THE INVERSION THEOREM AND PLANCHEREL’S THEOREM IN A BANACH SPACE

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1. Introduction. Let $G$ be a locally compact abelian group with Haar measure $\mu$, and let $X$ be a complex Banach space and $C$ be the set of complex numbers. A classic theorem due to Plancherel ([8], [10]) states that the Fourier transform maps $L_1(G, C) \cap L_2(G, C)$ onto a dense subset of $L_2(\hat{G}, C)$ (where $\hat{G}$ is the dual group of $G$ and has Haar measure $m$) in such a way that

$$\int_G \alpha(g)\overline{\beta(g)} \mu(dg) = \int_{\hat{g}} \hat{\alpha}(\gamma)\overline{\hat{\beta}(\gamma)} m(d\gamma)$$

for all $\alpha, \beta$ in $L_1(G, C) \cap L_2(G, C)$, where $\hat{\alpha}$ is the Fourier transform of $\alpha$, given by $\hat{\alpha}(\gamma) = \int_G (g, \gamma) \alpha(g) \mu(dg)$ for all $\gamma$ in $\hat{G}$. Here $(g, \gamma)$ denotes the action of the character $\gamma$ on $g$ in $G$. In this paper we extend this result to functions taking values in an inner product subspace of a Banach algebra.

Another well-known theorem ([8], [10]) states that if $\alpha$ is a positive definite element of $L_1(G, C) \cap L_2(G, C)$ then $\hat{\alpha}$ is in $L_1(\hat{G}, C)$ and

$$\alpha(g) = \int_{\hat{G}} (g, \gamma)\hat{\alpha}(\gamma) m(d\gamma)$$

for (almost) all $g$ in $G$. This inversion theorem is also generalized to functions assuming values in certain admissible Banach spaces.

Our work relies heavily on an extension of Bochner’s theorem established in [4]. We show that if $p$ is in $L_1(G, X) \cap L_2(G, X)$, if $p$ is positive definite (positivity is defined with respect to a particular cone in $X$), and if $p(0)$ satisfies a certain finiteness condition, then $\hat{p}$, the Fourier transform of $p$, is in $L_1(\hat{G}, X)$ and the inversion formula 1.1 given for $\alpha$ holds for $p$. A sharper theorem states that if $p$ is in $L_1(G, X) \cap L_2(G, X)$, if $p$ is positive definite, and if there is a real, finite, regular Borel measure $\lambda$ such that $\int_{\hat{G}} |\hat{\alpha}(\gamma)| \lambda(d\gamma)$ for all $\alpha$ in $L_1(G, C)$, then $\hat{p}$ is in $L_1(\hat{G}, X)$ and 1.1 is satisfied by $p$.

Using this theory we extend to infinite dimensions some results due to Hewitt and Wigner ([7]).

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1 For $1 \leq p \leq \infty$ $L^p(G, X)$ is the space of $\mu$-measurable functions $f$ mapping $G$ into $X$. For $1 \leq p < \infty$ we use the norm $\|f\|_p$, where $\|f\|_p = \left\{ \int_G |f(g)|^p \mu(dg) \right\}^{1/p}$, and for $p = \infty$ we use the norm $\|f\|_\infty$ which is the ($\mu$) essential supremum of $\|f(g)\|$ on $G$. $\|f\|$ denotes the norm in $X$. 

585
2. Bochner's theorem and dominated functions. Let $X$ be a Banach space, $X^*$ the dual of $X$ and $X^{**}$ the dual of $X^*$. For $\varphi$ in $X^*$ we denote the action of $\varphi$ on $x \in X$ by $(x, \varphi)$. Given a subset of $X^*$ we can define a cone of "positive" elements in $X$.

**Definition 2.1.** Let $\Phi$ be a subset of $X^*$. The subset $K_\Phi$ of $X$ given by

$$K_\Phi = \{ x \in X : (x, \varphi) \geq 0 \text{ for all } \varphi \in \Phi \}$$

is called the cone determined by $\Phi$.

Sometimes we write simply $K$ if $\Phi$ is fixed by the context. $X_\Phi$ is the set of "positive" elements.

Let $G$ be a $\sigma$-finite locally compact abelian group with Haar measure $\mu$ and let $\hat{G}$ be its dual group with Haar measure $m$.

**Definition 2.3.** Let $p$ be a map of $G$ into $X$. Then $p$ is $\Theta$-positive definite if it is measurable and if

$$\sum_{n=1}^{N} \sum_{m=1}^{N} c_n \overline{c}_m (p(g_n - g_m), \varphi) \geq 0$$

for any integer $N$, any $c_1, \ldots , c_N$ in $C$, any $g_1, \ldots , g_N$ in $G$, and all $\varphi$ in $\Phi$. If $p$ is in $L_{\infty}(G, X)$ the $p$ is integrally $\Phi$-positive definite if

$$\left( \int_G \int_G \alpha(g) \overline{\alpha(g')} p(g - g') d\mu d\mu, \varphi \right) \geq 0$$

for all $\alpha$ in $L_1(G, C)$ and all $\varphi$ in $\Phi$.

Next we impose a condition which relates $\Phi$ to the topology of $X$.

**Definition 2.6.** The family $\Phi$ is full if there is a $\rho > 0$ such that

$$\| x \| \leq \rho \sup \{ \| (x, \varphi) \| : \| \varphi \| : \varphi \in \Phi \}$$

for all $x$ in $X$.

The following two propositions examine the relationship between the two notions of positive-definiteness.

**Proposition 2.8.** If $\Phi$ is full and $p$ is $\Phi$-positive definite then $p$ is in $L_{\infty}(G, X)$ and $p(0)$ is in $K_\Phi$.

**Proof.** It is readily shown that for $g$ in $G$, $\varphi$ in $\Phi$, $| (p(g), \varphi) | \leq$
(\(p(0), \varphi\)) so that \(\|p(g)\| \leq \rho \|p(0)\|\).

**Proposition 2.9.** Let \(p\) be in \(L_\infty(G, X)\) such that one version of \(p\) is \(\omega X\)-continuous.\(^2\) Then \(p\) is \(\Phi\)-positive definite iff \(p\) is integrally \(\Phi\)-positive definite.

**Proof.** See [4] or [6].

We shall see shortly (Corollary 2.15) that all those elements of \(L_\infty(G, X)\) of interest to us have the continuity required in Proposition 2.9.

Next we recall some results from measure theory. Let \(S\) be a locally compact topological space and let \(\Sigma(S)\) be the Borel field of \(S\) (i.e. the smallest \(\sigma\)-field containing the closed sets of \(S\)).

**Definition 2.10.** A vector measure \(\nu\) is a weakly countably additive set function defined on \(\Sigma(S)\) and taking values in \(X\). \(\nu\) is weakly regular if the scalar measures \((\nu(\cdot), \varphi)\) are regular\(^3\) for all \(\varphi\) in \(X^*\). \(\nu\) is \(\Phi\)-positive if \((\nu(E), \varphi) \geq 0\) for all \(\varphi\) in \(\Phi\) and \(E\) in \(\Sigma(S)\).

**Definition 2.11.** A set function \(\nu^{**}\) mapping \(\Sigma(S)\) into \(X^{**}\) is weak-\(*\)-regular if \((\varphi, \nu^{**(\cdot)})\) is a regular scalar measure for all \(\varphi\) in \(X^*\). \(\nu^{**}\) is \(\Phi\)-positive if \((\varphi, \nu^{**(E)}) \geq 0\) for all \(\varphi\) in \(\Phi\), \(E\) in \(\Sigma(S)\).

The mapping \(f\) of \(G\) into \(X\) is \(\omega X\)-continuous if it is continuous when the weak topology is imposed on \(X\). \(G\) retains its usual topology.

**Theorem 2.12.** (A) If \(\nu\) is a weakly regular \(\Phi\)-positive vector measure defined on \(\Sigma(G)\) and if

\[
(2.13) \quad p(g) = \int_G (g, \gamma) \nu(d\gamma)
\]

then \(p\) is an integrally \(\Phi\)-positive definite element of \(L_\infty(G, X)\).

(B) If \(p\) is an integrally \(\Phi\)-positive definite element of \(L_\infty(G, X)\), then there is a set function \(\nu^{**}\) mapping \(\Sigma(G)\) into \(X^{**}\) such that

(i) \(\nu^{**}\) is weak-*-regular, \(\Phi\)-positive with finite semi-variation and (ii)

\[
(2.14) \quad (p(g), \varphi) = \int_G (g, \gamma)(\varphi, \nu^{**(d\gamma)})
\]

for all \(\varphi\) in \(X^*\) and almost all \(g\) in \(G\).

\(^2\) The mapping \(f\) of \(G\) into \(X\) is \(\omega X\)-continuous if it is continuous when the weak topology is imposed on \(X\). \(G\) retains its usual topology.

\(^3\) A scalar measure \(\lambda\) is regular if, given \(\varepsilon > 0\) and \(E \in \Sigma(S)\) with \(\|\lambda\| (E) < \infty\) (i.e. \(\lambda\) has finite variation on \(E\)), then there is a compact \(K \subseteq E\) and an open \(O \supset E\) such that \(\|\lambda\| (O - K) < \varepsilon\).
COROLLARY 2.15. If \( p \) is an integrally \( \Phi \)-positive definite element of \( L_\omega(G, X) \) then one version of \( p \) is \( \omega X \)-continuous. If \( p \) is given by 2.13, where \( \nu \) is a weakly regular \( \Phi \)-positive vector measure defined on \( \Sigma(\hat{G}) \), then \( p \) is a continuous map of \( G \) into \( X \).

Proof. This follows from the relevant regularity. See also [6].

With the aid of Theorem 2.12 we could prove a useful inversion theorem. However, a different version of Bochner's theorem will allow us to establish a sharper theorem. We require first the following.

DEFINITION 2.16. \( p \) in \( L_\omega(G, X) \) is dominated if there exists a finite, regular, positive Borel measure \( \lambda \), such that

\[
\left\| \int_G \alpha(g)p(g)\mu(g) \right\| \leq \int_{\hat{G}} |\hat{\alpha}(\gamma)| \lambda(d\gamma)
\]

for all \( \alpha \) in \( L_1(G, C) \), where \( \hat{\alpha} \) is the Fourier transform of \( \alpha \), i.e. \( \hat{\alpha}(\gamma) = \int_G (g, \gamma)\alpha(g)\mu(g) \). If \( R^+ \) is the set of nonnegative real numbers, we have

DEFINITION 2.18. Let \( \Phi \) be a subset of \( X \). Assume there is a function \( \varphi_0 \) mapping \( K_\Phi \) into \( R^+ \cup \{\infty\} \) in a linear manner such that \( \varphi_0 \) is uniformly positive on \( K_\Phi \), i.e. there exists \( k > 0 \) such that \( k(x, \varphi_0) \geq ||x|| \) for all \( x \) in \( K_\Phi \). Furthermore assume there are at most countable sequences \( \{c_i\} \), \( \{\varphi_i\} \) in \( \Phi \) such that \( \varphi_0(x, \varphi) = \sum_{i=1}^\infty c_i(x, \varphi_i) \) for all \( x \) in \( K_\Phi \). Then we say that the pair \((\Phi, X)\) is admissible. We let \( K_{\varphi_0} = \{x \in K_\Phi: (x, \varphi_0) < \infty\} \).

LEMMA 2.19. If \((\Phi, X)\) is admissible, if \( \Phi \) is full, and if \( p \in L_\omega(G, X) \) is integrally \( \Phi \)-positive definite with \( p(0) \) in \( K_{\varphi_0} \), then \( p \) is dominated.

In this lemma it is assumed we are talking about the \( \omega X \)-continuous version of \( p(\cdot) \) (Corollary 2.15).

Proof. Let \( \hat{\nu}(\alpha) = \int_{\hat{G}} \alpha(g)p(g)\mu(g) \) for all \( \alpha \) in \( L_1(G, C) \), then \( \hat{\nu}(\alpha, \varphi) = \int_{\hat{G}} \hat{\alpha}(\gamma)(\varphi, \nu^{**}(d\gamma)) \) for some weak-*-regular, \( \Phi \)-positive set function \( \nu^{**} \) given by Theorem 2.12. We can actually define \( \hat{\nu}(\cdot) \) mapping \( C_{\varphi_0}(\hat{G})^4 \) into \( X \) by \( \hat{\nu}(f)(\varphi, \nu^{**}(d\gamma)) \). Then \( \hat{\nu} \) is a

\[ C_{\varphi_0}(\hat{G})^4 \]

if \( \hat{G} \) is only locally compact.

\[ \text{For } a \in L_1(G, C), \hat{\nu}(a) = \nu(a) \in X. \text{ As } \{\hat{a} \in C_{\varphi_0}(\hat{G}): a \in L_1(G, C)\} \text{ is dense in } C_{\varphi_0}(\hat{G}), \text{ and as } \hat{\nu} \text{ is continuous, it can be extended uniquely, with range in } X. \]
bounded linear map, \( \| \hat{\psi}(f) \| \leq \| f \|_\infty \| \nu^{**} \| (\hat{G}) \).

If \( f \) is in \( C_0(\hat{G}) \) then \( f = f_1 - f_2 + \imath f_3 - \imath f_4 \) where \( f_i \) is in \( C_0(\hat{G}) \), \( f_i(\gamma) \geq 0 \), and each pair of functions \( (f_1, f_2), (f_3, f_4) \) has disjoint support. Hence \( f_i(\gamma) \leq |f(\gamma)| \), and \( \hat{\psi}(f_i) \) is in \( K \) so that \( \| \hat{\psi}(f_i) \| \leq k(\hat{\psi}(f_i), \varphi_0) = k \sum_{i=1}^n c_i(\hat{\psi}(f_i), \varphi_j) = k \sum_j c_j \int_\hat{G} f_i(\gamma)(\varphi_j, \nu^{**}(d\gamma)) \). Consider now the set function \( \lambda \) given by \( \lambda(E) = \sum_{i=1}^n c_i(\varphi_i, \nu^{**}(E)) \), \( E \in \Sigma(\hat{G}) \). Then \( \lambda(E) \geq 0 \) for all \( E \) in \( \Sigma(\hat{G}) \), and also \( \lambda \) is additive. Moreover \( \lambda(E) \leq (p(0), \varphi_0) < \infty \) as \( p(0) \) is in \( K_0 \).

\( \lambda \) is countably additive because \( \lambda(\bigcup_i E_i) = \sum_i \sum_j c_i(\varphi_i, \nu^{**}(E_j)) = \sum_i \sum_j c_i(\varphi_i, \nu^{**}(E_j)) = \sum_i \lambda(E_i) \), if the \( E_j \) are disjoint (note that \( c_i(\varphi_i, \nu^{**}(E_j)) \geq 0 \) for all \( i, j \)). Also \( \lambda \) is regular, for given \( \varepsilon > 0 \) and \( E \) in \( \Sigma(\hat{G}) \), there is a number \( N \) such that \( \sum_{i=1}^n c_i(\varphi_i, \nu^{**}(\hat{G})) < \varepsilon/2 \) and there is a compact \( K \subset E \) and an open \( O \supset E \) such that \( (\varphi_i, \nu^{**}(0 - K)) < \varepsilon/2 N c_i, i = 1, 2, \ldots, N \). Hence \( \lambda(O - K) < \varepsilon \).

Then \( \| \hat{\psi}(f) \| \leq \sum_{i=1}^n \| \hat{\psi}(f_i) \| \leq k \sum_i \int_\hat{G} f_i(\gamma) d\lambda \leq 4k \int_\hat{G} |f(\gamma)| d\lambda \). It follows that if \( \lambda' = 4k\lambda \) then \( \| \psi(\alpha) \| \leq \int_\hat{G} |\alpha(\gamma)| d\lambda' \). This establishes the lemma.

We can now state the alternate version of Bochner's theorem. Assume \( \Phi \) is full and countable

**Theorem 2.20.** \( p \) is a dominated, integrally \( \Phi \)-positive definite element of \( L_\omega(G, X) \) iff there is a weakly regular \( \Phi \)-positive vector measure \( \nu \) mapping \( \Sigma(\hat{G}) \) into \( X \) such that \( \nu \) has finite variation, i.e. \( \| \nu \| (\hat{G}) < \infty \), and such that for any \( \varphi \) in \( X^* \),

\[
(p(g), \varphi) = \int_\hat{G} (g, \gamma)(\nu(d\gamma), \varphi), \quad \text{a. e. } g.
\]

For the proof see [4]. Countability of \( \Phi \) is not required for the only if part.

**3. Inversion theorems.** If \( p \in L_1(G \times X) \) we recall that the Fourier transform of \( p \) is given by

\[
\hat{p}(\gamma) = \int_g (g, \gamma)p(g)\mu(dg).
\]

For convenience we let \( \mathcal{P} = \{ p \in L_\omega(G, X) \colon p \) is integrally \( \Phi \)-positive definite\} and \( \mathcal{P}_d = \{ p \in \mathcal{P} \colon p \) is dominated\}. We recall that if \( p \in \mathcal{P} \) then \( p \) is \( \omega X \)-continuous (Corollary 2.15). If \( (\Phi, X) \) is admissible then \( \mathcal{F}_0 \) is the set of functions \( p \) mapping \( G \) into \( X \) such that \( p \) is \( \omega X \)-continuous and such that \( p(0) \) is in \( K_0 \) where \( K_0 \) is defined in 2.18.

**Proposition 3.2.** (A) If \( p \in \text{span} \{ L_1(G, X) \cap \mathcal{P} \} \) and \( \phi \in
span \{\Phi\} then \((\hat{p}(\cdot), \varphi) \in L_1(\hat{G}, C)\) and (B) if the Haar measure of \(G\) is fixed then the Haar measure of \(\hat{G}\) can be so normalized that

\[(p(g), \varphi) = \int_{\hat{G}} (g, \gamma)(\hat{p}(\gamma), \varphi)m(d\gamma)\]

is valid for all \(p \in \text{span} \{L_1(G, X) \cap \mathcal{P}\}\) and all \(\varphi \in \text{span} \{\Phi\}\).

**Proof.** It is evident the results need only hold for \(p \in L_1(G, X) \cap \mathcal{P}, \varphi \in \Phi\). But this follows from the scalar inversion theorem ([10], p. 22).

A better result is the following.

**THEOREM 3.4.** Assume \(\Phi\) is full and \(G\) is \(\sigma\)-finite. (A) If \(p \in \text{span} \{L_1(G, X) \cap \mathcal{P}\}\) then \(\hat{p} \in L_1(\hat{G}, X)\), and (B) with \(\mu\) fixed, \(m\) can be so normalized that for each \(\varphi\) in \(X^*\)

\[(3.5) \quad (p(g), \varphi) = \left(\int_{\hat{G}} (g, \gamma)(\hat{p}(\gamma)m(d\gamma)\varphi\right) \quad a. e. g.\]

If \(\Phi\) is countable or if \(p\) is continuous (3.5) becomes

\[(3.6) \quad p(g) = \int_{\hat{G}} (g, \gamma)(\hat{p}(\gamma)m(d\gamma) \quad a. e. g.\]

**Proof.** Again we need only prove the results for \(p\) in \(L_1(G, X) \cap \mathcal{P}\). If \(p\) is in \(L_1(G, X)\) then \(\hat{p}\) is in \(C_b(\hat{G}, X)\), the space of continuous functions mapping \(\hat{G}\) into \(X\), which vanish at infinity if \(\hat{G}\) is only locally compact but not compact. As \(p\) is measurable and \(G\) is \(\sigma\)-finite, \(\hat{p}\) is essentially separably valued, and hence is measurable and a member of \(L_\infty(\hat{G}, X)\).

As \(p\) is in \(\mathcal{P}\), then by Theorem 2.20 there is a weakly regular \(\Phi\)-positive vector measure \(\nu\) with finite variation such that for any \(\varphi\) in \(\Phi\)

\[(3.7) \quad (p(g), \varphi) = \int_{\hat{G}} (g, \gamma)(\nu(d\gamma), \varphi), \quad a. e. g.\]

by Proposition 3.2. As both integrals are continuous, the equality hold for all \(g\). It follows, [10], that

\[(\nu(E), \varphi) = \int_{E} (\hat{p}(\gamma), \varphi)m(d\gamma)\]

\[= \left(\int_{E} (\hat{p}(\gamma)m(d\gamma), \varphi\right)\]
if \( m(E) < \infty \), as \( \hat{p} \) is bounded. Since \( \Phi \) is full, we have

\[
\nu(E) = \int_E \hat{p}(\gamma) m(d\gamma)
\]

if \( m(E) < \infty \). As \( \hat{p} \) is in \( C_0(\hat{G}, X) \) given \( n \) these exists a compact set \( K_n \) such that \( ||\hat{p}(\gamma)|| < 1/n \) if \( \gamma \) is in \( \hat{G} - K_n \). Let \( \chi_n(\cdot) \) be the indicator function of \( K_n \). Then

\[
\lim_{n \to \infty} \int_{\hat{G}} ||\chi_n(\gamma)\hat{p}(\gamma)|| m(d\gamma) = \lim_{n \to \infty} \int_{K_n} ||\hat{p}(\gamma)|| m(d\gamma)
\]

\[
= \lim_{n \to \infty} ||\nu|| (K_n)
\]

\[
= ||\nu|| (\hat{G}) < \infty.
\]

Also \( ||\chi_n(\gamma)\hat{p}(\gamma)|| \uparrow ||\hat{p}(\gamma)|| \) for each \( \gamma \) in \( \hat{G} \). Then by the monotone convergence theorem

\[
\lim_{n \to \infty} \int_{K_n} ||\hat{p}(\gamma)|| m(d\gamma) = \int_{\hat{G}} ||\hat{p}(\gamma)|| m(d\gamma) \leq ||\nu|| (\hat{G}).
\]

Hence \( \hat{p} \) is in \( L_1(\hat{G}, X) \), and for all measurable sets \( E \),

\[
\nu(E) = \int_E \hat{p}(\gamma) m(d\gamma).
\]

Since \( \Phi \) is full (3.5) now follows from (3.7).

If \( p \) is continuous, the set of measure zero where (3.5) does not hold is empty and (3.6) follows. If \( \Phi \) is countable, the union of these null sets (one for each \( \varphi \) in \( \Phi \)) is still a null set and again (3.6) holds.

**Corollary 3.8.** Assume \( \Phi \) is full, \( G \) is \( \sigma \)-finite, and \( (\Phi, X) \) is admissible.

(A) If \( p \) is in span \( \{L_1(G, X) \cap \mathcal{F} \cap \mathcal{T}_\gamma \} \) then \( \hat{p} \) is in \( L_1(\hat{G}, X) \).

(B) If \( \mu \) is fixed, \( m \) can be so normalized that for each \( \varphi \) in \( X^* \) (3.5) holds. If \( \Phi \) is countable or if \( p \) is continuous then (3.6) holds.

**Proof.** Apply Lemma 2.19 and Theorem 3.4.

4. The Plancherel theorem. As usual this theorem is set in a Hilbert space, and so we must first develop the necessary structure. Assume now that \( X \) is a Banach algebra with continuous involution \( x \rightarrow x^* \).

**Definition 4.1.** The triplet \( (\Phi, X, X_0) \) is strongly admissible if
(i) $\Phi, X$ is admissible, (ii), $X_0$ is a non-trivial subspace of $X$ such that $xx^*$ is in $K_0^6$ for all $x$ in $X_0$, and (iii) there exists $k_0 > 0$ such that if $x \in X_0$ then
\begin{equation}
 k_0 \|xx^*\| \geq \|x\|^2,
\end{equation}
We note that 4.2 is satisfied if $X$ is a C*-algebra. Now we have

**Proposition 4.3.** If $X$ is a Banach algebra and if $\Phi, X, X_0$ is strongly admissible then $X_0$ is a Hilbert space under the norm $\|\cdot\|_0 = \langle x, x \rangle_0$ and $\langle x, y \rangle_0 = (xy^*, \varphi_0)$.

**Proof.** $\varphi_0$ is only defined on $K$ and we do not know that if $x, y \in X_0$ then $xy^* \in K$. However we can extend $\varphi_0$ by setting $(xy^*, \varphi_0) = \sum c_i(xy_i^*, \varphi_i)$ where $\{c_i\}, \{\varphi_i\}$ define $\varphi_0$ on $K$. Then $|\langle x, y \rangle_0| = |(xy^*, \varphi)| = |\sum c_i(xy_i^*, \varphi_i)| \leq \sum |c_i||xx_i^*, \varphi_i|^1/2(yy_i^*, \varphi_i)^1/2$ where the last inequality follows because $\varphi_i$ is a positive functional. Hence we can define $\langle x, y \rangle_0$ for $x, y \in X_0$ and $|\langle x, y \rangle_0| \leq \|x\|_0 \|y\|_0$. It follows from 2.18 and 4.2 that $kk_0 \|x\|^2 \geq \|x\|^2$ and that $\|\cdot\|_0$ is a norm.

If $\{x_n\}$ is Cauchy in $\|\cdot\|_0$ then it is Cauchy in $\|\cdot\|$, so $x_n \to x \in X$. As $K$ is closed then $xx^* \in K$. Also $\{x_n\}$ is bounded in $\|\cdot\|_0$ because it is Cauchy, so $\sum c_i(x_nx_n^*, \varphi_i) \leq M$, hence $\sum c_i(xx_i^*, \varphi_i) \leq M$ or $x \in K_0$. Choose $m(\varepsilon)$ such that if $n, m > m(\varepsilon)$ then $\|x_n - x_m\|_0 < \varepsilon$. Then $\sum c_i[(x - x_m)[x - x_m]^*], \varphi_i) = \lim_{n \to \infty} \sum c_i[(x_n - x_m)[x_n - x_m]^*, \varphi_i) \leq \lim \sup_{n \to \infty} \sum c_i[(x_n - x_m)[x_n - x_m]^*, \varphi_i) < \varepsilon^2$ so that for $m > m(\varepsilon), \|x - x_m\|_0 < \varepsilon$, or $X_0$ is a Hilbert space.

If $X$ is a Banach algebra and $G$ is $\sigma$-finite, then $L_1(G, X)$ is also a Banach algebra ([5]). If $X$ has the involution $x \to x^*$, then we can define an involution on $L_1(G, X)$ as $p \to p^*$ where $p^*(g) = p(-g)$.

**Theorem 4.4.** If $G$ is $\sigma$-finite, $X$ is a Banach algebra with continuous involution, $\Phi$ is a full subset of $X^*$ and $(\Phi, X, X_0)$ is strongly admissible, then (i) if $\{e_\alpha\}$ is an orthonormal basis for $X_0$ and there exists $k_1$ such that $|\langle x, e_\alpha \rangle| \leq k_1 \|x\|$ for $x \in X_0$ and all $\alpha$, then the Fourier transform maps $L_1(G, X) \cap L_2(G, X_0)$ onto a dense subset of $L_2(\widehat{G}, X_0)$, (ii) for $q, r \in L_1(G, X) \cap L_2(G, X_0)$
\begin{equation}
 \int_G q(g)r(g)\mu(dg) = \int_{\widehat{G}} \hat{q}(\gamma)\hat{r}(\gamma)m(d\gamma),
\end{equation}
(iii) for $q, r \in L_1(G, X) \cap L_2(G, X_0)$
\begin{equation}
 \langle q, r \rangle = \langle \hat{q}, \hat{r} \rangle,
\end{equation}
where $\langle q, r \rangle = \int_G q(g)r(g)\mu(dg)$ and $\langle \hat{q}, \hat{r} \rangle = \int_{\widehat{G}} \hat{q}(\gamma)\hat{r}(\gamma)m(d\gamma)$.

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* $K_0$ is defined in 2.18.
Proof. We shall put

$$\| q \|_1 = \int G |q(g)|^2 \mu(dg) \quad \text{and} \quad \| q \|_2 = \left( \int G |q(g)|^4 \mu(dg) \right)^{1/2}$$

for $q \in L_1(G, X) \cap L_2(G, X_0)$. Let $p(g) = (q*q^*)(g)$. As $q \in L_1(G, X)$ so is $p$ with $\| p \|_1 \leq \| q \|_1$. It can also be shown that $p \in C_0(G, X_0)$ as $q \in L_2(G, X_0)$. Now $p(0) = \int G q(g)q(g)^* \mu(dg) \in K$ so

$$(p(0), \varphi_0) = \left( \int G q(g)q(g)^* \mu(dg), \varphi_0 \right)$$

$$= \sum_{i=1}^\infty c_i \int G \left( q(g)q(g)^*, \varphi_i \right) \mu(dg)$$

$$= \int G \left( q(g)q(g)^*, \varphi_0 \right) \mu(dg)$$

$$= \int G \| q(g) \|_\infty^2 \mu(dg)$$

$$= \| q \|_2^2 < \infty$$

using the monotone convergence theorem. Hence $p \in L_1(G, X) \cap \mathcal{F}_0$.

Now $C_0(G, X_0) \subset C_0(G, X)$ so $p \in L_\infty(G, X)$. Also

$$\int G \int G \alpha(g)\overline{\alpha(g')} p(g - g') \mu(dg) \mu(dg')$$

$$= \int G \left[ \int G \alpha(g)q(g - g'') \mu(dg) \right] \left[ \int G \alpha(g')q(g' - g''') \mu(dg''') \right]^* \mu(dg''')$$

$$= \int G q'(g)q'(g)^* \mu(dg)$$

using the Fubini and Tonelli theorems with $\alpha \in L_1(G, C)$, where $q' = \alpha*q \in L_1(G, X_0)$ ([5]) so $q'(g) \in X_0$ a.e. or $q'(g)q'(g)^* \in K_0$ a.e. Hence if $\varphi \in \Phi$ then

$$\left( \int G q'(g)q'(g)^* \mu(dg), \varphi \right) = \int G (q'(g)q'(g)^*, \varphi) \mu(dg) \geq 0$$

or $p \in \mathcal{P}$.

Consequently Corollary 3.8 yields $p(g) = \int_{\hat{G}} (g, \gamma) \hat{p}(\gamma) m(d\gamma)$. Then

$$\infty > \| q \|_2^2 = \langle q, q \rangle = \sum_{i=1}^\infty c_i \langle p(0), \varphi_i \rangle$$

$$= \sum_{i} c_i \int_{\hat{G}} \langle \hat{p}(\gamma), \varphi_i \rangle m(d\gamma)$$

$$= \int_{\hat{G}} \langle \hat{p}(\gamma), \varphi_0 \rangle m(d\gamma) = \langle \hat{q}, \hat{q} \rangle .$$

We have used the monotone convergence theorem again. Hence the
Fourier transform maps into $L_2(\hat{G}, X_0)$. By the usual expansion $\langle q, r \rangle = \langle \hat{q}, \hat{r} \rangle$. This establishes (iii).

Moreover $\int_G q(g)\overline{q}(g)\mu(dg) = p(0) = \int_G \hat{q}(\gamma)m(d\gamma) = \int_G \hat{q}(\gamma)\overline{q}(\gamma)*m(d\gamma)$. Also if $x, y$ are elements of a Banach algebra with involution then

$$4xy^* = (x + y)(x + y)^* - (x - y)(x - y)^* + i(x + iy)(x + iy)^* - i(x - iy)(x - iy)^*$$

so that (ii) is also proved.

We need only show that $Q = \{ \hat{q} \in L_2(\hat{G}, X_0) : q \in L_1(G, X) \cap L_2(G, X_0) \}$ is dense in $L_2(\hat{G}, X_0)$. As $\mu$ is translation invariant so is $L_1(G, X) \cap L_2(G, X_0)$ and hence $Q$ is invariant under multiplication by $(g, \cdot)$ for any $g \in G$. If $r \in L_2(\hat{G}, K_0)$ and $\langle q, r \rangle = 0$ for all $q \in Q$, then $\int_G (q(\gamma)r(\gamma)^*, \varphi_0)(g, \gamma)m(d\gamma) = 0$ for all $q \in Q$ and $g \in G$. As $(q(\cdot)r(\cdot)^*, \varphi_0) \in L_1(\hat{G}, C)$ it follows that $(q(\gamma)r(\gamma)^*, \varphi_0) = 0$ a.e. for every $q \in Q$, or $\langle q(\cdot), r(\gamma) \rangle_0 = 0$ a.e. As $L_1(G, X) \cap L_2(G, X_0)$ is invariant under multiplication by $(\cdot, \gamma), \gamma \in \hat{G}$, then $Q$ is invariant under translation.\(^7\) Hence to every $\gamma_0 \in \hat{G}$ there corresponds $q_0 \in Q$ such that $q_0(\gamma_0) \neq 0$, so $q_0(\gamma) \neq 0$ in a neighborhood of $\gamma_0$ as $q_0$ is continuous. If $\{e_\alpha\}$ is the basis of $X_0$ mentioned in the statement of part (i), then $q_0(\cdot) = \sum \alpha q_\alpha(\cdot)e_\alpha$ so there exists $\alpha_0$ such that $q_{\alpha_0}(\gamma) \neq 0$ in a neighborhood of $\gamma_0$. If $q_\alpha(\cdot) = \hat{p}_\alpha(\cdot)$ then $p = \sum \alpha p_\alpha e_\alpha$ and as $p \in L_1(G, C), p_\alpha \in L_2(G, C)$. By hypothesis $\| q_{\alpha}(\gamma) e_\alpha \| \leq k_1 \| x \|$ so $p_\alpha \in L_1(G, C)$ and $\hat{p}_\alpha(\cdot) = q_\alpha(\cdot)$. Hence $p_\alpha(\cdot) e_\alpha \in L_1(G, X) \cap L_2(G, X_0)$ for any $\alpha$ and $\hat{p}_\alpha(\cdot) e_\alpha = q_{\alpha_0}(\cdot) e_\alpha \in Q$. Since for each $\gamma$ in a neighborhood of $\gamma_0$, $\{q_{\alpha_0}(\gamma)e_\alpha\}_\alpha$ forms a complete set in $X_0$, and since $0 = \langle q_{\alpha_0}(\gamma)e_\alpha, r(\gamma) \rangle_0$, then $r(\gamma) = 0$ in a neighborhood of $\gamma_0$. But $\gamma_0$ was arbitrary so $r = 0$, or $Q$ is orthogonal only to 0 in $L_2(\hat{G}, X_0)$, a Hilbert space. Hence $Q$ is dense in $L_2(\hat{G}, X_0)$. This completes the proof.

**Corollary 4.8.** Under the assumptions of the theorem the Fourier transform can be extended in a unique manner to an isometry of $L_2(G, X_0)$ onto $L_2(\hat{G}, X_0)$.

**Proof.** We need only show $L_1(G, X) \cap L_2(G, X_0)$ is dense in $L_2(G, X_0)$. But $C_\alpha(G, X_0)$ is dense in $L_2(G, X_0)$ ([6]). Hence if $f \in L_2(G, X_0)$ then there exists $\{f_n\}_n \subset C_\alpha(G, X_0) \cap L_2(G, X_0)$ such that $\| f_n - f \|_2 \to 0$. Then $f_n \in C_\alpha(G, X)$ and $f_n$ is measurable so $f_n \in L_1(G, X)$.

**Remark.** The equality (4.5) holds for all $q, r \in L_2(G, X_0)$. Moreover, all results are correct assuming only that $\varphi_0$ is an arbitrary

---

\(^7\) By this we mean that $f_{\gamma_0}$ is in $Q$ for any $\gamma_0$ in $\hat{G}$ if $f$ is in $Q$ and $f_{\gamma_0}(\gamma) = f(\gamma + \gamma_0)$.

\(^8\) $C_\alpha(G, X_0)$ denotes the set of functions in $C_0(G, X_0)$ having compact support.
linear combination of $\varphi_i$'s, i.e. $\varphi_0 = \sum_{a \in A} c_a \varphi_a$.

5. Examples. Here we give some examples of admissible pairs and strongly admissible triplets.

**EXAMPLE 5.1.** Let $X = L_i([0,1], C)$ so $X$ is weakly complete, and let $\Phi$ consist of elements $\varphi_i$ such that

$$ (x, \varphi_i) = \int_0^1 \chi_i(t) x(t) dt \quad x \in X $$

where $\chi_i(\cdot)$ is the indicator function of one of a countable collection of sets $\{E_i\}$ dense in $\Sigma([0,1])$ under the usual Hausdorff metric. Assume $E_1 = [0,1]$. Then it can be shown ([4], [6]) that $\Phi$ is full and that $K$ is the cone of nonnegative (a.e.) functions. Let $(x, \varphi_0) = (x, \varphi_i) = \int_0^1 x(s) ds = ||x||$ for $x \in K$. Hence $(\Phi, X)$ is admissible and $K_0 = K$.

If $p$ is in $\mathcal{P}$ then $p(0)$ is in $K = K_0$ by Propositions 2.8 and 2.9 and by Corollary 2.15. So $p \in \mathcal{J}$ and the inversion theorem states that if $p \in \text{sp} \{L_i(G, L_\lambda([0,1], C)) \cap \mathcal{P} \}$ then $\hat{p} \in L_i(\hat{G}, L_i([0,1], C))$ and $p(g) = \int_0^1 (g, \gamma) \hat{p}(\gamma) m(d\gamma)$.

The author does not know of any nontrivial subspace $X_0$ which would make $(\Phi, X, X_0)$ strongly admissible.

**EXAMPLE 5.3.** Let $X = H$, a separable Hilbert space with a fixed orthonormal basis $\{e_i\}_\infty$. Let $H_0$ be the set of elements of $H$ with all but a finite number of components zero, with nonzero components being real, rational nonnegative, and with norm less than or equal to one. Then $\Phi = H_0$ is full ([4], [6]) and countable and $K_0 = \{ h \in H : h_i \geq 0 \}$.

Let $(h, \varphi_i) = \langle h, e_i \rangle$, $i = 1, 2, \cdots$ and $\varphi_0 = \sum_i \varphi_i$. Then $\varphi_0$ maps $K$ into $[0, \infty]$, and for $h$ in $K$

$$ (h, \varphi_0)^2 = (\Sigma h_i)^2 \geq \Sigma h_i^2 = ||h||^2 $$

so that $(\Phi, H)$ is admissible and $K_0 = \{ h \in K : \Sigma h_i < \infty \}$.

$H$ becomes a Banach algebra if we define $hk = \Sigma h_i k_i e_i$. Let $h^* = \Sigma h_i^* e_i$. For $h$ in $H$ $hh^*$ is in $K$ and $(hh^*, \varphi_0) = \Sigma h_i \bar{h}_i = ||h||^2$. We do not have $k ||hh^*|| \geq ||h||^2$ for some $k > 0$, but we do have $||h||_0 = ||h||$ which is sufficient to show that $X_0 = H$. Hence $(\Phi, H, H)$ is "strongly admissible," and the Plancherel theorem applies. Note that the condition $|\langle h, e_i \rangle| \leq ||h||$ also holds.

**EXAMPLE 5.4.** Let $X = \mathcal{L}(H, H)$, the linear bounded operators

\[ h_i = \langle h, e_i \rangle \]
mapping the separable Hilbert space \( H \) into itself. Let \( H_0 \) be a countable dense subset of the unit ball in \( H \) and let \( \Phi = \{ \varphi \in X^* : (T, \varphi) = \langle T h, h \rangle, T \in \mathcal{L}(H, H), h \in H_0 \} \). Let \( \{ e_i \} \) also be in \( H_0 \) for some orthonormal basis \( \{ e_i \} \). Then \( \Phi \) is full and countable and \( K_\varphi \) is the cone of positive operators ([4] or [6]). Let \( (T, \varphi_0) = \sum_i \langle T e_i, e_i \rangle \). So \( \varphi_0 = \sum_i \varphi_i \) is the trace, where \( (T, \varphi_i) = \langle T e_i, e_i \rangle \). Then \( \varphi_0 : K \to [0, \infty] \), \( (T, \varphi_0) = \text{tr} \| T \| \) if \( T \) is positive. Hence \( (\Phi, \mathcal{L}(H, H)) \) is admissible and \( K_\varphi \) is the cone of positive operators of finite trace and so a subset of the trace class.

We can see that in one case the condition \( p \in \mathcal{T}_0 \) is necessary for the inversion theorem to hold. Let \( G \) be the circle group so that \( \hat{G} \) is countable. Label its elements \( \gamma_1, \gamma_2, \ldots \), and let the set function \( \nu \) be given by

\[
\langle \nu([\gamma_n]) e_i, e_j \rangle = p_n \delta_{ni} \delta_{n,j}^{10} \quad i, j, n = 1, 2, \ldots
\]

where \( \infty > M \geq p_n \geq 0 \). \( \nu \) can be extended to a countably additive measure of finite semi-variation in the obvious way. Let \( p \) be given by

\[
p(t) = \sum_{n=1}^{\infty} e^{i t n} \nu([\gamma_n]).
\]

Then \( p \) is in \( \mathcal{P} \) (Theorem 2.12 (A)) and \( p \) is in \( L_1(G, X) \) because \( G \) is compact and \( \| p(t) \| \leq M \). If \( \hat{p} \) is to be in \( L_1(\hat{G}, X) \) then \( \| \nu \| (\hat{G}) \) must be finite or \( \sum_i^\infty p_n = \text{tr} \| p(0) \| < \infty. \)

Finally let \( X_0 = \mathcal{N} \); the Hilbert-Schmidt operators ([3]). Then for \( T \) in \( \mathcal{N} \), \( TT^* \) is in the trace class and is positive so that \( TT^* \) is in \( K_\varphi \). Also \( \mathcal{L}(H, H) \) is a \( C^* \)-algebra so \( (\Phi, \mathcal{L}(H, H), \mathcal{N}) \) is strongly admissible. A basis for \( \mathcal{N} \) is given by \( \{ T_{ij} \} \) where \( \langle T_{ij} e_k, e_l \rangle = \delta_{ik} \delta_{jl}, k, l = 1, 2, \ldots \). Then \( |\langle T, T_{ij} \rangle_0| = |\langle T e_i, e_j \rangle| \leq \| T \| \), and the condition in (i) of Theorem 4.4 also holds.

6. Fourier transforms on representations. In this section we apply the preceding theory to extend the inversion theorem and Plancherel's theorem to "Fourier transforms" defined for unitary representations in a separable Hilbert space. The case where \( H \) is finite dimensional has been treated by Hewitt and Wigner [7]. Let \( H \) be a separable complex Hilbert space, and let \( U(\cdot) \) be a continuous unitary representation of \( G \) in \( \mathcal{L}(H, H) \), i.e. \( U(g + g') = U(g)U(g') \), \( U(0) = I \), and \( V \) is a continuous mapping of \( G \) into the unitary operators on \( H \). It follows [9] that there exists a sequence \( \{ \gamma_i \} \) of characters, and a resolution \( \{ \pi_i \} \) of the identity in \( \mathcal{L}(H, H) \), such that

\[\delta_{ni} \text{ is the Kronecker delta.}\]
(6.1) \[ U(g) = \sum_i (g, \gamma_i)\pi_i . \]

(The summation is at most countable). If \( p \) is in \( L_1(G, \mathcal{L}(H, H)) \) define the transform

\[ \hat{p}(U) = \int_G p(g)U(-g)\mu(dg) . \]

We shall first consider the question of invertibility of this transform. As we shall see, it suffices to know \( \hat{p}(U) \) for all \( U \) corresponding to a fixed resolution \( \{\pi_i\} \).

From now on consider \( \{\pi_i\} \) fixed, and let us denote the set of subscripts by \( S \). Then \( S \) is at most countable, \( \sum_{i \in S} \pi_i = I \). Define \( \mathcal{R} = \prod_{i \in S} \hat{G}_i \), where \( \hat{G}_i = \hat{G} \) for all \( i \), with the product topology. Then \( \mathcal{R} \) can be considered as the set of all representations corresponding to \( \{\pi_i\} \), if we put

(6.5) \[ r \longmapsto \sum_{i \in S} (\cdot, \gamma_i)\pi_i = U(\cdot) \]

whenever \( r = \{\gamma_i\} \in \mathcal{R} \).

Let us now introduce a measure on \( \mathcal{R} \). Choose a symmetric neighborhood \( A \) of 0 in \( \hat{G} \) such that the closure of \( A \) is compact. Hence \( 0 < m(A) < \infty \). Assume \( m \) is normalized (relative to \( \mu \)) such that the inversion theorem holds. Now normalize \( \mu \) such that \( m(A) = 1 \). Note that if \( G \) is discrete and \( A = \hat{G} \), or if \( G \) is compact and \( A = \{0\} \), then the usual normalizations of \( \mu \) and \( m \) occur. For \( \alpha \) in \( \hat{G} \) and \( E \) in \( \Sigma(\hat{G}) \) define

\[ m_\alpha(E) = m[E \cap (A + \alpha)] . \]

Then \( m_\alpha(\cdot) \) is a probability measure on \( \hat{G} \), and by the Kolmogorov extension theorem, there exists a unique probability measure

\[ m^\infty_\alpha = m_\alpha \times m_\alpha \times \cdots \]

on \( \mathcal{R} \). We set \( \mathcal{R}^i = \prod_{j \in S \setminus \{i\}} \hat{G}_j \). For \( E \) in \( \Sigma(\mathcal{R}^i) \) write

\[ m^i_\alpha(E) = \int_{\mathcal{R}^i} \chi_{E \times \hat{G}}(r)m^\infty_\alpha(dr) \]

where it is understood we are integrating out \( \gamma_i \).

Now assume \( G \) is \( \sigma \)-finite and \( \Phi \) is a full, countable subset of \( \mathcal{L}(H, H)^* \). With the previous notation we have

**Theorem 6.4.** If \( p \) is in span \( \{L_1(G, \mathcal{L}(H, H)) \cap \mathcal{R}^i\} \), then

(6.5) \[ p(g) = \int_\Phi \int_{\mathcal{R}^i} \hat{p}(U)U(g)m^\infty_\alpha(dr)m(d\alpha) . \]
Proof. $\pi_i$ is a projection on the subspace $H_i$ of $H$. Moreover if we consider the equivalent spectral representation ([9], p. 247), then the subspaces are mutually orthogonal. Let us write $f(\alpha) = \hat{\rho}(\alpha) (g, \alpha)$, and $f^j(r) = f(\gamma_j)$ when $r = \{\gamma_j\}$. Then for $n$ finite, $\beta \in \hat{G}$ and $h \in H$,

\[
\left\| \sum_{i=1}^{n} \int_{\rho} f^i(r) m^\infty_\beta (dr) \pi_i h \right\|
\]

\[
= \left\| \sum_{i=1}^{n} \int_{\rho} f(\alpha) m^\infty_\beta (dr) m_\alpha (d\alpha) \pi_i h \right\|
\]

\[
= \left\| \sum_{i=1}^{n} \int_{\rho} f(\alpha) m_\beta (d\alpha) \pi_i h \right\|
\]

\[
\leq \left\| \int_{\rho} f(\alpha) m_\beta (d\alpha) \right\| \left\| h \right\|
\]

so that

\[
\left\| \sum_{i=1}^{n} \int_{\rho} f^i(r) m^\infty_\beta (dr) \pi_i \left\| \leq \left\| \sum_{i=1}^{n} \int_{\rho} f(\alpha) m_\beta (d\alpha) \right\| \leq \left\| \int_{\rho} \hat{\rho}(\alpha) \| \chi_{\lambda + \delta}(\alpha) m(d\alpha) \right\|
\]

As $\hat{\rho}$ is in $L_i$, and as $m(A) = 1$, then

\[
\int_{\rho} \int_{\rho} \| \hat{\rho}(\alpha) \| \chi_{\lambda + \delta}(\alpha) m(d\alpha) m(d\beta)
\]

\[
= \| \hat{\rho} \|_1 .
\]

Hence

\[
\sum_{i=1}^{n} \int_{\rho} f^i(r) m^\infty_\beta (dr) m_\beta (d\beta) \pi_i
\]

\[
= \int_{\rho} \sum_{i=1}^{n} \int_{\rho} f^i(r) m^\infty_\beta (dr) \pi_i m(d\beta) .
\]

Moreover

\[
\int_{\rho} f^i(r) m^\infty_\beta (dr) \pi_i
\]

\[
= \int_{\rho} \hat{\rho}(\gamma_i) (g, \gamma_i) m^\infty_\beta (dr) \pi_i
\]

\[
= \int_{\rho} \int_{\rho} p(g')(g - g', \gamma_i) \mu(dg') m^\infty_\beta (dr) \pi_i ,
\]

and

\[
\left\| \sum_{i=1}^{n} p(g')(g - g', \gamma_i) \pi_i h \right\|
\]

\[
\leq \| p(g') \| \left\| \sum_{i=1}^{n} (g - g', \gamma_i) \pi_i h \right\|
\]

\[
\leq \| p(g') \| \left\| h \right\|
\]
as \(|g, \gamma| = 1\) and the \(\pi_i\)'s are orthogonal projections. As \(p\) is in \(L_1\), and as \(m_i^\infty(\mathcal{B}) = 1\), then

\[
\sum_{i \in S} \int_{\mathcal{B}} f_i(r) m_i^\infty(dr) \pi_i
\]

(6.7)

\[
= \int_{\mathcal{B}} \sum_{i \in S} p(g')(g - g', \gamma_i) \pi_i \mu(dg') m_i^\infty(dr) .
\]

On the other hand

\[
\hat{p}(U) U(g) = \int_{\mathcal{B}} p(g') U(-g') \mu(dg') U(g)
\]

(6.8)

\[
= \int_{\mathcal{B}} p(g') U(g - g') \mu(dg')
\]

\[
= \int_{\mathcal{B}} p(g') \sum_{i \in S} (g - g', \gamma_i) \pi_i \mu(dg') .
\]

Hence we have shown that for each \(\beta, g\), \(\hat{p}(U) U(g)\) is integrable \(m_i^\infty(dr)\), and \(\int_{\mathcal{B}} \hat{p}(U) U(g) m_i^\infty(dr)\) is integrable \(m(d\beta)\), so that 6.5 makes sense.

Finally

\[
\int_{\mathcal{B}} \int_{\mathcal{B}} \hat{p}(U) U(g) m_i^\infty(dr) m(d\beta)
\]

\[
= \sum_{i \in S} \int_{\mathcal{B}} \int_{\mathcal{B}} f_i(r) m_i^\infty(dr) m(d\beta) \pi_i
\]

\[
= \sum_{i \in S} \int_{\mathcal{B}} \int_{\mathcal{B}} \hat{p}(\alpha)(g, \alpha) m_i^\infty(d\alpha) m(d\beta) \pi_i
\]

\[
= \sum_{i \in S} \int_{\mathcal{B}} \int_{\mathcal{B}} \chi_i(\alpha - \beta) \hat{p}(\alpha)(g, \alpha) m(d\beta) m(d\alpha) \pi_i
\]

\[
= \sum_{i \in S} \hat{p}(\alpha)(g, \alpha) m(d\alpha) \pi_i
\]

\[
= \sum_{i \in S} p(g) \pi_i
\]

\[
= p(g) .
\]

We have made use of 6.6, 6.7, 6.8, and the inversion theorem. The theorem is established.

Now consider the setting of Example 5.4.

**Theorem 6.9.** If \(p\) and \(q\) are in \(L_1[G_1, \mathcal{L}[H, H]] \cap L_e(G, \mathcal{N})\), then

\[
\int_{\mathcal{B}} p(g) q(g) \mu(dg) = \int_{\mathcal{B}} \int_{\mathcal{B}} \hat{p}(U) \hat{q}(U) m_i^\infty(dr) m(d\alpha)
\]
Proof. The proof is similar to the previous one except that Theorem 4.4 is used.

Further applications of this theory can be found in [6].

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UNIVERSITY OF BRITISH COLUMBIA
Max K. Agoston, An obstruction to finding a fixed point free map on a manifold . . . 543
Nadim A. Assad and William A. Kirk, Fixed point theorems for set-valued mappings of contractive type ................................................................. 553
John Winston Bunce, Characterizations of amenable and strongly amenable C*-algebras ................................................................. 563
Erik Maurice Ellentuck and Alfred Berry Manaster, The decidability of a class of AE sentence in the isols .................................................. 573
U. Haussmann, The inversion theorem and Plancherel's theorem in a Banach space ................................................................. 585
Peter Lawrence Falb and U. Haussmann, Bochner's theorem in infinite dimensions .................................................................................. 601
Peter Fletcher and William Lindgren, Quasi-uniformities with a transitive base .............................................................................. 619
Dennis Garbanati and Robert Charles Thompson, Classes of unimodular abelian group matrices .......................................................... 633
Kenneth Hardy and R. Grant Woods, On c-realcompact spaces and locally bounded normal functions ...................................................... 647
Manfred Knebusch, Alex I. Rosenberg and Roger P. Ware, Grothendieck and Witt rings of hermitian forms over Dedekind rings ........................................ 657
George M. Lewis, Cut loci of points at infinity .......................................................................................................................... 675
Jerome Irving Malitz and William Nelson Reinhardt, A complete countable L^0_\omega_1 theory with maximal models of many cardinalities .................................. 691
Wilfred Dennis Pepe and William P. Ziemer, Slices, multiplicity, and Lebesgue area .................................................................................. 701
Keith Pierce, Amalgamating abelian ordered groups ................................................................. 711
Stephen James Pride, Residual properties of free groups ................................................................. 725
Roy Martin Rakestraw, The convex cone of n-monotone functions .................................................................................. 735
T. Schwartzbauer, Entropy and approximation of measure preserving transformations .................................................................................. 753
Peter F. Stebe, Invariant functions of an iterative process for maximization of a polynomial .................................................................................. 765
Kondagunta Sundaresan and Wojbor Woyczynski, L-orthogonally scattered measures .................................................................................. 785
Kyle David Wallace, C_2-groups and \lambda-basic subgroups .................................................................................. 799
Barnet Mordecai Weinstock, Approximation by holomorphic functions on certain product sets in C^n .................................................................................. 811
Donald Steven Passman, Corrections to: “Isomorphic groups and group rings” .................................................................................. 823
Don David Porter, Correction to: “Symplectic bordism, Stiefel-Whitney numbers, and a Novikov resolution” .................................................................................. 825
John Ben Butler, Jr., Correction to: “Almost smooth perturbations of self-adjoint operators” .................................................................................. 825
Constantine G. Lascarides, Correction to: “A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer” .................................................................................. 826
George A. Elliott, Correction to: “An extension of some results of Takesaki in the reduction theory of von Neumann algebras” .................................................................................. 826
James Daniel Halpern, Correction to: “On a question of Tarski and a maximal theorem of Kurepa” .................................................................................. 827