CUT LOCI OF POINTS AT INFINITY

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In a $G$-space $R$ if $B$ is a co-ray to $A$ then the union of all co-rays to $A$ that contain $B$ is either a straight line or a co-ray to $A$ maximal in that it is properly contained in no other co-ray to $A$. In the latter case, the initial point of the maximal co-ray is a copoint to $A$. The concept of co-point is an analogue to that of minimum point in a sense made precise. On certain non-compact $G$-surfaces of finite connectivity, including those with non-positive curvature, we characterize the locus of co-points to a given ray and obtain bounds for the number of components of this locus, the number of co-rays emanating from a co-point and the number of co-points that are origins of more than two co-rays.

1. Introduction. A $G$-space can be described as a metric space any two of whose points can be joined by a segment, and in which any segment may be prolonged uniquely to a geodesic. The theory of $G$-spaces is found in Busemann [1] hereafter quoted as $GG$.

In a $G$-space, as in a Riemannian space, a minimum point $m$ of a point $p$ may be defined as a point for which no segment $T(p, m)$ can be prolonged beyond $m$. We shall be concerned with the analog to $m$ when $p$ lies at infinity in the sense made precise below.

A co-ray $B$ from a point $p$ to a ray $A$ is the limit of a converging sequence of segments $T(p_n, z_n)$ where $p_n \to p$ and $z_n$ tends to infinity on $A$. Obviously $B$ is also a co-ray to any ray contained in or containing $A$ as a sub-ray. Furthermore, the limit of a converging sequence of co-rays to a ray $A$ is likewise a co-ray to $A$. Less trivial is the fact (see $GG$, p. 136) that the co-ray to $A$ from any point of $B$ other than $p$ is unique and a sub-ray of $B$.

Given a ray $A$ in a straight space and a co-ray $B$ to $A$, the union of all co-rays to $A$ containing $B$ is an asymptote to the oriented straight line containing $A$ as a positive sub-ray. In an arbitrary non-compact $G$-space such a union is either an oriented straight line, any positive sub-ray of which is a co-ray to $A$, or it is a co-ray to $A$ that is not a proper sub-ray of any co-ray to $A$. This leads to the following terminology in non-compact $G$-spaces.

DEFINITION 1. Given a ray $A$ in a $G$-space

(a) An asymptote to $A$ is an oriented straight line any positive sub-ray of which is a co-ray to $A$.

(b) A maximal co-ray to $A$ is a co-ray to $A$ that is not a proper sub-ray of any co-ray to $A$. 
A co-point to $A$ is the origin of a maximal co-ray to $A$. We denote by $C(A)$ the set of co-points to $A$.

A point $p_\infty$ at infinity in a $G$-space is a maximal set of rays such that a co-ray to one ray in $p_\infty$ is a co-ray to each ray in $p_\infty$. For $A, B \in p_\infty$, $C(A) = C(B)$; hence the locus of co-points depends only on $p_\infty$. Nevertheless it is convenient to retain the notation $C(A)$. The concept of co-point is thus a natural analog to the concept of minimum point in the finite case.

The study of $C(A)$ was initiated by Nasu [4, 5, 6], who by “asymptote” means “maximal co-ray or asymptote” and uses “asymptotic conjugate point” instead of “co-point”.

It is our purpose to extend and clarify much of Nasu’s work. In § 6 we characterize $C(A)$ on certain $G$-surfaces of finite connectivity (including those with nonpositive curvature) obtaining bounds for the number of components of $C(A)$, the number of co-rays emanating from a point and the number of points that are origins of more than two co-rays to $A$.

2. Preliminaries. This section consists of results used in later proofs. We begin with a proposition of Nasu [4].

**Proposition 2.** Given a ray $A$ in a $G$-space and a point $p \in C(A)$, there exists for each $\delta > 0$ a positive $\varepsilon \leq \delta$ such that each co-ray to $A$ with origin exterior to $S(p, \delta)$ fails to intersect $S(p, \varepsilon)$. In particular, there is an $\varepsilon' > 0$ such that no asymptote to $A$ intersects $S(p, \varepsilon')$.

**Proof.** Otherwise there is a $\delta > 0$ and a sequence $p_n \to p$ such that each $p_n$ lies on a co-ray $B_n$ to $A$ whose origin $q_n \in S(p, \delta)$.

There is an index $N$ such that $pp_n < \delta/2$ for $n \geq N$. Thus for $n \geq N$, $q_n p_n \geq q_n p - pp_n > \delta/2$. Choose $q_n' \in B_n$ such that $(q_n q'_n p_n)$ and $q_n' p_n = \delta/4$. The co-ray $B'_n$ from $q'_n$ to $A$ is unique and a sub-ray of $B_n$. A sub-sequence of $B'_n$ converges to a co-ray $B$ to $A$ containing $p$ in its interior—a contradiction.

What follows is a modification of a theorem of Busemann [2, p. 18].

**Proposition 3.** For each $x \in S(p, \rho_\delta)$ let a ray $A(x)$ with origin $x$ be defined which depends continuously on $x$. If the spheres $K(p, \rho)$, $0 < \rho < \rho(p)$, are not contractible then, for some $x \in S(p, \rho_\delta)$, $x \neq p$ and $p \in A(x)$.

**Proof.** Let $0 < \delta < \min (\rho_\delta, \rho(p))$. $A(p)$ intersects $K = K(p, \delta)$ in exactly one point $w$ the antipode of which on $K$ we denote by $w'$. Let $u \in A(p)$ with $up = \delta/2$. The projection $P$ of $S(u, \delta/4)$ on $K$
by segments from $p$ is a proper sub-set of $K$ and in particular does not include $w'$.

For $0 < \varepsilon < \delta$, let $K_\varepsilon = (p, \varepsilon)$. Let $V = \{A(v) : v \in K_\varepsilon\}$ and let $z(t, v)$ represent $A(v)$ with $z(0, v) = v$. Choose $\varepsilon > 0$ so small that for $v \in V$, $z(\delta/2, v) \in S(u, \delta/4)$ and $T(z(\delta/2, v), v)$ lies in $S(p, \delta)$.

If $z(t\delta/2, v) \neq p$ for $0 \leq t \leq 1$, define $v_t$ by $(pz(t\delta/2, v)v_t)$ and $v_t p = \delta$.

For $t = 0$ we have $(pw_0v_0)$ and if $p \in V$ then $v_t$ traverses $K_\varepsilon$. For $t = 1$ we have $(pz(\delta/2, v)v_1)$ and $z(\delta/2, v) \in S(u, \delta/4)$. Hence $v_t \in P$.

The point $z(t\delta/2, v)$ depends continuously on both $t$ and $v$. Thus if $p \in V$ then $v_t$ defines a deformation of $K$ onto a proper sub-set of itself. This in turn can be deformed to a point thus contradicting the non-contractibility of $K$.

It follows that $p \in V$ yet $p \notin K_\varepsilon$ which proves the assertion.

Although it is not presently known whether the non-contractibility of small spheres holds in general, it is shown in Busemann [2] to hold in finite dimensional $G$-spaces.

The set $C(A)$ is not necessarily closed (see Nasu [4]). In the event that $C(A)$ is closed we have the following:

**Proposition 4.** Let $A$ be a ray in a $G$-space $R$ such that $C(A)$ is closed. Let $x_n \rightarrow x_0$ where $x_n$ and $x_0$ lie on maximal co-rays to $A$. If $x'_n$ and $x'_0$ are the co-points to $A$ determined by $x_n$ and $x_0$ respectively, then $x'_n \rightarrow x'_0$.

**Proof.** We show first that the sequence $x'_n$ is bounded. Otherwise there is a sub-sequence $x'_m$ such that $x_m x'_m \rightarrow \infty$. Then there are, for sufficiently large $m$, co-rays $B_m$ to $A$ containing $x_m$ whose initial points $q_m$ satisfy $x_0 q'_0 + 2 > x_m q_m > x_0 x'_0 + 1$.

Since $q_m$ is bounded, a sub-sequence $B_i$ of $B_m$ converges to a co-ray $B_0$ to $A$ containing $x_0$ with initial point $q_0$ satisfying $x_0 q_0 \geq x_0 x'_0 + 1$. This, however, is impossible since $x'_0 \in C(A)$. Therefore $x'_n$ is bounded.

If $x'_n$ does not converge to $x'_0$ then there is a sub-sequence $x'_j$ of $x'_n$ and a $\delta > 0$ such that each $x'_j \notin S(x'_0, \delta)$. Let $H_i$ be a maximal co-ray to $A$ containing $x_j$. Since $x'_n$ is bounded, a sub-sequence $H_k$ of $H_i$ converges to a co-ray $H$ to $A$ containing $x_0$. Hence the corresponding sequence $x'_k$ of co-points converges to the initial point of $H$ which, since $C(A)$ is closed, must be $x'_0$—a contradiction.

We conclude this section with the following separation property.

**Proposition 5.** Let $A$ be a ray. The complement of $C(A)$ has no bounded component, and no compact, sub-set of $C(A)$ separates the space.
Proof. Let \( p \in C(A) \) and let \( B \) be the co-ray from \( p \) to \( A \). Then \( B \cap C(A) = \emptyset \) and the component determined by \( p \) contains \( B \).

Suppose a compact sub-set \( K \) of \( C(A) \) separates the space \( R \). Then all points of \( A - \{ p \} \) lie in the same component of \( R - K \). Let \( p \) lie in a different component. Consider a sequence \( x_n \) on \( A \) with \( px_n \to \infty \) such that a sequence of segments \( T(p, x_n) \) converges to a co-ray \( B \) from \( p \) to \( A \). Each \( T(p, x_n) \) intersects \( K \) in a point \( y_n \), and, since \( K \) is compact, \( K \) contains an accumulation point \( y_0 \) of \( y_n \). It follows that \( y_0 \in K \cap B \) which is impossible.

3. The universal covering surface. While the preceding section concerned arbitrary \( G \)-spaces the remainder of this article is concerned with \( G \)-surfaces. In this section we generalize results of Nasu \([5, 6]\) proved under the stronger hypothesis of nonpositive curvature.

A tube in a \( G \)-surface \( R \) is a closed domain bounded by a geodesic polygon \( P \) and homeomorphic to a disk punctured at one point. A ray \( A \) in \( R \) is said to ultimately lie in a tube \( T \) if \( A \) or some sub-ray of \( A \) lies in \( T \).

**Theorem 6.** Let \( R \) be a \( G \)-space surface and \( A \) a ray in \( R \). If the universal covering surface \( \tilde{R} \) is straight and if \( A \) ultimately lies in a tube \( T \) then the number of co-rays to \( A \) from any point \( p \) is finite.

**Proof.** Assume without loss of generality that the initial point \( q \) of \( A \) is on \( P \), the polygon bounding \( T \), and is the only point in which \( A \) intersects \( P \). Assume further that \( p \) is exterior to \( T \). Let \( \lambda = \text{length of } P \), \( \gamma = \max \{ px : x \in P \} \) and \( 0 < \varepsilon < \rho P \). Consider the class of oriented geodesic polygons of the form \( T(q, p) \cup T(p, x) \cup T(x, z) \cup T(z, q) \) where \( T(q, p) \) is fixed, \( z \in A - \{ q \} \) and \( x \in S(p, \varepsilon) \). We show that the class of such polygons determines only a finite number of homotopy classes in \( R \).

Given such a polygon, there is a last point \( y \) in which \( T(x, z) \) intersects \( P \). Because \( T \) is homeomorphic to a punctured disk, there is a sub-arc \( P'(q, y) \) of \( P \) from \( q \) to \( y \) such that \( p'(q, y) \cup T(y, z) \cup T(z, q) \) is null homotopic. It follows that \( T(q, p) \cup T(p, x) \cup T(x, z) \cup T(z, q) \) is homotopic to \( T(q, p) \cup T(p, x) \cup T(x, y) \cup P'(y, q) \).

Fix \( \bar{p} \in \tilde{R} \) over \( p \), and hence fix \( T(\bar{q}, \bar{p}) \) over \( T(q, p) \). Let \( T(\bar{p}, \bar{x}) \) be the unique segment from \( \bar{p} \) over \( T(p, x) \), \( T(\bar{x}, \bar{y}) \) the unique segment from \( \bar{x} \) over \( T(x, y) \) and \( \bar{P}'(\bar{y}, \bar{q}) \) the unique geodesic polygon from \( \bar{y} \) over \( \tilde{P}'(y, q) \). The end-point \( \bar{q}^* \) of \( \bar{P}'(\bar{y}, \bar{q}) \) then lies over \( q \) and \( \frac{pq}{p}\bar{q} \leq px + xy + \text{length } \bar{P}'(\bar{y}, \bar{q}) = px + xy + \text{length } P'(y, q) \leq px + px + py + \lambda < 2\varepsilon + \gamma + \lambda \).
The point $\bar{q}$* so constructed are in one-to-one correspondence with the number of homotopy classes determined by the above class of geodesic polygons and are finite in number since they are all interior to $S(\bar{p}, 2\varepsilon + \gamma + \lambda)$.

Let $x_n \to p$ and let $z_n \in A - \{q\}$ be a sequence with $qz_n \to \infty$. Assume without loss of generality that $x_n p < \varepsilon$. Let $\Gamma_1, \cdots, \Gamma_k$ be the homotopy classes determined by the geodesic polygons $T(q, p) \cup T(p, x_n) \cup T(x_n, z_n) \cup T(z_n, q)$ where $T(q, p)$ is fixed, and let $\bar{q}_i, \cdots, \bar{q}_k \in R$ over $q$ be constructed as above. The end-points $\bar{z}_n$ of the unique geodesic polygons from $\bar{p}$ over $T(p, x_n) \cup T(x_n, z_n)$ then lie on one of the rays $\bar{A}_i, \cdots, \bar{A}_k$ over $A$ originating from $\bar{q}_i, \cdots, \bar{q}_k$.

If $T(x_n, z_n)$ converges to a co-ray $B$ to $A$ then $B$ is the image of a co-ray $\bar{B}$ from $\bar{p}$ to one of the rays $\bar{A}_i, \cdots, \bar{A}_k$. Since $\bar{R}$ is straight, the co-ray from $\bar{p}$ to any given ray is unique and the theorem follows.

We saw in the preceding proof that given $\bar{p} \in \bar{R}$ over $p$, the co-rays from $p$ to $A$ are the images of the co-rays from $\bar{p}$ to certain rays $\bar{A}_i, \cdots, \bar{A}_k$ over $A$. The following tells us that the choice of $\bar{A}_i$ is, to an extent, uniform.

**Theorem 7.** (Nasu [5]). Under the hypothesis of (6), if the asymptote relation in $\bar{R}$ is transitive and the co-rays from $p$ to $A$ are images of co-rays from $\bar{p} \in \bar{R}$ to rays $\bar{A}_i, \cdots, \bar{A}_m$ over $A$ then there is a positive $\beta_p < \rho(p)/2$ such that each co-ray to $A$ from $x \in S(p, \beta_p)$ is the image of a co-ray from $\bar{x} \in S(\bar{p}, \beta_p)$ over $x$ to one of the rays $\bar{A}_i, \cdots, \bar{A}_m$.

**Proof.** Assume otherwise. There is then a sequence $p_n \to p$ with $\rho p_n < \min(\rho(p)/2, pP/2)$ such that each $p_n$ is the origin of a co-ray $B_n$ to $A$ which is not the image of a co-ray from $\bar{p}_n \in S(\bar{p}, \rho(p)/2)$ over $p_n$ to any of the rays $\bar{A}_i, \cdots, \bar{A}_m$.

Assume without loss of generality that the co-rays $B_n$ converge to a co-ray $B$ from $p$ to $A$. Let $\gamma_n = \max\{p_n x: x \in P\}$. Each $B_n$ is the image of a co-ray from $\bar{p}_n$ to a ray $\bar{A}'_n$ over $A$ with initial point $\bar{q}'_n$ satisfying $\bar{p}_n \bar{q}'_n \leq \gamma_n + \lambda$ (since $\varepsilon$ in the proof of (6) can be made arbitrarily small). Also $\gamma_n \leq \gamma + pp_n$ hence $\bar{p}\bar{q}'_n \leq \bar{p}\bar{q}_n + \bar{p}\bar{q}_n \leq \bar{p}\bar{p}_n + \gamma + pp_n + \lambda = \gamma + \lambda + 2pp_n$. It follows that there are only a finite number of distinct points $\bar{q}_n$. We can therefore assume, by selecting an appropriate sub-sequence, that each $B_n$ is the image of the co-ray from $\bar{p}_n$ to $\bar{A} \neq \bar{A}_i, \cdots, \bar{A}_m$ over $A$.

$B$ is then the image of the co-ray $\bar{B}$ from $\bar{p}$ to $\bar{A}$. $\bar{B}$ is also a co-ray to one of the rays $\bar{A}_i$, say $\bar{A}_i$. It follows from the transitivity (and implied symmetry) of the asymptote relation that $\bar{A}$ and $\bar{A}_i$ are co-rays to each other. Then $\bar{B}_n$ is a co-ray to $\bar{A}_i$, a contradiction.
We note that an example due to Busemann (GG, pp. 265-66) shows the hypothesis that $A$ ultimately lie in a tube to be essential.

**Corollary 8.** Under the hypothesis of (7), if $p \in C(A)$ then $p$ is the origin of at least two co-rays to $A$. Furthermore, $C(A)$ is closed.

**Proof.** Assume that the co-ray $B$ from $p \in C(A)$ to $A$ is unique. It follows from (7) that the co-ray from each $x \in S(p, \beta_p)$ is unique. By (3) there is an $x \in S(p, \beta_p)$ such that $x \neq p$ and $p$ lies on the co-ray from $x$ to $A$—a contradiction.

On the other hand if $p \notin C(A)$ then the co-ray from $p$ to $A$ is unique and is thus unique for each $x \in S(p, \beta_p)$. Hence $S(p, \beta_p) \cap C(A) = \emptyset$ and the complement of $C(A)$ is open.

4. The local structure of $C(A)$. In this section we describe the local topological structure of $C(A)$. As in the previous section our results generalize results of Nasu [5, 6].

**Lemma 9.** Under the hypothesis of (7), if $p \in C(A)$ then there is $d > 0$ such that no point of $S(p, d)$, with the possible exception of $p$, is the origin of more than two co-rays to $A$.

**Proof.** Choose $y_p > 0$ such that $\beta_p < \gamma_p$, no asymptote to $A$ intersects $\tilde{S}(p, \gamma_p)$ and $\tilde{S}(p, \gamma_p)$ is homeomorphic to the closed unit disk in $E^2$. Denote by $B_i, 1 \leq i \leq m$, the maximal co-rays to $A$ from $p$ and by $x_i$ the intersection of $B_i$ with $K(p, \gamma_p) = \{x \mid px = \gamma_p\}$. Let the indexing be such that $x_{i+1}$ follows $x_i$ where $x_{m+1} = x_1$. The points $x_i$ partition $K(p, \gamma_p)$ into sub-arcs $K_i, 1 \leq i \leq m$, where $K_i$ has end-points $x_i$ and $x_{i+1}$. These arcs with the co-rays $B_i$ partition $\tilde{S}(p, \gamma_p)$ into closed simply connected regions $D_i, \ldots, D_m$ with non-empty mutually disjoint interiors such that each $D_i$ is bounded by $B_i \cap \tilde{S}(p, \gamma_p), K_i$ and $B_{i+1} \cap \tilde{S}(p, \gamma_p)$.

Choose $\bar{p} \in \bar{R}$ over $p$. Since $\gamma_p < \beta_p \leq \rho(p)/2$, the covering map sends $\tilde{S}(\bar{p}, \gamma_p)$ isometrically onto $\tilde{S}(p, \gamma_p)$. Let $\bar{B}_i$ with initial point $\bar{p}$ lie over $B_i$. $\bar{B}_i$ is then a co-ray to a ray $\bar{A}_i$ over $A$. $\tilde{S}(\bar{p}, \gamma_p)$ is partitioned into closed simply connected regions $\bar{D}_i$ over $\bar{D}_i$ where $\bar{D}_i$ is bounded by $\bar{B}_i \cap \tilde{S}(\bar{p}, \gamma_p), \bar{K}_i$ over $K_i$ and $\bar{B}_{i+1} \cap \tilde{S}(\bar{p}, \gamma_p)$.

For each $x \in S(p, \gamma_p)$ a co-ray from $x$ to $A$ is the image of a co-ray from $\bar{x} \in \tilde{S}(\bar{p}, \gamma_p)$ over $x$ to one of the rays $\bar{A}_i, 1 \leq i \leq m$. Since $\bar{R}$ has a transitive and hence symmetric asymptote relation, we can say that a co-ray from $x$ to $A$ is the image of the co-ray from $\bar{x}$ to one of the rays $\bar{B}_i, 1 \leq i \leq m$.

Consider $x \in D_i, x \neq p$. We assert that if $\gamma_p$ is sufficiently small then any co-ray from $x$ to $A$ is the image of the co-ray from $\bar{x} \in \bar{D}_i$.
over \( x \) to one of the rays \( \overline{B}_i \) or \( \overline{B}_{i+1} \). Assume otherwise and fix \( \gamma_p < \beta_p \). There is then a sequence of points \( x_n \rightarrow p \) in the interior of \( D_i \) such that each \( x_n \) is the origin of a co-ray \( H_n \) to \( A \) where \( H_n \) is the image of \( \overline{H}_n \), the co-ray from \( \overline{x}_n \in D \) over \( x_n \) to some \( \overline{B}_j, j \neq i, i + 1 \) (we can assume without loss of generality that each \( H_n \) is a co-ray to the same \( \overline{B}_j \)). A sub-sequence of the \( H_n \) then converges to \( \overline{B}_j \) which is impossible since \( \overline{B}_i \cup \overline{B}_{i+1} \) separates \( \overline{B}_j \) from each \( \overline{H}_n \). The assertion thus follows and hence the lemma.

Continuing in this manner, we prove the following result.

**Theorem 10.** Let \( R \) be a G-surface and \( A \) a ray in \( R \). If \( \overline{R} \) is straight and has a transitive asymptote relation, and if \( A \) ultimately lies in a tube, then for each \( p \in C(A) \) there is a closed region \( V \) containing \( p \) in its interior that is homeomorphic to a closed disk \( D \) in such a way that \( p \) corresponds to the center of \( D \) and \( C(A) \cap V \) to the union of a number of radii of \( D \) equal to the number of co-rays from \( p \) to \( A \).

**Proof.** We begin where the proof of (9) ends. Each \( x \in K_i \) determines a unique co-point \( \phi(x) \) to \( A \). It follows from (4) that the map \( \phi: K_i \rightarrow \phi(K_i) \) is continuous.

On \( K_i \) choose \( y_i \) so close to \( x_i \) that no point of the sub-arc \( K(x_i, y_i) \) of \( K_i \) joining \( x_i \) and \( y_i \) is a co-point to \( A \) and so that \( L_i = \phi[K(x_i, y_i)] \) is, with the exception of \( p = \phi(x_i) \), interior to \( D_i \). This is possible since \( \phi \) is continuous and \( C(A) \) is closed.

Let \( \overline{y}_i \in D_i \) lie over \( y_i \) and \( \overline{K}(\overline{x}_i, \overline{y}_i) \) over \( K(x_i, y_i) \) be the subarc of \( \overline{K}_i \) joining \( \overline{x}_i \) and \( \overline{y}_i \). By (7) if \( y_i \) is chosen sufficiently close to \( x_i \) then the co-ray from each \( \overline{x} \in \overline{K}(\overline{x}_i, \overline{y}_i) \) to \( \overline{B}_i \) lies over a co-ray to \( A \). Let \( \overline{H}_i \) be the co-ray from \( \phi(\overline{y}_i) \) over \( \phi(y_i) \) to \( \overline{B}_i \). Then \( \overline{H}_i \) lies over a co-ray to \( A \) from \( \phi(y_i) \).

\( \phi(y_i) \) is the origin of exactly two maximal co-rays to \( A \). Let \( U_i \) denote the remaining maximal co-ray to \( A \). Since \( \overline{H}_i \) is the co-ray from \( \phi(y_i) \) to \( \overline{B}_i \), the ray \( \overline{U}_i \) over \( U_i \) from \( \phi(y_i) \) is a co-ray to \( \overline{B}_{i+1} \).

Denote by \( z_i \) the intersection of \( U_i \) with \( K_i \). The choice of \( y_i \) guarantees that \( z_i \in K(x_i, y_i) \). Let \( K(z_i, x_{i+1}) \) be the sub-arc of \( K_i \) joining \( z_i \) and \( x_{i+1} \). It follows that \( K(x_i, y_i) \) and \( K(z_i, x_{i+1}) \) have no points in common. Let \( x \) be an interior point of \( K(x_i, y_i) \). \( \phi(x) \) is the origin of exactly two maximal co-rays to \( A \). If \( \phi(x) \in \overline{D}_i \) lies over \( \phi(x) \) then the co-ray \( \overline{H}_x \) from \( \phi(x) \) to \( \overline{B}_i \) and the co-ray \( \overline{U}_x \) from \( \phi(x) \) to \( \overline{B}_{i+1} \) lie over the maximal co-rays to \( A \) from \( \phi(x) \). \( y_i \) was chosen so that \( \overline{U}_x \) cannot intersect \( \overline{K}(\overline{x}_i, \overline{y}_i) \). Neither can \( \overline{U}_x \) intersect \( \overline{B}_i, \overline{B}_{i+1}, \overline{H}_i \) or \( \overline{U}_i \). \( \overline{U}_x \) must then intersect \( \overline{K}(\overline{z}, \overline{x}_{i+1}) \) over \( K(z_i, x_{i+1}) \) and \( U_x \) intersects \( K(z_i, x_{i+1}) \). It follows that \( \phi \) restricted to \( K(x_i, y_i) \) is one-to-one and \( L_i = \phi[K(x_i, y_i)] \) is an arc joining \( p \) and \( \phi(y_i) \).
We know that each \( x \in L_i \) is the origin of exactly two maximal co-rays to \( A \). One of these, \( H_x \), intersects \( K(z_i, x_{i+1}) \). With \( x \in L_i \) associate \( \sigma(x) = U_x \cap K(z_i, x_{i+1}) \).

The continuity of the map \( \sigma: L_i \rightarrow K(z_i, x_{i+1}) \) can be shown by a standard argument. \( \sigma(L_i) \) is then a connected sub-set of \( K(z_i, x_{i+1}) \) that contains both \( z_i \) and \( x_{i+1} \). Thus \( \sigma(L_i) = K(z_i, x_{i+1}) \) and \( L_i = \phi[K(z_i, x_{i+1})] \).

Consider the closed region \( V_i \) bounded by \( B_i \cap \bar{S}(p, \gamma_p), K(x_i, y_i), H_i \cap \bar{S}(p, \gamma_p), U_i \cap \bar{S}(p, \gamma_p), K(z_i, x_{i+1}) \) and \( B_{i+1} \cap \bar{S}(p, \gamma_p) \). \( V_i \cap C(A) = L_i \) and \( V = V_1 \cup \cdots \cup V_n \) is then the desired closed region.

We note that since \( \gamma_p > 0 \) can be arbitrarily small we can find such a \( V \) contained in any neighborhood of \( p \). This implies that \( C(A) \) is locally arc-wise connected and that the arc-wise connected components of \( C(A) \) are closed in \( C(A) \) and hence are closed in \( R \).

We conclude this section with some remarks on the applicability of the preceding results.

A \( G \)-surface \( R \) of finite connectivity can be regarded as a subspace of a compact manifold \( M \) of finite genus \( \gamma \). As such, it is obtained from \( M \) by excluding a finite number of points \( a_i \). There are simple closed pairwise disjoint geodesic polygons \( P_i \) in \( R \) each of which bounds a closed region \( M_i \) in \( M \) homeomorphic to a disk and containing \( a_i \), but no other \( a_j \), in its interior. The set \( T_i = M_i - \{a_i\} \) is then a tube in \( R \).

Each ray in \( R \) must ultimately lie in one of the tubes \( T_i \) and the preceding results apply to the extent that the universal covering surface has the appropriate properties. For example, if \( R \) has convex capsules then the universal covering surface \( \bar{R} \) is straight. If in addition \( \bar{R} \) has the divergence property then the asymptote relation is transitive. In particular this is the case when \( R \) has nonpositive curvature (see \( GG \), pp. 249-50).

5. The covering map and the co-ray relation. In view of (7) it is natural to ask when a co-ray to \( \bar{A} \) in \( \bar{R} \) over \( A \) lies over a co-ray to \( A \) in \( R \). In this section we present some partial answers to this question primarily for use in establishing our principal results in the following section.

Let \( R \) be a \( G \)-surface whose universal covering surface \( \bar{R} \) is straight and let \( A \) with origin \( q \) be a ray in \( R \) which ultimately lies in a tube \( T \). Fix \( p \) in \( R \) and \( \bar{A} \) in \( \bar{R} \) over \( A \) with origin \( \bar{q} \). It can be seen from the proof of (6), by applying covering motions of \( \bar{R} \) to the rays \( \bar{A}_i \) if necessary, that if \( B \) is a co-ray from \( p \) to \( A \) then there is a sequence \( x_n \rightarrow p \), a sequence \( z_n \) on \( A \) with \( qz_n \rightarrow \infty \) and a sequence of segments \( T(x_n, z_n) \rightarrow B \) such that if \( \bar{z}_n \) on \( \bar{A} \) lies over \( z_n \) then the segments \( T(\bar{x}_n, \bar{z}_n) \) over \( T(x_n, z_n) \) converge to a co-ray \( \bar{B} \) to \( \bar{A} \) over \( B \).
If \( \bar{p} \) is the origin of \( \bar{B} \) then \( \bar{p} \) lies over \( p \) and since \( \bar{x}_n\bar{z}_n = x_nz_n \) we have \( \alpha(\bar{A}, \bar{p}) = \alpha(A, p) \) (see GG, p. 31).

**Lemma 11.** Let \( A \) be a ray in a \( G \)-surface \( R \). If the universal covering surface \( \bar{R} \) is straight and if \( A \) ultimately lies in a tube then for any \( p \in R \) and any ray \( \bar{A} \) in \( \bar{R} \) over \( A \) there is no point \( \bar{p}_1 \in \bar{R} \) over \( p \) with \( \alpha(\bar{A}, \bar{p}_1) < \alpha(A, p) \).

**Proof.** Assume otherwise. Let \( \bar{p}, x_n, \bar{x}_n \) and \( \bar{z}_n \) be as above and let \( \bar{t}_n \to \bar{p}_1 \) where \( \bar{t}_n \) lies over \( x_n \). For sufficiently large \( n \), \( \bar{t}_n\bar{z}_n < \bar{x}_n\bar{z}_n \) which contradicts the assumption that \( T(\bar{x}_n, \bar{z}_n) \) lies over a segment.

We present now a sufficient condition for a co-ray to \( \bar{A} \) in \( \bar{R} \) to lie over a co-ray to \( A \) in \( R \).

**Theorem 12.** Let \( R \) be a \( G \)-surface whose universal covering surface is straight and let the ray \( A \) ultimately lie in a tube. For any \( p \in R \) and any ray \( A \) in \( R \) over \( A \), if \( p \in R \) lies over \( p \) and \( \alpha(\bar{A}, \bar{p}) = \alpha(A, p) \) then the co-ray \( B \) from \( p \) to \( A \) lies over a co-ray \( \bar{B} \) to \( A \).

**Proof.** We show first that \( \bar{B} \) lies over a ray \( B \). If \( \bar{x} \in \bar{B} \) then \( \bar{x} \) is a foot of \( \bar{x} \) on the limit sphere \( K_\infty(\bar{A}, \bar{p}) \) (see GG, p. 135). Since no points over \( p \) are interior to \( K_\infty(\bar{A}, \bar{p}) \), \( T(p, x) \) lies over a segment. Since \( x \in B \) is arbitrary, \( \bar{B} \) lies over a ray.

Let \( x \neq p \) be any point of \( B \) and let \( \bar{x} \) on \( \bar{B} \) lie over \( x \). Choose \( z_n \) on \( A \) with \( qz_n \to \infty \) and let \( \bar{z}_n \) on \( \bar{A} \) lie over \( z_n \). Since \( \bar{R} \) is straight, \( T(\bar{p}, \bar{z}_n) \) converges to \( \bar{B} \). For sufficiently large \( n \) we can choose \( \bar{z}_n \) in \( T(\bar{p}, \bar{z}_n) \) such that \( \bar{z}_n - z_n \) is a limit of \( \bar{z}_n - z_n \).

We then have the following:

(a) Limit \((\bar{p}z_n - pz_n) = 0 \) since \( \bar{z}_n \bar{q} = z_nq \) and limit \((\bar{p}z_n - \bar{z}_n\bar{q}) = \alpha(\bar{A}, \bar{p}) = \alpha(A, p) = \) limit \((pz_n - z_nq)\).

(b) Limit \((\bar{p}\bar{x}_n - px_n) = 0 \) since \( \bar{p}\bar{x}_n \to \bar{p}x = px \) and \( px_n \to px \).

(c) \( \bar{x}_n\bar{z}_n = \bar{p}\bar{z}_n - \bar{p}\bar{x}_n \) and \( x_nz_n \geq pz_n - px_n \).

From (c) we have \( 0 \leq \bar{x}_n\bar{z}_n - x_nz_n = (\bar{p}\bar{z}_n - \bar{p}\bar{x}_n - x_nz_n) \leq \bar{p}\bar{z}_n - \bar{p}\bar{x}_n - pz_n + px_n = (\bar{p}\bar{z}_n - pz_n) - (\bar{p}\bar{x}_n - px_n) \).

This inequality in conjunction with (a) and (b) yields limit \((\bar{x}_n\bar{z}_n - x_nz_n) = 0 \). We then have \( \alpha(A, x) = \) limit \((x_nz_n - z_nq) = \) limit \([(\bar{x}_n\bar{z}_n - z_nq) + (x_nz_n - \bar{x}_n\bar{z}_n)] = \) limit \((\bar{x}_n\bar{z}_n - z_nq) = \alpha(\bar{A}, \bar{x}) = \alpha(\bar{A}, \bar{p}) = \bar{p}x = \alpha(A, p) - px \). The assertion then follows from a result of Busemann (see GG, p. 136).

In his thesis (University of Southern California, 1970) the author believed he had carried the above line of reasoning further and obtained, under the hypothesis of (12), a negative answer to a still unsolved problem of Busemann: can a maximal co-ray be a proper sub-ray of another ray? Unfortunately this assertion with its implica-
tion of transitive co-rays in a certain class of G-surfaces was reported in Busemann [2, p. 89 (13) and p. 90 (15)] before an error in the proof was discovered by the author.

Let $P$ be a geodesic polygon that bounds $T$ the tube containing $A$. We may assume without loss of generality that $P$ contain $q$ but no other points of $A$. Given $\bar{q} \in \bar{R}$ over $q$, there is exactly one ray $\bar{A}$ over $A$ with origin $\bar{q}$ and exactly one geodesic polygon $\bar{P}$ over $P$ with initial point $\bar{q}$. The end-point $\bar{q}'$ of $\bar{P}$ also lies over $q$ and is the origin of exactly one ray $\bar{A}'$ over $A$. $\bar{A}$, $\bar{P}$ and $\bar{A}'$ bound a simply connected region $\bar{T}$ over $T$ on the interior of which the covering map is one-to-one.

**Proposition 13.** Let $\lambda$ denote the length of $P$ and hence of $\bar{P}$. If $\bar{p}$ lies in the interior of $\bar{T}$ with $\bar{p}A < \bar{p}\bar{P} - \lambda$ then the co-ray $\bar{B}$ from $\bar{p}$ to $\bar{A}$, the co-ray $\bar{B}'$ from $\bar{p}$ to $\bar{A}'$ or both lie over a co-ray to $A$.

**Proof.** Let $z_n$ be a sequence on $A$ with $qz_n \to \infty$ and let $\bar{z}_n$ on $\bar{A}$ lie over $z_n$. Assume that $\bar{z}_n''$ exterior to $\bar{T}$ also lies over $z_n$ and that $T(\bar{p}, \bar{z}_n'')$ lies over a segment. $T(\bar{p}, \bar{z}_n'')$ can intersect neither $\bar{A}$ nor $\bar{A}'$ and so must intersect $\bar{P}$. Since $T(\bar{z}_n, \bar{q})$ lies over the unique segment $T(\bar{z}_n, q)$, $\bar{p}\bar{z}_n'' \geq \bar{z}_n''\bar{P} + p\bar{P} \geq \bar{z}_n''\bar{q} - \lambda + \bar{p}\bar{P} > \bar{z}_n\bar{q} - \lambda + \bar{p}\bar{P}$ for all $n$. On the other hand for sufficiently large $n$, $\bar{p}\bar{z}_n \leq \bar{p}\bar{A} + \bar{z}_n\bar{q} < \bar{p}\bar{P} - \lambda + \bar{z}_n\bar{q}$. Thus for sufficiently large $n$, $T(\bar{p}, \bar{z}_n'')$ does not lie over a segment.

Let $\bar{z}_n''$ on $\bar{A}'$ lie over $z_n$. For sufficiently large $n$ either $T(\bar{p}, \bar{z}_n)$, $T(\bar{p}, \bar{z}_n'')$ or both lie over a segment. It follows that $\alpha(A, p) = \alpha(\bar{A}, \bar{p})$, $\alpha(A, p) = \alpha(\bar{A}', \bar{p})$ or both. The proposition then follows from (12).

Observe that the conflicting inequalities arise because $T(\bar{p}, \bar{z}_n'')$ intersects $\bar{P}$. This means that if $T(\bar{p}, \bar{z}_n'')$ lies over a segment then, for sufficiently large $n$, it does not intersect $\bar{P}$. Thus if $\bar{B}'$ lies over a co-ray to $A$, then $\bar{B}'$ does not intersect $\bar{P}$. Likewise if $\bar{B}$ lies over a co-ray to $A$ then $\bar{B}$ lies in $\bar{T}$.

**Definition 14.** The distance from co-ray to ray is weakly bounded if for a co-ray $B$ from $p$ to $A$ there is a sequence $x_n$ on $B$ with $x_np \to \infty$ such that $x_nA$ is bounded.

In particular the distance from co-ray to ray is weakly bounded in a straight space with convex capsules where, in fact, both $xA$ and $yB$ are bounded for $x \in B$ and $y \in A$. An example of Busemann (GG, p. 137) shows that the latter do not necessarily follow from the distance from co-ray to ray being weakly bounded.

**Proposition 15.** Let $R$ be a $G$-surface whose universal covering surface $\bar{R}$ is straight and has the distance from co-ray to ray weakly bounded. Let $A, \bar{A}, q, \bar{q} , T$ and $\bar{T}$ be as in (13). Let $\bar{B}$ be a co-ray to $\bar{A}$ in $\bar{R}$ such that a sub-ray of $\bar{B}$ lies in $\bar{T}$. If $\bar{B}$ lies over a co-ray
to A then there is a point $\bar{x}_n$ on $\bar{B}$ and a point $\bar{z}_n$ on $\bar{A}$ such that the sub-ray of $\bar{B}$ from $\bar{x}_n$, the sub-ray of $\bar{A}$ from $\bar{z}_n$ and $T(\bar{x}_n, \bar{z}_n)$ bound a sub-region of $\bar{T}$ the co-ray from each point of which to $\bar{A}$ lies over a co-ray to $\bar{A}$.

**Proof.** We may assume without loss of generality that the origin $\bar{p}$ of $\bar{B}$ is exterior to $\bar{T}$. There is a sequence $\bar{x}_n$ in $\bar{B}$ with $\bar{x}_n \bar{p} \to \infty$ and a constant $M > 0$ such that $\bar{x}_n \bar{A} < M$ for all $n$. Let $\bar{z}_n$ be a foot of $\bar{x}_n$ on $\bar{A}$. Then $\bar{z}_n \bar{q} \to \infty$ and $\bar{q} T(\bar{x}_n, \bar{z}_n) \to \infty$.

Choose $N$ such that for $n \geq N$ if $\bar{y} \in T(\bar{x}_n, \bar{z}_n)$ then $M < \bar{y} \bar{p} - \lambda$. For each $n \geq N$, if $\bar{y} \in T(\bar{x}_n, \bar{z}_n)$ then the co-ray $\bar{H}$ from $\bar{y}$ to $\bar{A}$ lies over a co-ray to $\bar{A}$. Otherwise the co-ray $\bar{H}'$ from $\bar{y}$ to $\bar{A}'$ lies over a co-ray to $\bar{A}$ in which case $\bar{H}'$ would either co-incide with $\bar{H}$ or intersect $\bar{B}$, which is impossible.

Consider the sub-region of $\bar{T}$ bounded by $T(\bar{x}_N, \bar{z}_N)$, the sub-ray of $\bar{B}$ from $\bar{x}_N$ and the sub-ray of $\bar{A}$ from $\bar{z}_N$. Let $\bar{y}$ be any point in the interior of this sub-region. For sufficiently large $n > N$, $\bar{y}$ is in the interior of the region bounded by $T(\bar{x}_N, \bar{z}_N)$, $T(\bar{x}_N, \bar{x}_n)$, $T(\bar{z}_N, \bar{z}_n)$ and $T(\bar{x}_n, \bar{z}_n)$. Any ray $\bar{H}$ from $\bar{y}$ that lies over a co-ray $\bar{A}$ must intersect one of these segments and so must be a co-ray to $\bar{A}$.

Slight modification of the preceding proof yields the following:

**Proposition 16.** Under the assumptions of (15), if $\bar{R}$ has a transitive asymptote relation and $\bar{A}$ and $\bar{A}'$ are co-rays to each other then there are points $\bar{z}$ and $\bar{z}'$ on $\bar{A}$ and $\bar{A}'$ respectively such that the sub-ray of $\bar{A}$ from $\bar{z}$, the sub-ray of $\bar{A}'$ from $\bar{z}'$ and $T(\bar{z}, \bar{z}')$ bound a sub-region of $\bar{T}$ the co-ray from any point of which to $\bar{A}$ lies over a co-ray to $\bar{A}$.

We also obtain a result of Nasu [5].

**Corollary 17.** Under the assumptions of (16), if $\bar{A}$ and $\bar{A}'$ are co-rays to each other then there is a sub-tube of $T$ disjoint from $C(\bar{A})$. If $\bar{A}$ and $\bar{A}'$ are not co-rays to each other then no sub-tube of $T$ is disjoint from $C(\bar{A})$.

**Proof.** The first assertion is a direct consequence of (8) and (16). To prove the second assertion let $\bar{a}(t)$, $0 \leq t \leq 1$, be any curve in $\bar{T}$ with $\bar{a}(0)$ in $\bar{A}$ and $\bar{a}(1)$ in $\bar{A}'$. There is then a largest value of $t$, $t_0$, such that for $0 \leq t \leq t_0$ the co-ray from $\bar{a}(t)$ to $\bar{A}$ lies over a co-ray to $\bar{A}$. Since $\bar{A}$ and $\bar{A}'$ are not co-rays to each other $0 < t_0 < 1$. Then $\bar{a}(t_0)$ is the origin of two rays lying over co-rays to $\bar{A}$.

6. The structure of $C(\bar{A})$ in a class $G$-surfaces. In this section
we analyze $C(A)$ in case $R$ is a $G$-surface of finite connectivity with a straight universal covering surface with a transitive asymptote relation and the distance from co-ray to ray weakly bounded. We have mentioned previously that this includes all $G$-surfaces of finite connectivity whose universal covering surface has transitive asymptotes, and hence includes all $G$-surfaces of finite connectivity with non-positive curvature.

The following consequence of (16) is basic to our analysis.

**Proposition 18.** Let $R$ be a $G$-surface of the above type. If $A$ is a ray in $R$ then $C(A)$ does not separate $R$.

**Proof.** Since $R$ has finite connectivity, $A$ lies ultimately in a tube $T$. Assume that the proposition is false. $C(A)$ then separates $R$ into at least two components. We consider two cases.

(i) The tube $T$ or a sub-tube is contained in one of the components. Consider a point $x$ in a different component. A co-ray $B$ from $x$ to $A$ has a sub-ray contained in $T$. Thus $B$ intersects $C(A)$ which is impossible.

(ii) None of the components of $R - C(A)$ contains a sub-tube of $T$. Assume without loss of generality that the initial point $q$ of $A$ is not in $C(A)$. Then $A$ lies in one of the components. Choose $x \in T$ in a component that does not contain $A$. Then the co-ray $B$ from $x$ to $A$ has a sub-ray contained in $T$.

Let $P$ be the simple closed geodesic polygon bounding $T$. We may assume that $q$ is the initial point of $P$. Choose $\bar{q} \in \bar{R}$ over $q$ and let $\bar{A}$ and $\bar{P}$, each with initial point $\bar{q}$, lie over $A$ and $P$. The endpoint $\bar{q}'$ of $\bar{P}$ lies over $q$. Let $\bar{A}'$ with initial point $\bar{q}'$ lie over $A$. Then $\bar{A}, \bar{P}$ and $\bar{A}'$ bound a simply connected region $\bar{T}$ over $T$.

Let $\bar{B}$ with a sub-ray in $\bar{T}$ lie over $B$. $\bar{B}$ is a co-ray to $\bar{A}'$ or $\bar{A}$, say $\bar{A}$ to be definite. It follows from (16) that there is a segment $T(\bar{x}_a, \bar{y}_b)$ in $\bar{T}$ joining $\bar{B}$ to $\bar{A}$ no point of which lies over a co-ray to $A$. Thus $C(A)$ does not separate $B$ and $A$ which is a contradiction.

It was mentioned at the end of §4 that a $G$-surface of finite connectivity can be regarded as a sub-space of a compact manifold $M$ of finite genus $\gamma$. As such it is obtained from $M$ by the removal of a finite number of points $a_i, 1 \leq i \leq N$, each of which corresponds to a tube $T_i$ bounded by $P_i$, a simple geodesic polygon in $R$.

**Definition 19.** Given a ray $A$ in $R$, a $G$-surface of finite connectivity, denote by $C^*(A)$ the closure of $C(A)$ relative to $M$. $C(A)$ is said to occupy a tube $T_j$ if the point $a_j$ in $M$ that determines $T_j$ is in $C^*(A)$. Similarly a component $C^*_i(A)$ of $C^*(A)$ occupies $T_j$ if $a_j$ is in $C^*_i(A)$. 
We note that $C^*(A)$ is obtained from $C(A)$ by adjoining those $a_j$ that corresponds to tubes occupied by $C(A)$.

A sufficiently small deleted neighborhood of $a_j$ in $C^*(A)$ can be thought of as a sub-tube of $T_j$. The next theorem extends (10) to include those $a_j$ in $C^*(A)$. Ultimately this allows us to assert that $C^*(A)$ is triangulable as a one dimensional simplicial complex, a fact from which we derive the principal results of this section.

**Theorem 20.** Let $R$ be a $G$-surface of finite connectivity with a straight universal covering surface $\bar{R}$ having a transitive asymptote relation and the distance from co-ray to ray weakly bounded. Let $A$ be a ray in $R$ and let $T$ be a tube occupied by $C(A)$. The geodesic polygon $P$ bounding $T$ can be chosen so that $C(A) \cap T$ consists of a finite number of disjoint unbounded arcs emanating from $P$.

**Proof.** We consider the case that $T$ is not the tube that contains $A$ or a sub-ray thereof. Let $y_n$ be an unbounded sequence in $T$. Let $B_n$ be a co-ray from $y_n$ to $A$. Each $B_n$ intersects $P$ in a first point $x_n$. Since $P$ is compact, the sequence $x_n$ is bounded and a sub-sequence of $B_n$ converges to an asymptote $L^+$ to $A$. Let $q$ be the first point in which $L^+$ intersects $P$ and let $H$ be the negative sub-ray of $L^+$ with origin $q$. We note that $H$ is contained in $T$ and has only $q$ in common with $P$.

In $\bar{R}$, the universe covering surface, choose a simply connected region $\bar{T}$ over $T$, bounded by $\bar{H}$ and $\bar{H}'$ over $H$ with initial points $\bar{q}$ and $\bar{q}'$ respectively, and by $\bar{P}$ over $P$ with $\bar{q}$ and $\bar{q}'$ as initial and final points. We note that the covering map is one-to-one on the interior of $\bar{T}$.

Assume without loss of generality that $\bar{P}$ is a segment. Then any ray that lies over a co-ray to $A$ can intersect $\bar{P}$ at most once. Furthermore any ray lying over a co-ray to $A$ that intersects $\bar{P}$ must originate from $\bar{T}$. This, with (10), implies that $\bar{P}$ contains at most a finite number of points that lie over co-points to $A$.

We show that $\bar{P}$ can be replaced by a geodesic polygon $\bar{P}'$ bounding a sub-region $\bar{T}'$ of $\bar{T}$ that lies over a sub-tube $T'$ of $T$ and is such that those points of $\bar{T}'$ that lie over $C(A) \cap T'$ form a finite number of unbounded disjoint arcs emanating from $\bar{P}'$.

Since $\bar{P}$ is compact it can be covered by a finite number of neighborhoods of the type in (10). It follows that there are a finite number of rays $\bar{A}_i, 1 \leq i \leq m$, over $A$, none a co-ray to any other, to one of which any ray from a point of $\bar{P}$ that lies over a co-ray to $A$ must be a co-ray. Furthermore, if $\bar{x} \in \bar{P}$ does not lie over a co-point to $A$, and if the co-ray from $\bar{x}$ to $\bar{A}_j$ lies over a co-ray to $A$ then by
(10) there is a sub-segment of $P$ the co-ray from any point of which to $A_j$ lies over a co-ray to $A$.

Hence $P$ can be partitioned into a finite number of non-overlapping segments $I_1, \ldots, I_k$ whose end-points are either end-points of $P$ or lie over co-points to $A$. For each $I_i$ there is an $A_{j(i)}$ the co-ray to which from any point of $I_i$ lies over a co-ray to $A$.

In $I_i$ suppose two points $\bar{x}$ and $\bar{y}$ the asymptotes through which to $A_{j(i)}$ lie over asymptotes to $A$ and are denoted by $\bar{B}(\bar{x})$ and $\bar{B}(\bar{y})$ respectively. A ray lying over a co-ray to $A$ in the strip bounded by $\bar{B}(\bar{x})$ and $\bar{B}(\bar{y})$ cannot intersect either $\bar{B}(\bar{x})$ or $\bar{B}(\bar{y})$. Such a ray must be a co-ray to $A_{j(i)}$. Thus the asymptotes to $A_{j(i)}$ through points of $I_i$ between $\bar{x}$ and $\bar{y}$ lie over asymptotes to $A$. This implies that those points of $I_i$ the asymptotes through which to $A_{j(i)}$ lie over asymptotes to $A$ form a sub-segment of $I_i$. We note that this sub-segment might consist of a single point or be empty.

It follows that $P$ contains a finite number of disjoint segments $K_0, \ldots, K_r$ whose points are the points of $P$ that lie on straight lines that lie over asymptotes to $A$. Let $\bar{x}_i$ and $\bar{y}_{i+1}$, $0 \leq i \leq r$, denote the end-points of $K_i$ indexed so that $\bar{q} = \bar{x}_0, \bar{q}' = \bar{y}_{r+1}$ and $K_{i+1}$ follows $K_i$ on $\bar{P}$. Let $J_i$, $1 \leq i \leq r$, denote the sub-segment of $\bar{P}$ joining $\bar{y}_i$ and $\bar{x}_i$. Then $\bar{P} = K_0 \cup J_0 \cup K_1 \cup \cdots \cup J_r \cup K_r$.

We will alter $\bar{P}$ by altering the segments $J_i$. Each point $\bar{z}$ in $J_i$ determines a unique point $\phi(\bar{z})$ in $\bar{T}$ that lies over a co-point to $A$. By (4) $\phi$ is continuous. Consider in $J_i$ a sequence $\bar{z}_n \rightarrow \bar{x}_i$. For sufficiently large $n$ the co-ray $\bar{B}_n$ from $\phi(\bar{z}_n)$ to $A_{j(i)}$ lies over a co-ray to $A$, and the sequence $\bar{B}_n$ converges to the asymptote to $A_{j(i)}$ through $\bar{x}_i$. Furthermore, since $C(A)$ is closed in $R$, $\phi(\bar{z}_n) \bar{x}_i \rightarrow \infty$.

$\phi(\bar{z}_n)$ is the origin of at least one other ray $\bar{B}'_n$ that lies over a co-ray to $A$. Since $\phi(\bar{z}_n)\bar{x}_i \rightarrow \infty$, a sub-sequence of $\bar{B}_n$ converges to an oriented straight line lying over an asymptote to $A$. The only possibility for the latter is the asymptote to $A_{j(i-1)}$ through $\bar{y}_i$.

Thus if $\bar{z}_i$ in $J_i$ is chosen sufficiently close to $x_i$ then $\phi(\bar{z}_i)$ is the origin of exactly two rays that lie over co-rays to $A$: the co-ray $\bar{B}_i$ to $A_{j(i)}$ and the co-ray $\bar{B}'_i$ to $A_{j(i-1)}$. Let $\bar{z}_i$ denote the intersection of $\bar{B}_i$ with $J_i$. If $\bar{z}_i$ is sufficiently close to $\bar{x}_i$ then the image of $T(\bar{z}_i, \bar{x}_i) - \{\bar{x}_i\}$ under $\phi$ coincides with that of $T(\bar{y}_i, \bar{z}_i) - \{\bar{y}_i\}$ under $\phi$, and their common image is an unbounded arc emanating from $\phi(\bar{z}_i)$.

We replace $J_i$ with $J'_i = T(\bar{y}_i, \bar{z}_i) \cup T(\bar{z}_i, \phi(\bar{z}_i)) \cup T(\phi(\bar{z}_i), \bar{z}_i) \cup T(\bar{z}_i, \bar{x}_i)$. When this is done for each $i$, $1 \leq i \leq r$, we have the desired geodesic polygon $\bar{P}'$.

The case in which $T$ contains $A$ or a sub-ray thereof is treated in a similar manner (although it involves a few more details).

Let $T_i = M_i - \{a_i\}$ where $M_i$ is homeomorphic to a closed disk
and contains $a_i$ in its interior. If $T_i$ is occupied by $C(A)$ it is clear from (20) that $P_i$ may be chosen so that $M_i$ is homeomorphic to a closed disk in such a way that $a_i$ corresponds to the center and $C^*(A) \cap M$ to a finite number of radii. On the other hand, if $T_i$ is not occupied by $C(A)$ then, by (17), $P_i$ may be chosen so that $T_i$ is disjoint from $C(A)$. If $\text{Int} T_i$ denotes the interior of $T_i$ then $R - \cup \text{Int} T_i$ is compact and $C(A) \cap (R - \cup \text{Int} T_i)$ can be covered by a finite number of neighborhoods of the type in (10). The following then holds.

**Corollary 21.** Let $A$ be a ray in $R$. $C^*(A)$ is triangulable as a one dimensional simplicial complex.

**Definition 22.** Given a ray $A$ in $R$ and $p$ a co-point to $A$ denote by $m(p)$ the number of co-rays from $p$ to $A$ minus two. If $m(p) > 0$ then $p$ is called a multiple co-point to $A$.

It is clear that in any triangulation of $C^*(A)$ the multiple co-points will be vertices. In particular (20) implies that $C(A)$ contains only a finite number of such points. We now state the principal result of this section.

**Theorem 23.** Let $R$ be an orientable (non-orientable) $G$-surface of finite connectivity with a straight universal covering surface $\bar{R}$ having a transitive asymptote relation and the distance from co-ray to ray weakly bounded. If $A$ is a ray in $R$, let $\pi(A)$ denote the number of components of $C(A)$, $\mu(A)$ the number of multiple co-points to $A$, $N$ the number of tubes in $R$ and $\gamma$ the genus of $M$, the compact surface of which $R$ is a subspace. It then follows that $\mu(A) \leq N - 2 + 2\gamma (\mu(A) \leq N - 2 + \gamma), \pi(A) \leq N - 1 + 2\gamma (\pi(A) \leq N - 1 + \gamma)$ and no co-point to $A$ is the origin of more than $N + 2\gamma (N + \gamma)$ co-rays to $A$.

**Proof.** We assume that $C(A) \neq \emptyset$. Let $\beta_0$ and $\beta_1$ be the first two Betti numbers of $C^*(A)$. The Euler-Poincaré characteristic of $C^*(A)$ is $\chi(C^*(A)) = \beta_0 - \beta_1$.

$\beta_0 = \sigma(A)$, the number of components of $C^*(A)$. $C^*(A)$ can be regarded as a subcomplex of $M$ which is likewise triangulable. Since $C(A)$ does not separate $R$, $C^*(A)$ does not separate $M$ and no one cycle in $C^*(A)$ bounds in $M$. Thus $\beta_1 \leq 2\gamma$. In the non-orientable case a bounding one cycle in $C^*(A)$ would correspond to a torsion element in $H_i(M)$ and the inequality is $\beta_1 \leq \gamma$.

Consider a component $C_t^*(A)$ of $C^*(A)$. Let $\pi_i$ be the number of components of $C(A)$ included in $C_t^*(A)$, let $A_i$ be the number of tubes occupied by $C_t^*(A)$ and let $p(i, j), 1 \leq j \leq \delta_i$, be the multiple co-points in $C_t^*(A)$. The Euler-Poincaré characteristic of $C^*(A)$ is $\chi(C_t^*(A)) = (A_i + \delta_i) - (\delta_i + \pi_i + \sum_j m(p(i, j))), 1 \leq j \leq \delta_i$. 
Summing over $i = 1, \ldots, \sigma(A)$ we obtain $\chi(C^*(A)) = \sum_i A_i - (\pi(A) + \sum_i \sum_j m(p(i, j))) = \beta_0 \geq \sigma(A) - 2\gamma$.

Then $2\gamma + N \geq 2\gamma + \sum A_i \geq \sigma(A) + \pi(A) + \sum_i \sum_j m(p(i, j))$. This yields two inequalities: $2\gamma + N \geq \sigma(A) + \pi(A) + \mu(A)$ and $2\gamma + N \geq \sigma(A) + \pi(A) + \mu(A) - 1 + \max m(p(i, j))$.

Finally we obtain $2\gamma + N \geq 1 + \pi(A), 2\gamma + N \geq 2 + \mu(A)$ and $2\gamma + N \geq 2 + \max m(p(i, j))$ which yield the desired results. In the non-orientable case $2\gamma$ is replaced by $\gamma$ in the preceding inequalities.

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