u)u^*: u \in U(A)\}$. Thus (d) implies (a) and the proof of Proposition 1 is complete.

REMARKS. (1) The equivalence of (a) and (c) shows that, to check strong amenability of A, it is only necessary to consider the A-module $A \hat{\otimes} A$; this gives another proof of the Proposition 7.15 of [5].

(2) In the notation of [6], a C^*-algebra A is amenable iff the first cohomology group $H_c^1(A, X^*)$ is zero for all Banach A-modules X. The reduction of dimension argument of [5, §1. a] then shows that all the cohomology groups $H_c^n(A, X^*)$ are zero. If A is strongly amenable, then the proof of (d) implies (a) above can be changed to show directly that $H_c^n(A, X^*)$ is zero for all n and all Banach A-modules X; this proof is similar to the proof of Theorem 3.3 in [6], with the map T taking the place of the invariant mean which is present in that theorem.

(3) In [1] the author used the existence of the function T to generalize the well-known Dixmier-Mackey theorem on amenable groups by proving that every continuous representation of a strongly amenable C^*-algebra on a Hibert space is similar to a *-representation. However, this fact can be proved in a more elementary fashion as follows: Let A be a strongly amenable C^*-algebra and let π be a continuous representation of A as bounded operators on a Hilbert space H. It suffices to assume A has an identity e and $\pi(e) = I$. Make $B(H)$ into a Banach A-module by the operations $aT = \pi(a)T$, $Ta = T\pi(a^*)^*$ for $a \in A$, $T \in B(H)$. Then $B(H)$ is the dual Banach A-module of the trace class operators. Define a bounded linear map D of A into $B(H)$ by $D(a) = \pi(a) - \pi(a^*)^*$. Then

$$\begin{aligned} aD(b) + D(a)b &= \pi(a)(\pi(b) - \pi(b^*)^*) + (\pi(a) - \pi(a^*)^*)\pi(b^*)^* \\ &= \pi(ab) - \pi((ab)^*)^* \\ &= D(ab) . \end{aligned}$$

Hence D is a derivation. Thus, since A is strongly amenable, there is an operator T in $\mathrm{co}\,\{D(u)u^*: u \in U(A)\}$ such that $D = -\delta(T)$. Then for $a \in A$, $\pi(a) - \pi(a^*)^* = -aT + Ta = T\pi(a^*)^* - \pi(a)T$, thus $\pi(a)(I + T) = (I + T)\pi(a^*)^*$. We let $R = T + I$. Now $D(u)u^* = \pi(u)\pi(u)^* - I$, thus $R \in \mathrm{co}\,\{\pi(u)\pi(u)^*: u \in U(A)\}$. For $x \in H$, $(\pi(u)\pi(u)^*x, x) = ||\pi(u)^*x||^2$, and $||x||^2 = ||\pi(u^*)^*\pi(u)^*x||^2 \leqq ||\pi||^2 ||\pi(u)^*x||^2$. Hence $(1/||\pi||^2)||x||^2 \leqq ||\pi(u)^*x||^2$. Thus R is positive and invertible. Let S be the positive square root of R. Then $\pi(a)S^2 = S^2\pi(a^*)^*$, and $S^{-1}\pi(a)S = S\pi(a^*)^*S^{-1}$. If we define $\pi_1(a) = S^{-1}\pi(a)S$, then π_1 is clearly a representation of A, and $\pi_1(a^*)^* = (S^{-1}\pi(a^*)S)^* = S\pi(a^*)^*S^{-1} = S^{-1}\pi(a)S = \pi_1(a)$. Hence π_1 is a *-representation.

We now give some equivalent conditions for a C^*-algebra to be amenable.

PROPOSITION 2. *Let A be a C^*-algebra with unit e. Then the* *following three statements are equivalent:*

(a) *A is amenable.*

(b) *There is a bounded linear map T of $(A \hat{\otimes} A)^*$ into $C = \{f \in (A \hat{\otimes} A)^*: af = fa \text{ all } a \in A\}$ such that T restricted to C is the identity on C and $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$ for all $a \in A$, $f \in (A \hat{\otimes} A)^*$.*

(c) *Let Y be a Banach A-module and X a two sided A-submodule of Y. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X$, $u \in U(A)$. Then there is a $h \in Y^*$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y$, $u \in U(A)$.*

Proof. *(a) implies (b)*: Let $Y = (A \hat{\otimes} A)^* \hat{\otimes} (A \hat{\otimes} A)$ and let Z, W and X be as in the proof of (g) implies (d) of Proposition 1. Let $F \in Y^*$ be defined by $F(f \otimes t) = f(t)$ and let D_1 be the inner derivation induced by F. Then for $a \in A, f \in (A \hat{\otimes} A)^*$, and $t \in (A \hat{\otimes} A)$, $D_1(a)(f \otimes t) = (af - fa)(t)$. Hence $D_1(a)$ is zero on W. A calculation using the fact that the two A-module operations on $(A \hat{\otimes} A)$ do not interact (see the comment at the end of § 1) shows that $D_1(a)$ is zero on Z. Hence there is a map D from A into $(Y/X)^*$ given by $D(\mathrm{a})(\bar{y}) = D_1(a)(y)$, where $\bar{y}$ is the coset in Y/X of an element $y \in Y$. It is easily seen that D is a derivation, hence since A is amenable there is an element $G_1 \in (Y/X)^*$ such that $D = \delta(G_1)$. Let $G \in Y^*$ be defined by $G(y) = G_1(\bar{y})$ for all $y \in Y$. Define a bounded linear map T_1 from $(A \hat{\otimes} A)^*$ into $(A \hat{\otimes} A)^*$ by $T_1(f)(t) = G(f \otimes t)$ for all $f \in (A \hat{\otimes} A)^*$, $t \in (A \hat{\otimes} A)$. Now $D(a) = aG_1 - G_1 a$ for all $a \in A$, thus

$$\begin{aligned} D(a)\overline{(f \otimes t)} &= D_1(a)(f \otimes t) \\ &= (af - fa)(t) \\ &= (aG_1 - G_1 a)\overline{(f \otimes t)} \\ &= G\overline{(f \otimes (ta - at)} \\ &= T_1(f)(ta - at) \\ &= (aT_1(f) - T_1(f)a)(t) . \end{aligned}$$

Hence $(af - fa)(t) = (aT_1(f) - T_1(f)a)(t)$ for all $t \in (A \hat{\otimes} A)$, and we thus have $f - T_1(f) \in C$. Let T be the bounded linear map from $(A \hat{\otimes} A)^*$ into C given by $T(f) = f - T_1(f)$. If $f \in C$, then $T_1(f)(t) = G_1\overline{(f \otimes t)} = 0$, thus $T(f) = f$ if $f \in C$. Similarly, since G is zero on Z, we have that $T(a \circ f) = a \circ T(f)$ and $T(f \circ a) = T(f) \circ a$. This completes the proof of (a) implies (b).

(b) implies (c): Let Y be a Banach A-module, X a submodule of Y, and let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X$, $u \in U(A)$. Let $f_1 \in Y^*$ be any extension of f and for each $y \in Y$ define an element $F(y)$ of $(A \hat{\otimes} A)^*$ by $F(y)(a \otimes b) = f_1(ayb)$. Then let $h \in Y^*$ be defined

by $h(y) = T(F(y))(e \otimes e)$. A calculation shows that for all $u \in U(A)$, $F(u^*yu) = u^* \circ F(y) \circ u$, so that $h(u^*yu) = h(y)$. Also, if $x \in X$, then it is easily seen that $F(x) \in C$, hence $h(x) = T(F(x))(e \otimes e) = F(x)(e \otimes e) = f_1(x) = f(x)$. Thus h has the desired properties.

(*c*) *implies* (*b*): The proof is essentially the same as the proof of (g) implies (d) in Proposition 1; we omit the details.

(*b*) *implies* (*a*): Again, the proof is essentially the same as the proof of (d) implies (a) in Proposition 1.

While we can not settle the question of whether every amenable C^*-algebra is strongly amenable, we think that the relationship between conditions (c) of Proposition 2 and conditions (f) and (g) of Proposition 1 may be useful in settling the question.

3. A Stone-Weierstrass type theorem. For A a C^*-algebra, let $ES(A)$ be the set of pure states of A. Let B be a C^*-subalgebra of A which separates $ES(A) \cup \{0\}$. The generalized Stone-Weierstrass question for C^*-algebras [9, section 4.7] asks when is A equal to B? Using a method introduced by Sakai [8], we can show that $A = B$ if A is separable and B is strongly amenable.

PROPOSITION 3. *Let A be a separable C^*-algebra. If B is a strongly amenable C^*-subalgebra of A which separates $ES(A) \cup \{0\}$, then $A = B$.*

Proof. By [8, Lemma 1] we can assume that A has an identity which is also in B. Then as in [8, proof of Proposition 2] if $B \neq A$, there is a $*$-representation π of A on a separable Hilbert space such that $(\pi(B))'' \neq (\pi(A))''$. Then by [12, Theorem 12.2] there is a Hilbert space H and a von Neumann algebra $D \subsetneqq B(H)$ such that D is $*$-anti-isomorphic to D' and such that $(\pi(B))''$ is $*$-isomorphic to D by a $*$-isomorphism S. Now $*$-anti-isomorphisms are clearly order isomorphisms and hence are ultraweakly continuous [2, A27]. Thus the image of $\pi(B)$ under the $*$-anti-isomorphism is weakly dense in D'. It was proven in [5, Section 7] that the weak closure of any $*$-representation of a strongly amenable C^*-algebra has Schwartz's Property P [10, Definition 1]; essentially the same proof shows that the weak closure of any $*$-anti-representation of a strongly amenable C^*-algebra has Property P. Hence, the von Neumann algebra D' has Property P. Thus by [10, Lemma 5] there is a linear norm-decreasing map P from $B(H)$ onto D which is the identity on D. Now consider S as a $*$-representation of $(\pi(B))''$ on H, then by [2, 2.10.2], there is a Hilbert space K containing H and a *-representation T of $(\pi(A))''$ on K such that $S(x) = T(x) \mid H$ for all $x \in (\pi(B))''$. Let p be the projection of K onto H, and define a linear norm-decreasing map R from $B(K)$ onto $B(H)$ by $Ry = py \mid H$ for all $y \in B(K)$. Then $S^{-1} \circ P \circ R \circ T$ is a linear norm-decreasing map from $(\pi(A))''$ onto $(\pi(B))''$ which is the identity on $(\pi(B))''$. Then by [8, Theorem 1], we have that $(S^{-1} \circ P \circ R \circ T)x = x$ for all $x \in (\pi(A))''$. Hence $(\pi(A))'' = (\pi(B))''$. This contradiction shows that $A = B$.

We remark that Sakai [9, 4.7.8] has proved Proposition 3 in the case when B is the uniform closure of an increasing directed set of Type I C^*-subalgebras. The author does not know of an example of a strongly amenable C^*-algebra which is not the uniform closure of an increasing directed set of Type I C^*-subalgebras.

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Received September 27, 1971. This research was partially supported by an NSF grant.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 43, No. 3 May, 1972