CHARACTERIZATIONS OF AMENABLE AND STRONGLY AMENABLE C*-ALGEBRAS

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In this paper it is proved that a C*-algebra $A$ is strongly amenable iff $A$ satisfies a certain fixed point property when acting on a compact convex set, or iff a certain Hahn-Banach type extension theorem is true for all Banach $A$-modules. It is proved that a C*-algebra $A$ is amenable iff $A$ satisfies a weaker Hahn-Banach type extension theorem.

A topological group $G$ is said to be amenable if there is a left invariant mean on the space of bounded continuous complex functions on $G$. A number of papers have been published which give equivalent definitions of amenability (for example, see the papers [4, 7, 11] or the book [3]). It has recently been proven that a locally compact group $G$ is amenable iff for all two-sided $L^i(G)$-modules $X$ and bounded derivations $D$ of $L^i(G)$ into $X^*$, we have that $D$ is the inner derivation induced by an element of $X^*$ [5, Theorem 2.5]. This result motivates the definition of amenable and strongly amenable C*-algebras [5, sections 5 and 7]. In §2 of this paper we give some conditions on a C*-algebra that are equivalent to amenability or strong amenability and are analogous to some of the known equivalent definitions of amenable groups. In §3 we show that the generalized Stone-Weierstrass theorem for separable C*-algebras is true when the C*-subalgebra in question is strongly amenable.

1. Preliminaries. Let $A$ be a C*-algebra. Then a complex Banach space $X$ is called a Banach $A$-module if it is a two-sided $A$-module and there exists a positive real number $M$ such that for all $a \in A$ and $x \in X$ we have

$$ ||ax|| \leq M ||a|| ||x|| $$

and

$$ ||xa|| \leq M ||x|| ||a||. $$

If $X$ is a Banach $A$-module, then the dual space $X^*$ becomes a Banach $A$-module if we define for $a \in A$, $f \in X^*$, and $x \in X$,

$$ (af)(x) = f(ax) $$

$$ (fa)(x) = f(ax) . $$

A derivation from $A$ into $X^*$ is a bounded linear map $D$ from $A$ into
$X^*$ such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If $f \in X^*$, the function $\delta(f)$ from $A$ into $X^*$ given by

$$\delta(f)(a) = af - fa$$

is called the inner derivation induced by $f$.

**DEFINITION 1.** [5, §5]. A C*-algebra $A$ is said to be **amenable** if every derivation from $A$ into $X^*$ is inner for all Banach $A$-modules $X$.

**DEFINITION 2.** [5, §7]. A C*-algebra $A$ is said to be **strongly amenable** if, whenever $X$ is a Banach $A$-module and $D$ is a derivation of $A$ into $X^*$, there is a $f \in \text{co } \{D(u)u^* : u \in U(A_e)\}$ with $D = -\delta(f)$, where $A_e$ is the C*-algebra obtained by adjoining the identity $e$ to $A$, $X$ is made into a unital $A_e$-module by defining $xe = ex = x$ for all $x \in K$, $D$ is extended to $A_e$ by defining $D(e) = 0$, $U(A_e)$ is the unitary group of $A_e$, and $\text{co } S$ denotes the $w^*$-closed convex hull of a set $S$ contained in $X^*$.

A C*-algebra $A$ is strongly amenable iff $A_e$ is strongly amenable, and a C*-algebra $A$ with identity is strongly amenable iff the definition is satisfied for all unital $A$-modules $X$ with $A_e$ replaced throughout by $A$ [5, §7]. The class of strongly amenable C*-algebras includes all C*-algebras which are GCR, uniformly hyperfinite, or the C*-group algebra of a locally compact amenable group [5, §7]. It is not known if there exist amenable C*-algebras which are not strongly amenable.

For a C*-algebra, let $A \hat{\otimes} A$ be the completion of the algebraic tensor product $A \otimes A$ in the greatest cross-norm. Then we can identify $(A \hat{\otimes} A)^*$ with the space of bounded bilinear functionals on $A \times A$ [13, p. 438]. We see that $A \hat{\otimes} A$ becomes a Banach $A$-module if we define for $a, b, c \in A$,

$$a(b \otimes c) = ab \otimes c$$

$$(b \otimes c)a = b \otimes ca .$$

Hence $(A \hat{\otimes} A)^*$ becomes a Banach $A$-module under the dual action: If $f \in (A \hat{\otimes} A)$ and $a, b, c \in A$,

$$(af)(b \otimes c) = f(ab \otimes c)$$

$$(fa)(b \otimes c) = f(ab \otimes c) .$$

We can also make $A \hat{\otimes} A$ and $(A \hat{\otimes} A)^*$ into Banach $A$-modules by defining for $f \in (A \hat{\otimes} A)^*$ and $a, b, c \in A$:

$$a \circ (b \otimes c) = b \otimes ac$$

$$(b \otimes c) \circ a = ba \otimes c$$
Note that the two operations on $A \otimes A$ do not interact; that is, if $a, b, c, d \in A$,

$$a \circ (b \otimes c) = b(a \otimes c)$$

and so forth.

2. Amenable and strongly amenable $C^*$-algebras.

Proposition 1. Let $A$ be a $C^*$-algebra with unit $e$. Then the following seven statements are equivalent:

(a) $A$ is strongly amenable.

(b) For all unital Banach $A$-modules $X$ and $f \in X^*$, there exists $g \in \sigma\{ufu^*: u \in U(A)\}$ such that $ag = ga$ for all $a \in A$.

(c) For any $f \in (A \hat{\otimes} A)^*$ there exists $g \in \sigma\{ufu^*: u \in U(A)\}$ such that $ag = ga$ for all $a \in A$.

(d) There is a linear map $T$ of $(A \hat{\otimes} A)^*$ into $C = \{g \in (A \hat{\otimes} A)^*: ag = ga \text{ all } a \in A\}$ such that $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$, and $T(f) \in \sigma\{ufu^*: u \in U(A)\}$ for all $a \in A, f \in (A \hat{\otimes} A)^*$.

(e) Let $X$ be a Banach $A$-module, $S$ a $w^*$-closed convex subset of $X^*$ such that $usu^* \in S$ for all $s \in S, u \in U(A)$. Then there exists an element $s \in S$ such that $usu^* = s$ for all $u \in U(A)$.

(f) Let $Y$ be a Banach $A$-module and $X$ a subspace of $Y$ such that $uxu^* \in X$ for all $x \in X, u \in U(A)$. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends $f$, there is an $h \in \sigma\{ugu^*: u \in U(A)\}$ such that $h$ extends $f$ and $h(uyu^*) = h(y)$ for all $y \in Y$ and $u \in U(A)$.

(g) Let $Y$ be a Banach $A$-module and $X$ a two-sided $A$-submodule of $Y$. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends $f$, there is an $h \in \sigma\{ugu^*: u \in U(A)\}$ such that $h$ extends $f$ and $h(uyu^*) = h(y)$ for all $y \in Y$ and $u \in U(A)$.

Before proving the proposition, we make some remarks. The implications (a) implies (b) and (b) implies (d) were proven in [1]. The map $T$ in (d) takes the place of the invariant mean that is present in amenable groups. The condition in (e) is a fixed point property; it is known that a locally compact group is amenable iff it has a certain fixed point property [7]. The condition (f) and (g) might be called the strong invariant extension property for subspaces,
and the strong invariant extension property for submodules respectively. A locally compact group is amenable iff it has a certain Hahn-Banach type extension property similar to (f) and (g) [11].

Proof of Proposition 1. (a) implies (b): Let \( f \in X^* \) and \( \delta(f) \) be the inner derivation induced by \( f \). Then there is a \( a \in \sigma(\delta(f)(u)u^* : u \in U(A)) \) such that \( \delta(f) = -\delta(g) \). But \( \delta(f)(u)u^* = uf u^* - f \), hence \( f + g \in \sigma(uf u^* : u \in U(A)) \). Also \( \delta(f)(a) = -\delta(g)(a) \) for all \( a \in A \), thus \( (f + g)a = a(f + g) \) for all \( a \in A \).

(b) implies (c): Clear.

c) implies (d): The proof is an adaptation to the present situation of a proof of J. Schwartz [10, Lemma 5]. Let \( A \) be the set of all linear mappings \( T \) of \( (A \otimes A)^* \) into \( (A \otimes A)^* \) such that \( T(f) \in \sigma(uf u^* : u \in U(A)) \) and \( T(a_\circ f) = a_\circ T(f) \), \( T(f_\circ a) = T(f) \circ a \) for all \( f \in (A \otimes A)^* \), \( a \in A \). The set \( A \) is nonempty since the identity map is in \( A \). We order \( A \) by defining \( \alpha \leq \beta \) if for all \( f \in (A \otimes A)^* \),

\[
\sigma(uf u^* : u \in U(A)) \subseteq \sigma(uf u^* : u \in U(A)).
\]

Then \( \leq \) defines a quasi-order on \( A \). Suppose \( \{T_\alpha : \alpha \in \Delta \} \) is a totally ordered subset of \( A \). We have \( \|T_\alpha(f)\| \leq \|f\| \), thus for all \( d \in A \otimes A \), \( |T_\alpha(f)(d)| \leq \|f\| \|d\| \) and \( \{T_\alpha(f)(d) : \alpha \in \Delta \} \) is a bounded function on \( \Delta \). Let \( \text{LIM} \) be a Banach limit on the directed set \( \Delta \) (see [10, p. 21] for information on Banach limits). Then set \( T(f)(d) = \text{LIM} T_\alpha(f)(d) \) for all \( f \in (A \otimes A)^* \) and \( d \in A \otimes A \). Then \( T \) is a bounded linear map from \( (A \otimes A)^* \) into \( (A \otimes A)^* \). An easy calculation shows that \( T(a_\circ f) = a_\circ T(f) \) and \( T(f_\circ a) = T(f) \circ a \). We show that \( T(f) \in \sigma(uf u^* : u \in U(A)) \) and \( T \supseteq T_\alpha \) for all \( \alpha \in \Delta \). If \( \beta \geq \alpha \) and \( f \in (A \otimes A)^* \) then

\[
T_\beta(f) \in \sigma(uf u^* : u \in U(A)) = K.
\]

Suppose, for contradiction, that \( T(f) \notin K \). Then by the strong separation theorem, there exists \( d \in A \otimes A \), \( \lambda \) real and \( \varepsilon > 0 \) such that for all \( g \in K \),

\[
\text{Re } T(f)(d) \leq \lambda < \lambda + \varepsilon \leq \text{Re } g(d).
\]

Hence \( \text{Re } T(f)(d) \leq \lambda < \lambda + \varepsilon \leq \text{Re } T_\beta(f)(d) \) for all \( \beta \geq \alpha \). But applying the Banach limit to this equation we obtain \( \text{Re } T(f)(d) < \text{Re } T(f)(d) \). Hence \( T(f) \in K \). Thus

\[
\sigma(uf u^* : u \in U(A)) \subseteq \sigma(uf u^* : u \in U(A))
\]

for all \( \alpha \in \Delta \). Hence \( T \in A \) and \( T \supseteq T_\alpha \). Hence \( A \) is inductive, so by Zorn's lemma \( A \) has a maximal element \( T \). We show that \( T(f) \in C \) for all \( f \in (A \otimes A)^* \). If \( g \in (A \otimes A)^* \) is such that \( T(g) \in C \), then \( \sigma(uf u^* : u \in U(A)) \) contains more than one element. Since we
assumed (c), \( C \cap \co \{uT(g)u^*: u \in U(A)\} \) is nonempty. Let \( \sum \lambda u T(g)u^* \) be a net indexed by a directed set \( J \) which converge \( w^* \) to an element \( h \) of \( C \) (we suppress all indices in the sum). Define for \( f \in (A \otimes A)^* \) and \( d \in (A \otimes A) \),

\[
T'(f)(d) = \lim \sum \lambda u T(f)u^*(d) .
\]

Then \( T' \) is a bounded linear map from \( (A \otimes A)^* \) to \( (A \otimes A)^* \) and another application of the strong separation theorem shows that \( T'(f) \in \co \{uT(f)u^*: u \in U(A)\} \). If we show that \( T'(a \circ f) = a \circ T'(f) \) and \( T'(f \circ a) = T'(f) \circ a \), we will know that \( T' \in A \) and \( T' \geq T \). But

\[
T'(a \circ f)(b \otimes c) = \lim \sum \lambda u T(a \circ f)u^*(b \otimes c)
\]

\[
= \lim \sum \lambda (a \circ T(f))(u^* b \otimes c u)
\]

\[
= \lim \sum \lambda T(f)(u^* ba \otimes c u)
\]

\[
= T'(f)(ba \otimes c)
\]

\[
= (a \circ T'(f))(b \otimes c) .
\]

Hence \( T'(a \circ f) = a \circ T'(f) \) and likewise \( T'(f \circ a) = T'(f) \circ a \). But the net \( \sum \lambda u T(g)u^*(d) \) has the actual limit \( h(d) \). Thus \( T'(g) = h \) and \( \co \{uT'(g)u^*: u \in U(A)\} = \{h\} \), and \( \{h\} \) is properly contained in \( \co \{uT(g)u^*: u \in U(A)\} \), hence it is not true that \( T \geq T' \). But this contradicts the maximality of \( T \), and we have that \( T(f) \in C \) for all \( f \). The completes the proof.

(d) implies (e): Let \( x \in X \) and fix \( s \in S \). Define a bounded bilinear function \( F(x, s) \) on \( A \times A \) by \( F(x, s)(a, b) = s(abx) \) for all \( a, b \in A \). We consider \( F(x, s) \) as an element of \( (A \otimes A)^* \). Let \( G(s)(x) = T(F(x, s))(e \otimes e) \). Then clearly \( G(s) \in X^* \). Now if \( u \in U(A) \), then

\[
F(u^* x u, s)(a \otimes b) = s(u^* x ub)
\]

\[
= F(x, s)(u^* \circ (a \otimes b) \circ u^*)
\]

\[
= (u^* \circ F(x, s) \circ u)(a \otimes b) .
\]

Thus \( F(u^* x u, s) = u^* \circ F(x, s) \circ u \). Hence for all \( x \in X \) and \( u \in U(A) \), by using the properties of the map \( T \),

\[
(uG(s)u^*)(x) = G(s)(u^* x u)
\]

\[
= T(F(u^* x u, s))(e \otimes e)
\]

\[
= T(u^* \circ F(x, s) \circ u)(e \otimes e)
\]

\[
= (uT(F(x, s))u^*)(e \otimes e)
\]

\[
= T(F(x, s))(e \otimes e)
\]

\[
= G(s)(x) .
\]

Thus \( uG(s)u^* = G(s) \) for all \( u \in U(A) \). We will be done when we show that \( G(s) \in S \). If \( G(s) \notin S \), then there exists \( x \in X \), a real number
\( \lambda \) and \( \varepsilon > 0 \) such that for all \( t \in S \) we have
\[
\text{Re } G(s)(x) \leq \lambda < \lambda + \varepsilon \leq \text{Re } t(x) .
\]
Now \( T(F(x, s)) \in \text{co } \{ uF(x, s)u^*: u \in U(A) \} \), and \( uF(x, s)u^*(e \otimes e) = (usu^*)(x) \). Since \( usu^* \in S \) for all \( u \in U(A) \), this implies that \( \text{Re } G(s)(x) \leq \lambda + \varepsilon \). This contradiction proves that \( G(s) \in S \).

\((e) \text{ implies } (f)\): Let \( X, Y \) and \( f \in X^* \) be as in \((f)\). Let \( g \in Y^* \) be any extension of \( f \) and let \( S = \text{co } \{ ugu^*: u \in U(A) \} \). Then \( S \) is \( w^*\)-closed convex subset of \( Y^* \) and if \( s \in S \) then \( usu^* \in S \) for all \( u \in U(A) \). Hence by \((e)\) there is an element \( h \in S \) such that \( uhu^* = h \) for all \( u \in U(A) \). Since \( uXu^* \subseteq X \) for all \( u \in U(A) \), \( f(uxu^*) = f(x) \) for all \( x \in X \) and \( g \) extends \( f \), it is easily seen that \( h \) extends \( f \).

\((f) \text{ implies } (g)\): Clear.

\((g) \text{ implies } (e)\): Given \( g \in (A \hat{\otimes} A)^* \), let \( Y = A \hat{\otimes} A, X = \{0\}, f = 0 \) and apply \((g)\). Thus there is an \( h \in \text{co } \{ ugu^*: u \in U(A) \} \) such that \( h(uyu^*) = h(y) \) for all \( y \in A \hat{\otimes} A \) and \( u \in A \); that is, \( ah = ha \) for all \( a \in A \).

\((d) \text{ implies } (a)\): Let \( D: A \to X^* \) be a derivation. Let \( x \in X \) and define a bounded bilinear functional \( f(x) \) on \( A \times A \) by \( f(x)(b, c) = D(b)(xc) \). Then define an element \( G \in X^* \) by \( G(x) = T(f(x))(e \otimes e) \). We will show that \( D = \delta(G) \) and \( -G \in \text{co } \{ D(\varphi_\varphi u^*: \varphi \in i7(A) \} \). For \( x \in X \) and \( a \in A \), define a bounded bilinear functional \( g(x, a) \) on \( A \times A \) by \( g(x, a)(b, c) = D(a)(xcb) \). Then a computation shows that \( a \circ f(x) = f(ax) + g(x, a) \) and \( f(ax) = f(x) \circ a \). If \( a \in A, x \in X, \) then
\[
(\delta(G)(a))(x) = (aG - Ga)(x)
= \quad \begin{array}{l}
G(xa - ax)
\quad = T(f(xa - ax))(e \otimes e)
\quad = T(f(x)a - a \circ f(x) + g(x, a))(e \otimes e)
\quad = T(f(x))(e \otimes a) - T(f(x))(a \otimes e) + T(g(x, a))(e \otimes e)
\quad = T(g(x, a))(e \otimes e) .
\end{array}
\]
The last equality is true because \( T \) maps into \( C \). Now for \( u \in U(A), (ug(x, a)u^*)(e \otimes e) = g(x, a)(u^* \otimes u) = D(a)(x) \), and \( T(g(x, a)) \) is in \( \text{co } \{ ugu^*: u \in U(A) \} \), hence \( (\delta(G)(a))(x) = D(a)(x) \) for all \( x \in X \) and \( a \in A \), thus \( D = \delta(G) \). An application of the strong separation theorem shows that \( -G \in \text{co } \{ D(\varphi_\varphi u^*: \varphi \in i7(A) \} \). Thus \((d) \text{ implies } (a)\) and the proof of Proposition 1 is complete.

**Remarks.** (1) The equivalence of \((a) \) and \((c) \) shows that, to
check strong amenability of $A$, it is only necessary to consider the $A$-module $A \otimes A$; this gives another proof of the Proposition 7.15 of [5].

(2) In the notation of [6], a $C^*$-algebra $A$ is amenable iff the first cohomology group $H^1(A, X^*)$ is zero for all Banach $A$-modules $X$. The reduction of dimension argument of [5, §1. a] then shows that all the cohomology groups $H^n(A, X^*)$ are zero. If $A$ is strongly amenable, then the proof of (d) implies (a) above can be changed to show directly that $H^*_n(A, X^*)$ is zero for all $n$ and all Banach $A$-modules $X$; this proof is similar to the proof of Theorem 3.3 in [6], with the map $T$ taking the place of the invariant mean which is present in that theorem.

(3) In [1] the author used the existence of the function $T$ to generalize the well-known Dixmier-Mackey theorem on amenable groups by proving that every continuous representation of a strongly amenable $C^*$-algebra on a Hilbert space is similar to a *-representation. However, this fact can be proved in a more elementary fashion as follows: Let $A$ be a strongly amenable $C^*$-algebra and let $\pi$ be a continuous representation of $A$ as bounded operators on a Hilbert space $H$. It suffices to assume $A$ has an identity $e$ and $\pi(e) = I$. Make $B(H)$ into a Banach $A$-module by the operations $aT = \pi(a)T$, $Ta = T\pi(a^*)^*$ for $a \in A$, $T \in B(H)$. Then $B(H)$ is the dual Banach $A$-module of the trace class operators. Define a bounded linear map $D$ of $A$ into $B(H)$ by $D(a) = \pi(a) - \pi(a^*)^*$. Then
\[
 aD(b) + D(a)b = \pi(a)(\pi(b) - \pi(b^*)^*) + (\pi(a) - \pi(a^*)^*)\pi(b^*)^* = \pi(ab) - \pi((ab)^*)^* = D(ab).
\]
Hence $D$ is a derivation. Thus, since $A$ is strongly amenable, there is an operator $T$ in $\text{co} \{D(u)u^* : u \in U(A)\}$ such that $D = -\delta(T)$. Then for $a \in A$, $\pi(a) - \pi(a^*)^* = -aT + Ta = T\pi(a^*)^* - \pi(a)T$, thus $\pi(a)(I + T) = (I + T)\pi(a^*)^*$. We let $R = T + I$. Now $D(u)u^* = \pi(u)\pi(u^*)^* - I$, thus $R \in \text{co} \{\pi(u)\pi(u^*) : u \in U(A)\}$. For $x \in H$, $(\pi(u)\pi(u^*)x, x) = ||\pi(u^*)^*x||^2$, and $||x||^2 = ||\pi(u^*)^*\pi(u)x||^2 \leq ||\pi||^2||\pi(u)^*x||^2$. Hence $(1/||\pi||^2)||x||^2 \leq ||\pi(u)^*x||^2$. Thus $R$ is positive and invertible. Let $S$ be the positive square root of $R$. Then $\pi(a)S^* = S^*\pi(a^*)^*$, and $S^{-1}\pi(a)S = S\pi(a^*)^*S^{-1}$. If we define $\pi_1(a) = S^{-1}\pi(a)S$, then $\pi_1$ is clearly a representation of $A$, and $\pi_1(a^*)^* = (S^{-1}\pi(a^*)^*)S^* = S\pi(a^*)^*S^{-1} = S^{-1}\pi(a)S = \pi_1(a)$. Hence $\pi_1$ is a *-representation.

We now give some equivalent conditions for a $C^*$-algebra to be amenable.

PROPOSITION 2. Let $A$ be a $C^*$-algebra with unit $e$. Then the
following three statements are equivalent:

(a) $A$ is amenable.

(b) There is a bounded linear map $T$ of $(A \hat{\otimes} A)^*$ into $C = \{f \in (A \hat{\otimes} A)^*: af = fa$ all $a \in A\}$ such that $T$ restricted to $C$ is the identity on $C$ and $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$ for all $a \in A$, $f \in (A \hat{\otimes} A)^*$.

(c) Let $Y$ be a Banach $A$-module and $X$ a two sided $A$-submodule of $Y$. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X$, $u \in U(A)$. Then there is a $h \in Y^*$ such that $h$ extends $f$ and $h(uyu^*) = h(y)$ for all $y \in Y$, $u \in U(A)$.

Proof. (a) implies (b): Let $Y = (A \hat{\otimes} A)^* \otimes (A \hat{\otimes} A)$ and let $Z$, $W$ and $X$ be as in the proof of (g) implies (d) of Proposition 1. Let $F \in Y^*$ be defined by $F(f \otimes t) = f(t)$ and let $D_i$ be the inner derivation induced by $F$. Then for $a \in A$, $f \in (A \hat{\otimes} A)^*$, and $t \in (A \hat{\otimes} A)$, $D_i(a)(f \otimes t) = (af - fa)(t)$. Hence $D_i(a)$ is zero on $W$. A calculation using the fact that the two $A$-module operations on $(A \hat{\otimes} A)$ do not interact (see the comment at the end of §1) shows that $D_i(a)$ is zero on $Z$. Hence there is a map $D$ from $A$ into $(Y/X)^*$ given by $D(a)(\bar{y}) = D_i(a)(y)$, where $\bar{y}$ is the coset in $Y/X$ of an element $y \in Y$. It is easily seen that $D$ is a derivation, hence since $A$ is amenable there is an element $G_i \in (Y/X)^*$ such that $D = \delta(G_i)$. Let $G \in Y^*$ be defined by $G(y) = G_i(\bar{y})$ for all $y \in Y$. Define a bounded linear map $T_i$ from $(A \hat{\otimes} A)^*$ into $(A \hat{\otimes} A)^*$ by $T_i(f)(t) = G(f \otimes t)$ for all $f \in (A \hat{\otimes} A)^*$, $t \in (A \hat{\otimes} A)$. Now $D(a) = aG_i - G_i a$ for all $a \in A$, thus

$$D(a)(f \otimes t) = D_i(a)(f \otimes t)$$

$$= (af - fa)(t)$$

$$= (aG_i - G_i a)(f \otimes t)$$

$$= G(f \otimes (ta - at))$$

$$= T_i(f)(ta - at)$$

$$= (aT_i(f) - T_i(f)a)(t).$$

Hence $(af - fa)(t) = (aT_i(f) - T_i(f)a)(t)$ for all $t \in (A \hat{\otimes} A)$, and we thus have $f - T_i(f) \in C$. Let $T$ be the bounded linear map from $(A \hat{\otimes} A)^*$ into $C$ given by $T(f) = f - T_i(f)$. If $f \in C$, then $T_i(f)(t) = G_i(f \otimes t) = 0$, thus $T(f) = f$ if $f \in C$. Similarly, since $G$ is zero on $Z$, we have that $T(a \circ f) = a \circ T(f)$ and $T(f \circ a) = T(f) \circ a$. This completes the proof of (a) implies (b).

(b) implies (c): Let $Y$ be a Banach $A$-module, $X$ a submodule of $Y$, and let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X$, $u \in U(A)$. Let $f_i \in Y^*$ be any extension of $f$ and for each $y \in Y$ define an element $F(y)$ of $(A \hat{\otimes} A)^*$ by $F(y)(a \otimes b) = f_i(ayb)$. Then let $h \in Y^*$ be defined
by \( h(y) = T(F(y))(e \otimes e) \). A calculation shows that for all \( u \in U(A) \), 
\[
F(u^{*}yu) = u^{*}F(y)u \text{, so that } h(u^{*}yu) = h(y). 
\]
Also, if \( x \in X \), then it is easily seen that \( F(x) \in C \), hence \( h(x) = T(F(x))(e \otimes e) = F(x)(e \otimes e) = f_{1}(x) = f(x) \). Thus \( h \) has the desired properties.

\[ (c) \text{ implies } (b): \text{ The proof is essentially the same as the proof of } (g) \text{ implies } (d) \text{ in Proposition 1; we omit the details.} \]

\[ (b) \text{ implies } (a): \text{ Again, the proof is essentially the same as the proof of } (d) \text{ implies } (a) \text{ in Proposition 1.} \]

While we can not settle the question of whether every amenable C*-algebra is strongly amenable, we think that the relationship between conditions (c) of Proposition 2 and conditions (f) and (g) of Proposition 1 may be useful in settling the question.

3. A Stone-Weierstrass type theorem. For \( A \) a C*-algebra, let \( ES(A) \) be the set of pure states of \( A \). Let \( B \) be a C*-subalgebra of \( A \) which separates \( ES(A) \cup \{0\} \). The generalized Stone-Weierstrass question for C*-algebras [9, section 4.7] asks when is \( A \) equal to \( B^{-1} \)? Using a method introduced by Sakai [8], we can show that \( A = B \) if \( A \) is separable and \( B \) is strongly amenable.

Proposition 3. Let \( A \) be a separable C*-algebra. If \( B \) is a strongly amenable C*-subalgebra of \( A \) which separates \( ES(A) \cup \{0\} \), then \( A = B \).

Proof. By [8, Lemma 1] we can assume that \( A \) has an identity which is also in \( B \). Then as in [8, proof of Proposition 2] if \( B \neq A \), there is a *-representation \( \pi \) of \( A \) on a separable Hilbert space such that \( (\pi(B))'' \neq (\pi(A))'' \). Then by [12, Theorem 12.2] there is a Hilbert space \( H \) and a von Neumann algebra \( D \subseteq B(H) \) such that \( D \) is *-anti-isomorphic to \( D' \) and such that \( (\pi(B))'' \) is *-isomorphic to \( D \) by a *-isomorphism \( S \). Now *-anti-isomorphisms are clearly order isomorphisms and hence are ultraweakly continuous [2, A27]. Thus the image of \( \pi(B) \) under the *-anti-isomorphism is weakly dense in \( D' \). It was proven in [5, Section 7] that the weak closure of any *-representation of a strongly amenable C*-algebra has Schwartz's Property P [10, Definition 1]; essentially the same proof shows that the weak closure of any *-anti-representation of a strongly amenable C*-algebra has Property P. Hence, the von Neumann algebra \( D' \) has Property P. Thus by [10, Lemma 1] there is a linear norm-decreasing map \( P \) from \( B(H) \) onto \( D \) which is the identity on \( D \). Now consider \( S \) as a *-representation of \( (\pi(B))'' \) on \( H \), then by [2, 2.10.2], there is
a Hilbert space \( K \) containing \( H \) and a *-representation \( T \) of \((\pi(A))''\) on \( K \) such that \( S(x) = T(x) | i\mathbb{I} \) for all \( x \in (\pi(B))'' \). Let \( p \) be the projection of \( K \) onto \( H \), and define a linear norm-decreasing map \( R \) from \( B(K) \) onto \( B(H) \) by \( R_y = py | H \) for all \( y \in B(K) \). Then \( S^{-1}p^\circ R^\circ T \) is a linear norm-decreasing map from \((\pi(A))''\) onto \((\pi(B))''\) which is the identity on \((\pi(B))''\). Then by [8, Theorem 1], we have that \((S^{-1}p^\circ R^\circ T)x = x \) for all \( x \in (\pi(A))'' \). Hence \((\pi(A))'' = (\pi(B))''\). This contradiction shows that \( A = B \).

We remark that Sakai [9, 4.7.8] has proved Proposition 3 in the case when \( B \) is the uniform closure of an increasing directed set of Type I \( C^* \)-subalgebras. The author does not know of an example of a strongly amenable \( C^* \)-algebra which is not the uniform closure of an increasing directed set of Type I \( C^* \)-subalgebras.

**References**


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