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1. Introduction. Let G be a locally compact abelian group with Haar measure μ, and let X be a complex Banach space and C be the set of complex numbers. A classic theorem due to Plancherel ([8], [10]) states that the Fourier transform maps $L_1(G, C) \cap L_2(G, C)$[1] onto a dense subset of $L_2(\hat{G}, C)$ ($\hat{G}$ is the dual group of G and has Haar measure m) in such a way that $\int_G \alpha(g)\overline{\beta(g)}\mu(dg) = \int_{\hat{G}} \hat{\alpha}(\gamma)\overline{\hat{\beta}(\gamma)}m(d\gamma)$ for all α, β in $L_1(G, C) \cap L_2(G, C)$ where $\hat{\alpha}$ is the Fourier transform of α, given by $\hat{\alpha}(\gamma) = \int_G \overline{(g, \gamma)}\alpha(g)\mu(dg)$ for all γ in $\hat{G}$. Here (g, γ) denotes the action of the character γ on g in G. In this paper we extend this result to functions taking values in an inner product subspace of a Banach algebra.

Another well-known theorem ([8], [10]) states that if α is a positive definite element of $L_1(G, C) \cap L_\infty(G, C)$ then $\hat{\alpha}$ is in $L_1(\hat{G}, C)$ and

$$\alpha(g) = \int_{\hat{G}} (g, \gamma)\hat{\alpha}(\gamma)m(d\gamma) \tag{1.1}$$

for (almost) all g in G. This inversion theorem is also generalized to functions assuming values in certain admissible Banach spaces.

Our work relies heavily on an extension of Bochner's theorem established in [4]. We show that if p is in $L_1(G, X) \cap L_\infty(G, X)$, if p is positive definite (positivity is defined with respect to a particular cone in X), and if $p(0)$ satisfies a certain finiteness condition, then $\hat{p}$, the Fourier transform of p, is in $L_1(\hat{G}, X)$ and the inversion formula 1.1 given for α holds for p. A sharper theorem states that if p is in $L_1(G, X) \cap L_\infty(G, X)$, if p is positive definite, and if there is a real, finite, regular Borel measure λ such that $\left\|\int_G \alpha(g)p(g)\mu(dg)\right\| \leq \int_{\hat{G}} |\hat{\alpha}(\gamma)|\lambda(d\gamma)$ for all α in $L_1(G, C)$, then $\hat{p}$ is in $L_1(\hat{G}, X)$ and 1.1 is satisfied by p.

Using this theory we extend to infinite dimensions some results due to Hewitt and Wigner ([7]).

[1] For $1 \leq p \leq \infty$ $L_p(G, X)$ is the space of μ-measurable functions f mapping G into X. For $1 \leq p < \infty$ we use the norm $\|\cdot\|_p$, where $\|f\|_p = \left\{\int_G \|f(g)\|^p \mu(dg)\right\}^{1/p}$, and for $p = \infty$ we use the norm $\|f\|_\infty$ which is the (μ) essential supremum of $\|f(g)\|$ on G. $\|\cdot\|$ denotes the norm in X.

2. Bochner's theorem and dominated functions. Let X be a Banach space, X^* the dual of X and X^{**} the dual of X^*. For φ in X^* we denote the action of φ on $x \in X$ by (x, φ). Given a subset of X^* we can define a cone of "positive" elements in X.

DEFINITION 2.1. Let Φ be a subset of X^*. The subset K_Φ of X given by

$$K_\Phi = \{x \in X\colon (x, \varphi) \geqq 0 \text{ for all } \varphi \in \Phi\} \tag{2.2}$$

is called the cone determined by Φ.

Sometimes we write simply K if Φ is fixed by the context. X_Φ is the set of "positive" elements.

Let G be a σ-finite locally compact abelian group with Haar measure μ and let $\hat{G}$ be its dual group with Haar measure m.

DEFINITION 2.3. Let p be a map of G into X. Then p is Φ-positive definite if it is measurable and if

$$\sum_{n=1}^{N} \sum_{m=1}^{N} c_n \bar{c}_m (p(g_n - g_m), \varphi) \geqq 0 \tag{2.4}$$

for any integer N, any $c_1, \cdots, c_N$ in C, any $g_1, \cdots, g_N$ in G, and all φ in Φ. If p is in $L_\infty(G, X)$ the p is integrally Φ-positive definite if

$$\left(\int_G \int_G \alpha(g)\overline{\alpha(g')}p(g - g')d\mu d\mu, \varphi\right) \geqq 0 \tag{2.5}$$

for all α in $L_1(G, C)$ and all φ in Φ.

Next we impose a condition which relates Φ to the topology of X.

DEFINITION 2.6. The family Φ is full if there is a $\rho > 0$ such that

$$||x|| \leqq \rho \sup\{|(x, \varphi)|/||\varphi||\colon \varphi \in \Phi\} \tag{2.7}$$

for all x in X.

The following two propositions examine the relationship between the two notions of positive-definiteness.

PROPOSITION 2.8. *If Φ is full and p is Φ-positive definite then p is in $L_\infty(G, X)$ and $p(0)$ is in K_Φ.*

Proof. It is readily shown that for g in G, φ in Φ, $|(p(g), \varphi)| \leqq$

$(p(0), \varphi)$ so that $\| p(g) \| \leqq \rho \| p(0) \|$.

PROPOSITION 2.9. *Let p be in $L_\infty(G, X)$ such that one version of p is ωX-continuous.*[2] *Then p is Φ-positive definite iff p is integrally Φ-positive definite.*

Proof. See [4] or [6].

We shall see shortly (Corollary 2.15) that all those elements of $L_\infty(G, X)$ of interest to us have the continuity required in Proposition 2.9.

Next we recall some results from measure theory. Let S be a locally compact topological space and let $\Sigma(S)$ be the Borel field of S (i.e. the smallest σ-field containing the closed sets of S).

DEFINITION 2.10. A vector measure ν is a weakly countably additive set function defined on $\Sigma(S)$ and taking values in X. ν is weakly regular if the scalar measures $(\nu(\cdot), \varphi)$ are regular[3] for all φ in X^*. ν is Φ-positive if $(\nu(E), \varphi) \geqq 0$ for all φ in Φ and E in $\Sigma(S)$.

DEFINITION 2.11. A set function ν^{**} mapping $\Sigma(S)$ into X^{**} is weak-*-regular if $(\varphi, \nu^{**}(\cdot))$ is a regular scalar measure for all φ in X^*. ν^{**} is Φ-positive if $(\varphi, \nu^{**}(E)) \geqq 0$ for all φ in Φ, E in $\Sigma(S)$.

If ν is a vector measure we denote its variation on a measurable set E by $\| \nu \| (E)$ and its semi-variation by $| \nu | (E)$ ([2], [1]). The following theorem, an extension of Bochner's theorem, is essential to our work. The proof is given in [4]. We assume Φ is full.

THEOREM 2.12. (A) *If ν is a weakly regular Φ-positive vector measure defined on $\Sigma(\hat{G})$ and if*

$$p(g) = \int_{\hat{G}} (g, \gamma)\nu(d\gamma) \tag{2.13}$$

then p is an integrally Φ-positive definite element of $L_\infty(G, X)$.

(B) *If p is an integrally Φ-positive definite element of $L_\infty(G, X)$, then there is a set function ν^{**} mapping $\Sigma(\hat{G})$ into X^{**} such that* (i) *ν^{**} is weak-*-regular, Φ-positive with finite semi-variation and* (ii)

$$(p(g), \varphi) = \int_{\hat{G}} (g, \gamma)(\varphi, \nu^{**}(d\gamma)) \tag{2.14}$$

for all φ in X^* and almost all g in G.

[2] The mapping f of G into X is ωX-continuous if it is continuous when the weak topology is imposed on X. G retains its usual topology.

[3] A scalar measure λ is regular if, given $\varepsilon > 0$ and $E \in \Sigma(S)$ with $\| \lambda \| (E) < \infty$ (i.e. λ has finite variation on E), then there is a compact $K \subset E$ and an open $O \supset E$ such that $\| \lambda \| (O - K) < \varepsilon$.

COROLLARY 2.15. *If p is an integrally Φ-positive definite element of $L_\infty(G, X)$ then one version of p is ωX-continuous. If p is given by* 2.13, *where ν is a weakly regular Φ-positive vector measure defined on $\Sigma(\hat{G})$, then p is a continuous map of G into X.*

Proof. This follows from the relevant regularity. See also [6].

With the aid of Theorem 2.12 we could prove a useful inversion theorem. However, a different version of Bochner's theorem will allow us to establish a sharper theorem. We require first the following.

DEFINITION 2.16. p in $L_\infty(G, X)$ is dominated if there exists a finite, regular, positive Borel measure λ, such that

$$\left\|\int_G \alpha(g)p(g)\mu(dg)\right\| \leqq \int_{\hat{G}} |\hat{\alpha}(\gamma)|\lambda(d\gamma) \tag{2.17}$$

for all α in $L_1(G, C)$, where $\hat{\alpha}$ is the Fourier transform of α, i.e. $\hat{\alpha}(\gamma) = \int_G \overline{(g, \gamma)}\alpha(g)\mu(dg)$. If R^+ is the set of nonnegative real numbers, we have

DEFINITION 2.18. Let Φ be a subset of X. Assume there is a function φ_0 mapping K_Φ into $R^+ \cup \{\infty\}$ in a linear manner such that φ_0 is uniformly positive on K_Φ, i.e. there exists $k > 0$ such that $k(x, \varphi_0) \geqq ||x||$ for all x in K_Φ. Furthermore assume there are at most countable sequences $\{c_i\}$ in R^+ and $\{\varphi_i\}$ in Φ such that $(x, \varphi_0) = \sum_{i=1}^\infty c_i(x, \varphi_i)$ for all x in K_Φ. Then we say that the pair (Φ, X) is admissible. We let $K_0 = \{x \in K_\Phi \colon (x, \varphi_0) < \infty\}$.

LEMMA 2.19. *If (Φ, X) is admissible, if Φ is full, and if $p \in L_\infty(G, X)$ is integrally Φ-positive definite with $p(0)$ in K_0, then p is dominated.*

In this lemma it is assumed we are talking about the ωX-continuous version of $p(\cdot)$ (Corollary 2.15).

Proof. Let $\psi(\alpha) = \int_G \alpha(g)p(g)\mu(dg)$ for all α in $L_1(G, C)$, then $(\psi(\alpha), \varphi) = \int_{\hat{G}} \hat{\alpha}(\gamma)(\varphi, \nu^{**}(d\gamma))$ for some weak-*-regular, Φ-positive set function ν^{**} given by Theorem 2.12. We can actually define $\hat{\psi}(\cdot)$ mapping $C_0(\hat{G})$[4] into X by $(\hat{\psi}(f), \varphi) = \int_{\hat{G}} f(\gamma)(\varphi, \nu^{**}(d\gamma))$.[5] Then $\hat{\psi}$ is a

[4] $C_0(\hat{G})$ is the space of continuous functions mapping $\hat{G}$ into C, which vanish at ∞ if $\hat{G}$ is only locally compact.

[5] For α in $L_1(G, C)$, $\hat{\psi}(\hat{\alpha}) = \psi(\alpha) \in X$. As $\{\hat{\alpha} \in C_0(\hat{G}) \colon \alpha \in L_1(G, C)\}$ is dense in $C_0(\hat{G})$, and as $\hat{\psi}$ is continuous, it can be extended uniquely, with range in X.

bounded linear map, $\| \hat{\psi}(f) \| \leqq \| f \|_\infty |\nu^{**}|(\hat{G})$.

If f is in $C_0(\hat{G})$ then $f = f_1 - f_2 + if_3 - if_4$ where f_i is in $C_0(\hat{G})$, $f_i(\gamma) \geqq 0$, and each pair of functions (f_1, f_2), (f_3, f_4) has disjoint support. Hence $f_i(\gamma) \leqq |f(\gamma)|$, and $\hat{\psi}(f_i)$ is in K_Φ so that $\| \hat{\psi}(f_i) \| \leqq k(\hat{\psi}(f_i), \varphi_0) = k \sum_{j=1}^\infty c_j(\hat{\psi}(f_i), \varphi_j) = k \sum_j c_j \int_{\hat{G}} f_i(\gamma)(\varphi_j, \nu^{**}(d\gamma))$. Consider now the set function λ given by $\lambda(E) = \sum_{i=1}^\infty c_i(\varphi_i, \nu^{**}(E))$, $E \in \Sigma(\hat{G})$. Then $\lambda(E) \geqq 0$ for all E in $\Sigma(\hat{G})$, and also λ is additive. Moreover $\lambda(E) \leqq (p(0), \varphi_0) < \infty$ as $p(0)$ is in K_0.

λ is countably additive because $\lambda(\bigcup_j E_j) = \sum_i \sum_j c_i(\varphi_i, \nu^{**}(E_j)) = \sum_j \sum_i c_i(\varphi_i, \nu^{**}(E_j)) = \sum_j \lambda(E_j)$, if the E_j are disjoint (note that $c_i(\varphi_i, \nu^{**}(E_j)) \geqq 0$ for all i, j). Also λ is regular, for given $\varepsilon > 0$ and E in $\Sigma(\hat{G})$, there is a number N such that $\sum_{N+1}^\infty c_i(\varphi_i, \nu^{**}(\hat{G})) < \varepsilon/2$ and there is a compact $K \subset E$ and an open $O \supset E$ such that $(\varphi_i, \nu^{**}(0 - K)) < \varepsilon/2Nc_i$, $i = 1, 2, \cdots, N$. Hence $\lambda(O - K) < \varepsilon$.

Then $\| \hat{\psi}(f) \| \leqq \sum_{i=1}^4 \| \hat{\psi}(f_i) \| \leqq k \sum_i \int_{\hat{G}} f_i(\gamma) d\lambda \leqq 4k \int_{\hat{G}} |f(\gamma)| d\lambda$. It follows that if $\lambda' = 4k\lambda$ then $\| \psi(\alpha) \| \leqq \int_{\hat{G}} |\alpha(\gamma)| d\lambda'$. This establishes the lemma.

We can now state the alternate version of Bochner's theorem. Assume Φ is full and countable

THEOREM 2.20. *p is a dominated, integrally Φ-positive definite element of $L_\infty(G, X)$ iff there is a weakly regular Φ-positive vector measure ν mapping $\Sigma(\hat{G})$ into X such that ν has finite variation,* i.e. *$\| \nu \| (\hat{G}) < \infty$, and such that for any φ in X^*,*

$$(p(g), \varphi) = \int_{\hat{G}} (g, \gamma)(\nu(d\gamma), \varphi) , \qquad a.\ e.\ g. \tag{2.21}$$

For the proof see [4]. Countability of Φ is not required for the only if part.

3. Inversion theorems. If $p \in L_1(G \cdot X)$ we recall that the Fourier transform of p is given by

$$\hat{p}(\gamma) = \int_G \overline{(g, \gamma)} p(g) \mu(dg) . \tag{3.1}$$

For convenience we let $\mathscr{P} = \{p \in L_\infty(G, X): p$ is integrally Φ-positive definite$\}$ and $\mathscr{P}_d = (p \in \mathscr{P}: p$ is dominated$\}$. We recall that if $p \in \mathscr{P}$ then p is ωX-continuous (Corollary 2.15). If (Φ, X) is admissible then $\mathscr{T}_0$ is the set of functions p mapping G into X such that p is ωX-continuous and such that $p(0)$ is in K_0 where K_0 is defined in 2.18.

PROPOSITION 3.2. (A) *If $p \in$ span $\{L_1(G, X) \cap \mathscr{P}\}$ and if $\varphi \in$*

span $\{\Phi\}$ *then* $(\hat{p}(\cdot), \varphi) \in L_1(\hat{G}, C)$ *and* (B) *if the Haar measure of* G *is fixed then the Haar measure of* $\hat{G}$ *can be so normalized that*

$$(3.3) \qquad (p(g), \varphi) = \int_{\hat{G}} (g, \gamma)(\hat{p}(\gamma), \varphi)m(d\gamma)$$

is valid for all $p \in$ *span* $\{L_1(G, X) \cap \mathscr{P}\}$ *and all* $\varphi \in$ *span* $\{\Phi\}$.

Proof. It is evident the results need only hold for $p \in L_1(G, X) \cap \mathscr{P}$, $\varphi \in \Phi$. But this follows from the scalar inversion theorem ([10], p. 22).

A better result is the following.

THEOREM 3.4. *Assume* Φ *is full and* G *is* σ*-finite.* (A) *If* $p \in$ *span* $\{L_1(G, X) \cap \mathscr{P}_d\}$ *then* $\hat{p} \in L_1(\hat{G}, X)$, *and* (B) *with* μ *fixed,* m *can be so normalized that for each* φ *in* X^*

$$(3.5) \qquad (p(g), \varphi) = \left(\int_{\hat{G}} (g, \gamma)\hat{p}(\gamma)m(d\gamma)\varphi\right) \qquad a.\ e.\ g.$$

If Φ *is countable or if* p *is continuous* (3.5) *becomes*

$$(3.6) \qquad p(g) = \int_{\hat{G}} (g, \gamma)\hat{p}(\gamma)m(d\gamma) \qquad a.\ e.\ g.$$

Proof. Again we need only prove the results for p in $L_1(G, X) \cap \mathscr{P}_d$. If p is in $L_1(G, X)$ then $\hat{p}$ is in $C_0(\hat{G}, X)$, the space of continuous functions mapping $\hat{G}$ into X, which vanish at infinity if $\hat{G}$ is only locally compact but not compact. As p is measurable and G is σ-finite, $\hat{p}$ is essentially separably valued, and hence is measurable and a member of $L_\infty(\hat{G}, X)$.

As p is in $\mathscr{P}_d$, then by Theorem 2.20 there is a weakly regular Φ-positive vector measure ν with finite variation such that for any φ in Φ

$$(3.7) \qquad \begin{aligned} (p(g), \varphi) &= \int_{\hat{G}} (g, \gamma)(\nu(d\gamma), \varphi)\ , \qquad a.\ e.\ g. \\ &= \int_{\hat{G}} (g, \gamma)(\hat{p}(\gamma), \varphi)m(d\gamma) \end{aligned}$$

by Proposition 3.2. As both integrals are continuous, the equality hold for all g. It follows, [10], that

$$\begin{aligned} (\nu(E), \varphi) &= \int_E (\hat{p}((\gamma), \varphi)m(d\gamma) \\ &= \left(\int_E (\hat{p}(\gamma)m(d\gamma), \varphi\right) \end{aligned}$$

if $m(E) < \infty$, as $\hat{p}$ is bounded. Since Φ is full, we have

$$\nu(E) = \int_E \hat{p}(\gamma) m(d\gamma)$$

if $m(E) < \infty$. As $\hat{p}$ is in $C_0(\hat{G}, X)$ given n these exists a compact set K_n such that $\| \hat{p}(\gamma) \| < 1/n$ if γ is in $\hat{G} - K_n$. Let $\chi_n(\cdot)$ be the indicator function of K_n. Then

$$\begin{aligned}
&\lim_{n \to \infty} \int_{\hat{G}} \| \chi_n(\gamma) \hat{p}(\gamma) \| m(d\gamma) \\
&\quad = \lim_{n \to \infty} \int_{K_n} \| \hat{p}(\gamma) \| m(d\gamma) \\
&\quad = \lim_{n \to \infty} \| \nu \| (K_n) \\
&\quad = \| \nu \| (\hat{G}) < \infty \, .
\end{aligned}$$

Also $\| \chi_n(\gamma) \hat{p}(\gamma) \| \uparrow \| \hat{p}(\gamma) \|$ for each γ in $\hat{G}$. Then by the monotone convergence theorem

$$\lim_{n \to \infty} \int_{K_n} \| \hat{p}(\gamma) \| m(d\gamma) = \int_{\hat{G}} \| \hat{p}(\gamma) \| m(d\gamma) \leqq \| \nu \| (\hat{G}) \, .$$

Hence $\hat{p}$ is in $L_1(\hat{G}, X)$, and for all measurable sets E,

$$\nu(E) = \int_E \hat{p}(\gamma) m(d\gamma) \, .$$

Since Φ is full (3.5) now follows from (3.7).

If p is continuous, the set of measure zero where (3.5) does not hold is empty and (3.6) follows. If Φ is countable, the union of these null sets (one for each φ in Φ) is still a null set and again (3.6) holds.

COROLLARY 3.8. *Assume Φ is full, G is σ-finite, and (Φ, X) is admissible.*

(A) *If p is in span $\{L_1(G, X) \cap \mathscr{P} \cap \mathscr{T}_0\}$ then $\hat{p}$ is in $L_1(\hat{G}, X)$.*

(B) *If μ is fixed, m can be so normalized that for each φ* in X^* (3.5) *holds. If Φ is countable or if p is continuous then* (3.6) *holds.*

Proof. Apply Lemma 2.19 and Theorem 3.4.

4. The Plancherel theorem. As usual this theorem is set in a Hilbert space, and so we must first develop the necessary structure. Assume now that X is a Banach algebra with continuous involution $x \to x^*$.

DEFINITION 4.1. The triplet (Φ, X, X_0) is strongly admissible if

(i) (Φ, X) is admissible, (ii), X_0 is a non-trivial subspace of X such that xx^* is in K_0[6] for all x in X_0, and (iii) there exists $k_0 > 0$ such that if $x \in X_0$ then

$$k_0 \| xx^* \| \geqq \| x \|^2 , \tag{4.2}$$

We note that 4.2 is satisfied if X is a C^*-algebra. Now we have

PROPOSITION 4.3. *If X is a Banach algebra and if (Φ, X, X_0) is strongly admissible then X_0 is a Hilbert space under the norm $\|\cdot\|_0$ where $\| x \|_0^2 = \langle x, x\rangle_0$ and $\langle x, y\rangle_0 = (xy^*, \varphi_0)$.*

Proof. φ_0 is only defined on K and we do not know that if $x, y \in X_0$ then $xy^* \in K$. However we can extend φ_0 by setting $(xy^*, \varphi_0) = \sum_{i=1}^{\infty} c_i(xy^*, \varphi_i)$ where $\{c_i\}$, $\{\varphi_i\}$ define φ_0 on K. Then $|\langle x, y\rangle_0| = |(xy^*, \varphi)| = |\sum_{i=1}^{\infty} c_i(xy^*, \varphi_i)| \leqq \sum_1^{\infty} c_i(xx^*, \varphi_i)^{1/2}(yy^*, \varphi_i)^{1/2}$ where the last inequality follows because φ_i is a positive functional. Hence we can define $\langle x, y\rangle_0$ for $x, y \in X_0$ and $|\langle x, y\rangle_0| \leqq \| x \|_0 \| y \|_0$. It follows from 2.18 and 4.2 that $kk_0 \| x \|_0^2 \geqq \| x \|^2$ and that $\|\cdot\|_0$ is a norm.

If $\{x_n\}$ is Cauchy in $\|\cdot\|_0$ then it is Cauchy in $\|\cdot\|$, so $x_n \to x \in X$. As K is closed then $xx^* \in K$. Also $\{x_n\}$ is bounded in $\|\cdot\|_0$ because it is Cauchy, so $\sum_{i=1}^{\infty} c_i(x_n x_n^*, \varphi_i) \leqq M$, hence $\sum_{i=1}^{\infty} c_i(xx^*, \varphi_i) \leqq M$ or $x \in K_0$. Choose $m(\varepsilon)$ such that if $n, m > m(\varepsilon)$ then $\| x_n - x_m \|_0 < \varepsilon$. Then $\sum_{i=1}^{N} c_i([x - x_m][x - x_m]^*, \varphi_i) = \lim_{n\to\infty} \sum_{i=1}^{N} c_i([x_n - x_m][x_n - x_m]^*, \varphi_i) \leqq \limsup_{n\to\infty} \sum_{i=1}^{\infty} c_i([x_n - x_m][x_n - x_m]^*, \varphi_i) < \varepsilon^2$ so that for $m > m(\varepsilon)$, $\| x - x_m \|_0 < \varepsilon$, or X_0 is a Hilbert space.

If X is a Banach algebra and G is σ-finite, then $L_1(G, X)$ is also a Banach algebra ([5]). If X has the involution $x \to x^*$, then we can define an involution on $L_1(G, X)$ as $p \to p^*$ where $p^*(g) = p(-g).^*$

THEOREM 4.4. *If G is σ-finite, X is a Banach algebra with continuous involution, Φ is a full subset of X^* and (Φ, X, X_0) is strongly admissible, then* (i) *if $\{e_\alpha\}$ is an orthonormal basis for X_0 and there exists k_1 such that $|\langle x, e_\alpha\rangle_0| \leqq k_1 \| x \|$ for $x \in X_0$ and all α, then the Fourier transform maps $L_1(G, X) \cap L_2(G, X_0)$ onto a dense subset of $L_2(\hat{G}, X_0)$,* (ii) *for $q, r \in L_1(G, X) \cap L_2(G, X_0)$*

$$\int_G q(g)r(g)\mu(dg) = \int_{\hat{G}} \hat{q}(\gamma)\hat{r}(\gamma)m(d\gamma) , \tag{4.5}$$

(iii) *for $q, r \in L_1(G, X) \cap L_2(G, X_0)$*

$$\langle q, r\rangle = \langle \hat{q}, \hat{r}\rangle , \tag{4.6}$$

where $\langle q, r\rangle = \int_G \langle q(g), r(g)\rangle_0 \mu(dg)$ and $\langle \hat{q}, \hat{r}\rangle = \int_{\hat{G}} \langle \hat{q}(\gamma), \hat{r}(\gamma)\rangle_0 m(d\gamma)$.

[6] K_0 is defined in 2.18.

Proof. We shall put

$$\| q \|_1 = \int_G \| q(g) \| \mu(dg) \qquad \text{and} \qquad \| q \|_2 = \left\{ \int_G \| q(g) \|_0^2 \mu(dg) \right\}^{1/2}$$

for $q \in L_1(G, X) \cap L_2(G, X_0)$. Let $p(g) = (q * q^*)(g)$. As $q \in L_1(G, X)$ so is p with $\| p \|_1 \leqq \| q \|_1^2$. It can also be shown that $p \in C_0(G, X_0)$ as $q \in L_2(G, X_0)$. Now $p(0) = \int_G q(g) q(g)^* \mu(dg) \in K$ so

$$\begin{aligned}(p(0), \varphi_0) &= \left(\int_G q(g) q(g)^* \mu(dg), \varphi_0 \right) \\ &= \sum_{i=1}^{\infty} c_i \int_G (q(g) q(g)^*, \varphi_i) \mu(dg) \\ &= \int_G (q(g) q(g)^*, \varphi_0) \mu(dg) \\ &= \int_G \| q(g) \|_0^2 \mu(dg) \\ &= \| q \|_2^2 < \infty\end{aligned}$$

using the monotone convergence theorem. Hence $p \in L_1(G, X) \cap \mathscr{T}_0$.

Now $C_0(G, X_0) \subset C_0(G, X)$ so $p \in L_\infty(G, X)$. Also

$$\begin{aligned}&\int_G \int_G \alpha(g) \overline{\alpha(g')} p(g - g') \mu(dg) \mu(dg') \\ &\quad = \int_G \left[\int_G \alpha(g) q(g - g'') \mu(dg) \right] \left[\int_G \alpha(g') q(g' - g'') \mu(dg') \right]^* \mu(dg'') \\ &\quad = \int_G q'(g) q'(g)^* \mu(dg)\end{aligned}$$

using the Fubini and Tonelli theorems with $\alpha \in L_1(G, C)$, where $q' = \alpha * q \in L_2(G, X_0)$ ([5]) so $q'(g) \in X_0$ a.e. or $q'(g) q'(g)^* \in K_0$ a.e. Hence if $\varphi \in \Phi$ then

$$\left(\int_G q'(g) q'(g)^* \mu(dg), \varphi \right) = \int_G (q'(g) q'(g)^*, \varphi) \mu(dg) \geqq 0$$

or $p \in \mathscr{P}$.

Consequently Corollary 3.8 yields $p(g) = \int_{\hat{G}} (g, \gamma) \hat{p}(\gamma) m(d\gamma)$. Then

$$\begin{aligned}\infty > \| q \|_2^2 = \langle q, q \rangle &= \sum_{i=1}^{\infty} c_i (p(0), \varphi_i) \\ &= \sum_i c_i \int_{\hat{G}} (\hat{p}(\gamma), \varphi_i) m(d\gamma) \\ &= \int_{\hat{G}} (\hat{p}(\gamma), \varphi_0) m(d\gamma) = \langle \hat{q}, \hat{q} \rangle .\end{aligned}$$

We have used the monotone convergence theorem again. Hence the

Fourier transform maps into $L_2(\hat{G}, X_0)$. By the usual expansion $\langle q, r\rangle = \langle \hat{q}, \hat{r}\rangle$. This establishes (iii).

Moreover $\int_G q(g)q(g)^*\mu(dg) = p(0) = \int_{\hat{G}} \hat{p}(\gamma)m(d\gamma) = \int_{\hat{G}} \hat{q}(\gamma)\hat{q}(\gamma)^*m(d\gamma)$. Also if x, y are elements of a Banach algebra with involution then

$$\begin{aligned}4xy^* &= (x+y)(x+y)^* - (x-y)(x-y)^* \\ &\quad + i(x+iy)(x+iy)^* - i(x-iy)(x-iy)^* \end{aligned} \tag{4.7}$$

so that (ii) is also proved.

We need only show that $Q=\{\hat{q} \in L_2(\hat{G}, X_0): q \text{ in } L_1(G, X) \cap L_2(G, X_0)\}$ is dense in $L_2(\hat{G}, X_0)$. As μ is translation invariant so is $L_1(G, X) \cap L_2(G, X_0)$ and hence Q is invariant under multiplication by $(g, \cdot)$ for any $g \in G$. If $r \in L_2(\hat{G}, K_0)$ and $\langle q, r\rangle = 0$ for all $q \in Q$, then $\int_{\hat{G}} (q(\gamma)r(\gamma)^*, \varphi_0)(g, \gamma)m(d\gamma) = 0$ for all $q \in Q$ and $g \in G$. As $(q(\cdot)r(\cdot)^*, \varphi_0) \in L_1(\hat{G}, C)$ it follows that $(q(\gamma)r(\gamma)^*, \varphi_0) = 0$ a.e. for every $q \in Q$, or $\langle q(\gamma), r(\gamma)\rangle_0 = 0$ a.e. As $L_1(G, X) \cap L_2(G, X_0)$ is invariant under multiplication by $(\cdot, \gamma)$, $\gamma \in \hat{G}$, then Q is invariant under translation.[7] Hence to every $\gamma_0 \in \hat{G}$ there corresponds $q_0 \in Q$ such that $q_0(\gamma_0) \neq 0$, so $q_0(\gamma) \neq 0$ in a neighborhood of γ_0 as q_0 is continuous. If $\{e_\alpha\}$ is the basis of X_0 mentioned in the statement of part (i), then $q_0(\cdot) = \sum_\alpha q_\alpha(\cdot)e_\alpha$ so there exists α_0 such that $q_{\alpha_0}(\gamma) \neq 0$ in a neighborhood of γ_0. If $q_0(\cdot) = \hat{p}(\cdot)$ then $p = \sum_\alpha p_\alpha e_\alpha$ and as $p \in L_2(G, X_0)$, $p_\alpha \in L_2(G, C)$. By hypothesis $|\langle x, e_\alpha\rangle_0| \leqq k_1 \|x\|$ so $p_\alpha \in L_1(G, C)$ and $\hat{p}_\alpha(\gamma) = q_\alpha(\gamma)$. Hence $p_{\alpha_0}(\cdot)e_\alpha \in L_1(G, X) \cap L_2(G, X_0)$ for any α and $\hat{p}_{\alpha_0}(\cdot)e_\alpha = q_{\alpha_0}(\cdot)e_\alpha \in Q$. Since for each γ in a neighborhood of γ_0, $\{q_{\alpha_0}(\gamma)e_\alpha\}_\alpha$ forms a complete set in X_0, and since $0 = \langle q_{\alpha_0}(\gamma)e_\alpha, r(\gamma)\rangle_0$, then $r(\gamma) = 0$ in a neighborhood of γ_0. But γ_0 was arbitrary so $r = 0$, or Q is orthogonal only to 0 in $L_2(\hat{G}, X_0)$, a Hilbert space. Hence Q is dense in $L_2(\hat{G}, X_0)$. This completes the proof.

COROLLARY 4.8. *Under the assumptions of the theorem the Fourier transform can be extended in a unique manner to an isometry of $L_2(G, X_0)$ onto $L_2(\hat{G}, X_0)$.*

Proof. We need only show $L_1(G, X) \cap L_2(G, X_0)$ is dense in $L_2(G, X_0)$. But $C_c(G, X_0)$[8] is dense in $L_2(G, X_0)$ ([6]). Hence if $f \in L_2(G, X_0)$ then there exists $\{f_n\}_1^\infty \subset C_c(G, X_0) \cap L_2(G, X_0)$ such that $\|f_n - f\|_2 \to 0$. Then $f_n \in C_c(G, X)$ and f_n is measurable so $f_n \in L_1(G, X)$.

REMARK. The equality (4.5) holds for all $q, r \in L_2(G, X_0)$. Moreover, all results are correct assuming only that φ_0 is an arbitrary

---

[7] By this we mean that f_{γ_0} is in Q for any γ_0 in $\hat{G}$ if f is in Q and $f_{\gamma_0}(\gamma) = f(\gamma + \gamma_0)$.

[8] $C_c(G, X_0)$ denotes the set of functions in $C_0(G, X_0)$ having compact support.

linear combination of φ_i's, i.e. $\varphi_0 = \sum_{\alpha \in A} c_\alpha \varphi_\alpha$.

5. **Examples.** Here we give some examples of admissible pairs and strongly admissible triplets.

EXAMPLE 5.1. Let $X = L_1([0, 1], C)$ so X is weakly complete, and let Φ consist of elements φ_i such that

$$(x, \varphi_i) = \int_0^1 \chi_i(t)x(t)dt \qquad x \in X \tag{5.2}$$

where $\chi_i(\cdot)$ is the indicator function of one of a countable collection of sets $\{E_i\}$ dense in $\Sigma([0, 1])$ under the usual Hausdorff metric. Assume $E_1 = [0, 1]$. Then it can be shown ([4], [6]) that Φ is full and that K is the cone of nonnegative (a.e.) functions. Let $(x, \varphi_0) = (x, \varphi_1) = \int_0^1 x(s)ds = \|x\|_1$ for $x \in K$. Hence (Φ, X) is admissible and $K_0 = K$.

If p is in $\mathscr{P}$ then $p(0)$ is in $K = K_0$ by Propositions 2.8 and 2.9 and by Corollary 2.15. So $p \in \mathscr{T}_0$ and the inversion theorem states that if $p \in \operatorname{sp}\{L_1(G, L_1([0, 1], C)) \cap \mathscr{P}\}$ then $\hat{p} \in L_1(\hat{G}, L_1([0, 1], C))$ and $p(g) = \int_{\hat{G}} (g, \gamma)\hat{p}(\gamma)m(d\gamma)$.

The author does not know of any nontrivial subspace X_0 which would make (Φ, X, X_0) strongly admissible.

EXAMPLE 5.3. Let $X = H$, a separable Hilbert space with a fixed orthonormal basis $\{e_i\}_1^\infty$. Let H_0 be the set of elements of H with all but a finite number of components zero, with nonzero components being real, rational nonnegative, and with norm less than or equal to one. Then $\Phi = H_0$ is full ([4], [6]) and countable and $K_\Phi = \{h \in H: h_i \geqq 0\}$.[9] Let $(h, \varphi_i) = \langle h, e_i\rangle$, $i = 1, 2, \cdots$ and $\varphi_0 = \sum_1^\infty \varphi_i$. Then φ_0 maps K into $[0, \infty]$, and for h in K

$$(h, \varphi_0)^2 = (\Sigma h_i)^2 \geqq \Sigma h_i^2 = \|h\|^2$$

so that (Φ, H) is admissible and $K_0 = \{h \in K: \sum_1^\infty h_i < \infty\}$.

H becomes a Banach algebra if we define $hk = \sum_i^\infty h_i k_i e_i$. Let $h^* = \Sigma \bar{h}_i e_i$. For h in H hh^* is in K and $(hh^*, \varphi_0) = \sum_i h_i \bar{h}_i = \|h\|^2$. We do not have $k\|hh^*\| \geqq \|h\|^2$ for some $k > 0$, but we do have $\|h\|_0 = \|h\|$ which is sufficient to show that $X_0 = H$. Hence (Φ, H, H) is "strongly admissible," and the Plancherel theorem applies. Note that the condition $|\langle h, e_i\rangle| \leqq \|h\|$ also holds.

EXAMPLE 5.4. Let $X = \mathscr{L}(H, H)$, the linear bounded operators

[9] $h_i = \langle h, e_i\rangle$

mapping the separable Hilbert space H into itself. Let H_0 be a countable dense subset of the unit ball in H and let $\Phi = \{\varphi \in X^*: (T, \varphi) = \langle Th, h\rangle, T \in \mathscr{L}(H, H), h \in H_0\}$. Let $\{e_i\}$ also be in H_0 for some orthonormal basis $\{e_i\}$. Then Φ is full and countable and K_Φ is the cone of positive operators ([4] or [6]). Let $(T, \varphi_0) = \sum_1^\infty \langle Te_i, e_i\rangle$. So $\varphi_0 = \sum_1^\infty \varphi_i$ is the trace, where $(T, \varphi_i) = \langle Te_i, e_i\rangle$. Then $\varphi_0: K \rightarrow [0, \infty]$, $(T, \varphi_0) = \operatorname{tr} T \geqq \|T\|$ if T is positive. Hence $(\Phi, \mathscr{L}(H, H))$ is admissible and K_0 is the cone of positive operators of finite trace and so a subset of the trace class.

We can see that in one case the condition $p \in \mathscr{T}_0$ is necessary for the inversion theorem to hold. Let G be the circle group so that $\hat{G}$ is countable. Label its elements $\gamma_1, \gamma_2, \cdots$, and let the set function ν be given by

$$\langle \nu(\{\gamma_n\})e_i, e_j\rangle = p_n \delta_{ni} \delta_{nj},^{10} \qquad i, j, n = 1, 2, \cdots \tag{5.5}$$

where $\infty > M \geqq p_n \geqq 0$. ν can be extended to a countably additive measure of finite semi-variation in the obvious way. Let p be given by

$$p(t) = \sum_{n=1}^{\infty} e^{it\gamma_n} \nu(\{\gamma_n\}) . \tag{5.6}$$

Then p is in $\mathscr{P}$ (Theorem 2.12 (A)) and p is in $L_1(G, X)$ because G is compact and $\|p(t)\| \leqq M$. If $\hat{p}$ is to be in $L_1(\hat{G}, X)$ then $\|\nu\|(\hat{G})$ must be finite or $\sum_1^\infty p_n = \operatorname{tr} p(0) < \infty$.

Finally let $X_0 = \mathscr{N}$; the Hilbert-Schmidt operators ([3]). Then for T in $\mathscr{N}$, TT^* is in the trace class and is positive so that TT^* is in K_0. Also $\mathscr{L}(H, H)$ is a C^*-algebra so $(\Phi, \mathscr{L}(H, H), \mathscr{N})$ is strongly admissible. A basis for $\mathscr{N}$ is given by $\{T_{ij}\}$ where $\langle T_{ij}e_k, e_l\rangle = \delta_{ik}\delta_{jl}$, $k, l = 1, 2, \cdots$. Then $|\langle T, T_{ij}\rangle_0| = |\langle Te_i, e_j\rangle| \leqq \|T\|$, and the condition in (i) of Theorem 4.4 also holds.

6. Fourier transforms on representations. In this section we apply the preceding theory to extend the inversion theorem and Plancherel's theorem to "Fourier transforms" defined for unitary representations in a separable Hilbert space. The case where H is finite dimensional has been treated by Hewitt and Wigner [7]. Let H be a separable complex Hilbert space, and let $U(\cdot)$ be a continuous unitary representation of G in $\mathscr{L}(H, H)$, i.e. $U(g + g') = U(g)U(g')$, $U(0) = I$, and V is a continuous mapping of G into the unitary operators on H. It follows [9] that there exists a sequence $\{\gamma_i\}$ of characters, and a resolution $\{\pi_i\}$ of the identity in $\mathscr{L}(H, H)$, such that

[10] δ_{ni} is the Kronecker delta.

(6.1) $$U(g) = \sum_i (g, \gamma_i)\pi_i \, .$$

(The summation is at most countable). If p is in $L_1(G, \mathscr{L}(H, H))$ define the transform

(6.2) $$\hat{p}(U) = \int_G p(g)U(-g)\mu(dg) \, .$$

We shall first consider the question of invertibility of this transform. As we shall see, it suffices to know $\hat{p}(U)$ for all U corresponding to a fixed resolution $\{\pi_i\}$.

From now on consider $\{\pi_i\}$ fixed, and let us denote the set of subscripts by S. Then S is at most countable, $\sum_{i \in S} \pi_i = I$. Define $\mathscr{R} \equiv \prod_{i \in S} \hat{G}_i$, where $\hat{G}_i \equiv \hat{G}$ for all i, with the product topology. Then $\mathscr{R}$ can be considered as the set of all representations corresponding to $\{\pi_i\}$, if we put

(6.5) $$r \longleftrightarrow \sum_{i \in S} (\cdot, \gamma_i)\pi_i = U(\cdot)$$

whenever $r = \{\gamma_i\} \in \mathscr{R}$.

Let us now introduce a measure on $\mathscr{R}$. Choose a symmetric neighborhood A of 0 in $\hat{G}$ such that the closure of A is compact. Hence $0 < m(A) < \infty$. Assume m is normalized (relative to μ) such that the inversion theorem holds. Now normalize μ such that $m(A) = 1$. Note that if G is discrete and $A = \hat{G}$, or if G is compact and $A = \{0\}$, then the usual normalizations of μ and m occur. For α in $\hat{G}$ and E in $\Sigma(\hat{G})$ define

$$m_\alpha(E) = m[E \cap (A + \alpha)] \, .$$

Then $m_\alpha(\cdot)$ is a probability measure on $\hat{G}$, and by the Kolmogorov extension theorem, there exists a unique probability measure

$$m_\alpha^\infty = m_\alpha \times m_\alpha \times \cdots$$

on $\mathscr{R}$. We set $\mathscr{R}^i = \prod_{j \in S - \{i\}} \hat{G}_j$. For E in $\Sigma(\mathscr{R}^i)$ write

$$m_\alpha^i(E) = \int_{\mathscr{R}} \chi_{E \times \hat{G}}(r) m_\alpha^\infty(dr)$$

where it is understood we are integrating out γ_i.

Now assume G is σ-finite and Φ is a full, countable subset of $\mathscr{L}(H, H)^*$. With the previous notation we have

THEOREM 6.4. *If p is in span $\{L_1(G, \mathscr{L}(H, H)) \cap \mathscr{P}_d\}$, then*

(6.5) $$p(g) = \int_{\hat{G}} \int_{\mathscr{R}} \hat{p}(U)U(g) m_\alpha^\infty(dr) m(d\alpha) \, .$$

Proof. π_i is a projection on the subspace H_i of H. Moreover if we consider the equivalent spectral representation ([9], p. 247), then the subspaces are mutually orthogonal. Let us write $f(\alpha) = \hat{p}(\alpha)(g, \alpha)$, and $f^j(r) \equiv f(\gamma_j)$ when $r = \{\gamma_i\}$. Then for n finite, $\beta \in \hat{G}$ and h in H,

$$\begin{aligned}
&\left\|\sum_{i=1}^{n}\int_{\mathscr{R}} f^i(r)m_\beta^\infty(dr)\pi_i h\right\| \\
&\quad = \left\|\sum_{i=1}^{n}\int_{\hat{G}}\int_{\mathscr{R}^i} f(\alpha)m_\beta^i(dr)m_\beta(d\alpha)\pi_i h\right\| \\
&\quad = \left\|\sum_{i=1}^{n}\int_{\hat{G}} f(\alpha)m_\beta(d\alpha)\pi_i h\right\| \\
&\quad \leqq \left\|\int_{\hat{G}} f(\alpha)m_\beta(d\alpha)\right\| \, \|h\|
\end{aligned}$$

so that

$$\begin{aligned}
\left\|\sum_{i=1}^{n}\int_{\mathscr{R}} f_i(r)m_\beta^\infty(dr)\pi_i\right\| &\leqq \left\|\int_{\hat{G}} f(\alpha)m_\beta(d\alpha)\right\| \\
&\leqq \int_{\hat{G}} \|\hat{p}(\alpha)\|\, \chi_{A+\beta}(\alpha)m(d\alpha) \, .
\end{aligned}$$

As $\hat{p}$ is in L_1, and as $m(A) = 1$, then

$$\begin{aligned}
&\int_{\hat{G}}\int_{\hat{G}} \|\hat{p}(\alpha)\|\, \chi_{A+\beta}(\alpha)m(d\alpha)m(d\beta) \\
&\quad = \|\hat{p}\|_1 \, .
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i\in S}\int_{\hat{G}}\int_{\mathscr{R}} f^i(r)m_\beta^\infty(dr)m(d\beta)\pi_i \\
&\quad = \int_{\hat{G}}\sum_{i\in S}\int_{\mathscr{R}} f^i(r)m_\beta^\infty(dr)\pi_i m(d\beta) \, .
\end{aligned} \tag{6.6}$$

Moreover

$$\begin{aligned}
&\int_{\mathscr{R}} f^i(r)m_\beta^\infty(dr)\pi_i \\
&\quad = \int_{\mathscr{R}} \hat{p}(\gamma_i)(g, \gamma_i)m_\beta^\infty(dr)\pi_i \\
&\quad = \int_{\mathscr{R}}\int_G p(g')(g - g', \gamma_i)\mu(dg')m_\beta^\infty(dr)\pi_i \, ,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} p(g')(g - g', \gamma_i)\pi_i h\right\| \\
&\quad \leqq \|p(g')\| \left\|\sum_{i=1}^{n} (g - g', \gamma_i)\pi_i h\right\| \\
&\quad \leqq \|p(g')\| \, \|h\|
\end{aligned}$$

as $|(g, \gamma)| = 1$ and the π_i's are orthogonal projections. As p is in L_1, and as $m_\beta^\infty(\mathscr{R}) = 1$, then

$$\sum_{i \in S} \int_{\mathscr{R}} f_i(r) m_\beta^\infty(dr)\pi_i \tag{6.7}$$
$$= \int_{\mathscr{R}} \int_G \sum_{i \in S} p(g')(g - g', \gamma_i)\pi_i \mu(dg') m_\beta^\infty(dr) .$$

On the other hand

$$\hat{p}(U)U(g) = \int_G p(g')U(-g')\mu(dg')U(g) \tag{6.8}$$
$$= \int_G p(g')U(g - g')\mu(dg')$$
$$= \int_G p(g') \sum_{i \in S} (g - g', \gamma_i)\pi_i \mu(dg') .$$

Hence we have shown that for each β, g, $\hat{p}(U)U(g)$ is integrable $m_\beta^\infty(dr)$, and $\int_{\mathscr{R}} \hat{p}(U)U(g)m_\beta^\infty(dr)$ is integrable $m(d\beta)$, so that 6.5 makes sense.

Finally

$$\int_{\hat{G}} \int_{\mathscr{R}} \hat{p}(U)U(g)m_\beta^\infty(dr)m(d\beta)$$
$$= \sum_{i \in S} \int_{\hat{G}} \int_{\mathscr{R}} f^i(r) m_\beta^\infty(dr)m(d\beta)\pi_i$$
$$= \sum_{i \in S} \int_{\hat{G}} \int_{\hat{G}} \int_{\mathscr{A}^i} \hat{p}(\alpha)(g, \alpha) m_\beta^i(dr) m_\beta(d\alpha) m(d\beta)\pi_i$$
$$= \sum_{i \in S} \int_{\hat{G}} \int_{\hat{G}} \hat{p}(\alpha)(g, \alpha) m_\beta(d\alpha) m(d\beta)\pi_i$$
$$= \sum_{i \in S} \int_{\hat{G}} \int_{\hat{G}} \chi_A(\alpha - \beta)\hat{p}(\alpha)(g, \alpha) m(d\beta) m(d\alpha)\pi_i$$
$$= \sum_{i \in S} \int_{\hat{G}} \hat{p}(\alpha)(g, \alpha) m(d\alpha)\pi_i$$
$$= \sum_{i \in S} p(g)\pi_i$$
$$= p(g) .$$

We have made use of 6.6, 6.7, 6.8, and the inversion theorem. The theorem is established.

Now consider the setting of Example 5.4.

THEOREM 6.9. *If p and q are in $L_1[G_1, \mathscr{L}[H, H)] \cap L_2(G, \mathscr{N})$, then*

$$\int_G p(g)q(g)^* \mu(dg) = \int_{\hat{G}} \int_{\mathscr{R}} \hat{p}(U)\hat{q}(U)^* m_\alpha^\infty(dr) m(d\alpha)$$

Proof. The proof is similar to the previous one except that Theorem 4.4 is used.

Further applications of this theory can be found in [6].

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