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**CUT LOCI OF POINTS AT INFINITY**

GEORGE M. LEWIS

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In a  $G$ -space  $R$  if  $B$  is a co-ray to  $A$  then the union of all co-rays to  $A$  that contain  $B$  is either a straight line or a co-ray to  $A$  maximal in that it is properly contained in no other co-ray to  $A$ . In the latter case, the initial point of the maximal co-ray is a copoint to  $A$ . The concept of co-point is an analogue to that of minimum point in a sense made precise. On certain non-compact  $G$ -surfaces of finite connectivity, including those with non-positive curvature, we characterize the locus of co-points to a given ray and obtain bounds for the number of components of this locus, the number of co-rays emanating from a co-point and the number of co-points that are origins of more than two co-rays.

1. Introduction. A  $G$ -space can be described as a metric space any two of whose points can be joined by a segment, and in which any segment may be prolonged uniquely to a geodesic. The theory of  $G$ -spaces is found in Busemann [1] hereafter quoted as  $GG$ .

In a  $G$ -space, as in a Riemannian space, a minimum point  $m$  of a point  $p$  may be defined as a point for which no segment  $T(p, m)$  can be prolonged beyond  $m$ . We shall be concerned with the analog to  $m$  when  $p$  lies at infinity in the sense made precise below.

A co-ray  $B$  from a point  $p$  to a ray  $A$  is the limit of a converging sequence of segments  $T(p_n, z_n)$  where  $p_n \rightarrow p$  and  $z_n$  tends to infinity on  $A$ . Obviously  $B$  is also a co-ray to any ray contained in or containing  $A$  as a sub-ray. Furthermore, the limit of a converging sequence of co-rays to a ray  $A$  is likewise a co-ray to  $A$ . Less trivial is the fact (see  $GG$ , p. 136) that the co-ray to  $A$  from any point of  $B$  other than  $p$  is unique and a sub-ray of  $B$ .

Given a ray  $A$  in a straight space and a co-ray  $B$  to  $A$ , the union of all co-rays to  $A$  containing  $B$  is an asymptote to the oriented straight line containing  $A$  as a positive sub-ray. In an arbitrary non-compact  $G$ -space such a union is either an oriented straight line, any positive sub-ray of which is a co-ray to  $A$ , or it is a co-ray to  $A$  that is not a proper sub-ray of any co-ray to  $A$ . This leads to the following terminology in non-compact  $G$ -spaces.

DEFINITION 1. Given a ray  $A$  in a  $G$ -space

(a) An asymptote to  $A$  is an oriented straight line any positive sub-ray of which is a co-ray to  $A$ .

(b) A maximal co-ray to  $A$  is a co-ray to  $A$  that is not a proper sub-ray of any co-ray to  $A$ .

(c) A co-point to  $A$  is the origin of a maximal co-ray to  $A$ . We denote by  $C(A)$  the set of co-points to  $A$ .

A point  $p_\infty$  at infinity in a  $G$ -space is a maximal set of rays such that a co-ray to one ray in  $p_\infty$  is a co-ray to each ray in  $p_\infty$ . For  $A, B \in p_\infty$ ,  $C(A) = C(B)$ ; hence the locus of co-points depends only on  $p_\infty$ . Nevertheless it is convenient to retain the notation  $C(A)$ . The concept of co-point is thus a natural analog to the concept of minimum point in the finite case.

The study of  $C(A)$  was initiated by Nasu [4, 5, 6], who by "asymptote" means "maximal co-ray or asymptote" and uses "asymptotic conjugate point" instead of "co-point".

It is our purpose to extend and clarify much of Nasu's work. In §6 we characterize  $C(A)$  on certain  $G$ -surfaces of finite connectivity (including those with nonpositive curvature) obtaining bounds for the number of components of  $C(A)$ , the number of co-rays emanating from a point and the number of points that are origins of more than two co-rays to  $A$ .

**2. Preliminaries.** This section consists of results used in later proofs. We begin with a proposition of Nasu [4].

**PROPOSITION 2.** *Given a ray  $A$  in a  $G$ -space and a point  $p \in C(A)$ , there exists for each  $\delta > 0$  a positive  $\varepsilon \leq \delta$  such that each co-ray to  $A$  with origin exterior to  $S(p, \delta)$  fails to intersect  $S(p, \varepsilon)$ . In particular, there is an  $\varepsilon' > 0$  such that no asymptote to  $A$  intersects  $S(p, \varepsilon')$ .*

*Proof.* Otherwise there is a  $\delta > 0$  and a sequence  $p_n \rightarrow p$  such that each  $p_n$  lies on a co-ray  $B_n$  to  $A$  whose origin  $q_n \notin S(p, \delta)$ .

There is an index  $N$  such that  $pp_n < \delta/2$  for  $n \geq N$ . Thus for  $n \geq N$ ,  $q_n p_n \geq q_n p - pp_n > \delta/2$ . Choose  $q'_n \in B_n$  such that  $(q_n q'_n p_n)$  and  $q'_n p_n = \delta/4$ . The co-ray  $B'_n$  from  $q'_n$  to  $A$  is unique and a sub-ray of  $B_n$ . A sub-sequence of  $B'_n$  converges to a co-ray  $B$  to  $A$  containing  $p$  in its interior—a contradiction.

What follows is a modification of a theorem of Busemann [2, p. 18].

**PROPOSITION 3.** *For each  $x \in S(p, \rho_0)$  let a ray  $A(x)$  with origin  $x$  be defined which depends continuously on  $x$ . If the spheres  $K(p, \rho)$ ,  $0 < \rho < \rho(p)$ , are not contractible then, for some  $x \in S(p, \rho_0)$ ,  $x \neq p$  and  $p \in A(x)$ .*

*Proof.* Let  $0 < \delta < \min(\rho_0, \rho(p))$ .  $A(p)$  intersects  $K = K(p, \delta)$  in exactly one point  $w$  the antipode of which on  $K$  we denote by  $w'$ .

Let  $u \in A(p)$  with  $up = \delta/2$ . The projection  $P$  of  $S(u, \delta/4)$  on  $K$

by segments from  $p$  is a proper sub-set of  $K$  and in particular does not include  $w'$ .

For  $0 < \varepsilon < \delta$ , let  $K_\varepsilon = (p, \varepsilon)$ . Let  $V = \{A(v) : v \in K_\varepsilon\}$  and let  $z(t, v)$  represent  $A(v)$  with  $z(0, v) = v$ . Choose  $\varepsilon > 0$  so small that for  $v \in V$ ,  $z(\delta/2, v) \in S(u, \delta/4)$  and  $T(z(\delta/2, v), v)$  lies in  $S(p, \delta)$ .

If  $z(t\delta/2, v) \neq p$  for  $0 \leq t \leq 1$ , define  $v_t$  by  $(pz(t\delta/2, v)v_t)$  and  $v_t p = \delta$ .

For  $t = 0$  we have  $(pvv_0)$  and if  $p \in V$  then  $v_0$  traverses  $K$  as  $v$  traverses  $K_\varepsilon$ . For  $t = 1$  we have  $(pz(\delta/2, v)v_1)$  and  $z(\delta/2, v) \in S(u, \delta/4)$ . Hence  $v_1 \in P$ .

The point  $z(t\delta/2, v)$  depends continuously on both  $t$  and  $v$ . Thus if  $p \notin V$  then  $v_t$  defines a deformation of  $K$  onto a proper sub-set of itself. This in turn can be deformed to a point thus contradicting the non-contractibility of  $K$ .

It follows that  $p \in V$  yet  $p \notin K_\varepsilon$  which proves the assertion.

Although it is not presently known whether the non-contractibility of small spheres holds in general, it is shown in Busemann [2] to hold in finite dimensional  $G$ -spaces.

The set  $C(A)$  is not necessarily closed (see Nasu [4]). In the event that  $C(A)$  is closed we have the following:

**PROPOSITION 4.** *Let  $A$  be a ray in a  $G$ -space  $R$  such that  $C(A)$  is closed. Let  $x_n \rightarrow x_0$  where  $x_n$  and  $x_0$  lie on maximal co-rays to  $A$ . If  $x'_n$  and  $x'_0$  are the co-points to  $A$  determined by  $x_n$  and  $x_0$  respectively, then  $x'_n \rightarrow x'_0$ .*

*Proof.* We show first that the sequence  $x'_n$  is bounded. Otherwise there is a sub-sequence  $x'_m$  such that  $x_m x'_m \rightarrow \infty$ . Then there are, for sufficiently large  $m$ , co-rays  $B_m$  to  $A$  containing  $x_m$  whose initial points  $q_m$  satisfy  $x_0 x'_0 + 2 > x_m q_m > x_0 x'_0 + 1$ .

Since  $q_m$  is bounded, a sub-sequence  $B_i$  of  $B_m$  converges to a co-ray  $B_0$  to  $A$  containing  $x_0$  with initial point  $q_0$  satisfying  $x_0 q_0 \geq x_0 x'_0 + 1$ . This, however, is impossible since  $x'_0 \in C(A)$ . Therefore  $x'_n$  is bounded.

If  $x'_n$  does not converge to  $x'_0$  then there is a sub-sequence  $x'_j$  of  $x'_n$  and a  $\delta > 0$  such that each  $x'_j \notin S(x'_0, \delta)$ . Let  $H_j$  be a maximal co-ray to  $A$  containing  $x_j$ . Since  $x'_n$  is bounded, a sub-sequence  $H_k$  of  $H_j$  converges to a co-ray  $H$  to  $A$  containing  $x_0$ . Hence the corresponding sequence  $x'_k$  of co-points converges to the initial point of  $H$  which, since  $C(A)$  is closed, must be  $x'_0$ —a contradiction.

We conclude this section with the following separation property.

**PROPOSITION 5.** *Let  $A$  be a ray. The complement of  $C(A)$  has no bounded component, and no compact, sub-set of  $C(A)$  separates the space.*

*Proof.* Let  $p \in C(A)$  and let  $B$  be the co-ray from  $p$  to  $A$ . Then  $B \cap C(A) = \emptyset$  and the component determined by  $p$  contains  $B$ .

Suppose a compact sub-set  $K$  of  $C(A)$  separates the space  $R$ . Then all points of  $A - \{p\}$  lie in the same component of  $R - K$ . Let  $p$  lie in a different component. Consider a sequence  $x_n$  on  $A$  with  $px_n \rightarrow \infty$  such that a sequence of segments  $T(p, x_n)$  converges to a co-ray  $B$  from  $p$  to  $A$ . Each  $T(p, x_n)$  intersects  $K$  in a point  $y_n$ , and, since  $K$  is compact,  $K$  contains an accumulation point  $y_0$  of  $y_n$ . It follows that  $y_0 \in K \cap B$  which is impossible.

**3. The universal covering surface.** While the preceding section concerned arbitrary  $G$ -spaces the remainder of this article is concerned with  $G$ -surfaces. In this section we generalize results of Nasu [5, 6] proved under the stronger hypothesis of nonpositive curvature.

A tube in a  $G$ -surface  $R$  is a closed domain bounded by a geodesic polygon  $P$  and homeomorphic to a disk punctured at one point. A ray  $A$  in  $R$  is said to ultimately lie in a tube  $T$  if  $A$  or some sub-ray of  $A$  lies in  $T$ .

**THEOREM 6.** *Let  $R$  be a  $G$ -space surface and  $A$  a ray in  $R$ . If the universal covering surface  $\bar{R}$  is straight and if  $A$  ultimately lies in a tube  $T$  then the number of co-rays to  $A$  from any point  $p$  is finite.*

*Proof.* Assume without loss of generality that the initial point  $q$  of  $A$  is on  $P$ , the polygon bounding  $T$ , and is the only point in which  $A$  intersects  $P$ . Assume further that  $p$  is exterior to  $T$ . Let  $\lambda = \text{length of } P, \gamma = \max \{px: x \in P\}$  and  $0 < \varepsilon < pP$ . Consider the class of oriented geodesic polygons of the form  $T(q, p) \cup T(p, x) \cup T(x, z) \cup T(z, q)$  where  $T(q, p)$  is fixed,  $z \in A - \{q\}$  and  $x \in S(p, \varepsilon)$ . We show that the class of such polygons determines only a finite number of homotopy classes in  $R$ .

Given such a polygon, there is a last point  $y$  in which  $T(x, z)$  intersects  $P$ . Because  $T$  is homeomorphic to a punctured disk, there is a sub-arc  $P'(q, y)$  of  $P$  from  $q$  to  $y$  such that  $p'(q, y) \cup T(y, z) \cup T(z, q)$  is null homotopic. It follows that  $T(q, p) \cup T(p, x) \cup T(x, z) \cup T(z, q)$  is homotopic to  $T(q, p) \cup T(p, x) \cup T(x, y) \cup P'(y, q)$ .

Fix  $\bar{p} \in \bar{R}$  over  $p$ , and hence fix  $T(\bar{q}, \bar{p})$  over  $T(q, p)$ . Let  $T(\bar{p}, \bar{x})$  be the unique segment from  $\bar{p}$  over  $T(p, x)$ ,  $T(\bar{x}, \bar{y})$  the unique segment from  $\bar{x}$  over  $T(x, y)$  and  $\bar{P}'(\bar{y}, \bar{q}^*)$  the unique geodesic polygon from  $\bar{y}$  over  $\bar{P}'(y, q)$ . The end-point  $\bar{q}^*$  of  $\bar{P}'(\bar{y}, \bar{q})$  then lies over  $q$  and  $\overline{pq} \leq \overline{p\bar{x}} + \overline{x\bar{y}} + \text{length } \bar{P}'(\bar{y}, \bar{q}) = px + xy + \text{length } P'(y, q) \leq px + px + py + \lambda < 2\varepsilon + \gamma + \lambda$ .

The point  $\bar{q}^*$  so constructed are in one-to-one correspondence with the number of homotopy classes determined by the above class of geodesic polygons and are finite in number since they are all interior to  $S(\bar{p}, 2\varepsilon + \gamma + \lambda)$ .

Let  $x_n \rightarrow p$  and let  $z_n \in A - \{q\}$  be a sequence with  $qz_n \rightarrow \infty$ . Assume without loss of generality that  $x_n p < \varepsilon$ . Let  $\Gamma_1, \dots, \Gamma_k$  be the homotopy classes determined by the geodesic polygons  $T(q, p) \cup T(p, x_n) \cup T(x_n, z_n) \cup T(z_n, q)$  where  $T(q, p)$  is fixed, and let  $\bar{q}_1, \dots, \bar{q}_k \in \bar{R}$  over  $q$  be constructed as above. The end-points  $\bar{z}_n$  of the unique geodesic polygons from  $\bar{p}$  over  $T(p, x_n) \cup T(x_n, z_n)$  then lie on one of the rays  $\bar{A}_1, \dots, \bar{A}_k$  over  $A$  originating from  $\bar{q}_1, \dots, \bar{q}_k$ .

If  $T(x_n, z_n)$  converges to a co-ray  $B$  to  $A$  then  $B$  is the image of a co-ray  $\bar{B}$  from  $\bar{p}$  to one of the rays  $\bar{A}_1, \dots, \bar{A}_k$ . Since  $\bar{R}$  is straight, the co-ray from  $\bar{p}$  to any given ray is unique and the theorem follows.

We saw in the preceding proof that given  $\bar{p} \in \bar{R}$  over  $p$ , the co-rays from  $p$  to  $A$  are the images of the co-rays from  $\bar{p}$  to certain rays  $\bar{A}_1, \dots, \bar{A}_k$  over  $A$ . The following tells us that the choice of  $\bar{A}_i$  is, to an extent, uniform.

**THEOREM 7.** (Nasu [5]). *Under the hypothesis of (6), if the asymptote relation in  $\bar{R}$  is transitive and the co-rays from  $p$  to  $A$  are images of co-rays from  $\bar{p} \in \bar{R}$  to rays  $\bar{A}_1, \dots, \bar{A}_m$  over  $A$  then there is a positive  $\beta_p < \rho(p)/2$  such that each co-ray to  $A$  from  $x \in S(p, \beta_p)$  is the image of a co-ray from  $\bar{x} \in S(\bar{p}, \beta_p)$  over  $x$  to one of the rays  $\bar{A}_1, \dots, \bar{A}_m$ .*

*Proof.* Assume otherwise. There is then a sequence  $p_n \rightarrow p$  with  $pp_n < \min(\rho(p)/2, pP/2)$  such that each  $p_n$  is the origin of a co-ray  $B_n$  to  $A$  which is not the image of a co-ray from  $\bar{p}_n \in S(\bar{p}, \rho(p)/2)$  over  $p_n$  to any of the rays  $\bar{A}_1, \dots, \bar{A}_m$ .

Assume without loss of generality that the co-rays  $B_n$  converge to a co-ray  $B$  from  $p$  to  $A$ . Let  $\gamma_n = \max\{p_n x : x \in P\}$ . Each  $B_n$  is the image of a co-ray from  $\bar{p}_n$  to a ray  $\bar{A}'_n$  over  $A$  with initial point  $\bar{q}'_n$  satisfying  $\bar{p}_n \bar{q}'_n \leq \gamma_n + \lambda$  (since  $\varepsilon$  in the proof of (6) can be made arbitrarily small). Also  $\gamma_n \leq \gamma + pp_n$  hence  $\bar{p} \bar{q}'_n \leq \bar{p} \bar{p}_n + \bar{p} \bar{q}'_n \leq \bar{p} \bar{p}_n + \gamma + pp_n + \lambda = \gamma + \lambda + 2pp_n$ . It follows that there are only a finite number of distinct points  $\bar{q}'_n$ . We can therefore assume, by selecting an appropriate sub-sequence, that each  $B_n$  is the image of the co-ray from  $\bar{p}_n$  to  $\bar{A} \neq \bar{A}_1, \dots, \bar{A}_m$  over  $A$ .

$B$  is then the image of the co-ray  $\bar{B}$  from  $\bar{p}$  to  $\bar{A}$ .  $\bar{B}$  is also a co-ray to one of the rays  $\bar{A}_i$ , say  $\bar{A}_1$ . It follows from the transitivity (and implied symmetry) of the asymptote relation that  $\bar{A}$  and  $\bar{A}_1$  are co-rays to each other. Then  $\bar{B}_n$  is a co-ray to  $\bar{A}_1$ -a contradiction.

We note that an example due to Busemann (*GG*, pp. 265–66) shows the hypothesis that  $A$  ultimately lie in a tube to be essential.

**COROLLARY 8.** *Under the hypothesis of (7), if  $p \in C(A)$  then  $p$  is the origin of at least two co-rays to  $A$ . Furthermore,  $C(A)$  is closed.*

*Proof.* Assume that the co-ray  $B$  from  $p \in C(A)$  to  $A$  is unique. It follows from (7) that the co-ray from each  $x \in S(p, \beta_p)$  is unique. By (3) there is an  $x \in S(p, \beta_p)$  such that  $x \neq p$  and  $p$  lies on the co-ray from  $x$  to  $A$ —a contradiction.

On the other hand if  $p \notin C(A)$  then the co-ray from  $p$  to  $A$  is unique and is thus unique for each  $x \in S(p, \beta_p)$ . Hence  $S(p, \beta_p) \cap C(A) = \emptyset$  and the complement of  $C(A)$  is open.

**4. The local structure of  $C(A)$ .** In this section we describe the local topological structure of  $C(A)$ . As in the previous section our results generalize results of Nasu [5, 6].

**LEMMA 9.** *Under the hypothesis of (7), if  $p \in C(A)$  then there is a  $\gamma_p > 0$  such that no point of  $\bar{S}(p, \gamma_p)$ , with the possible exception of  $p$ , is the origin of more than two co-rays to  $A$ .*

*Proof.* Choose  $\gamma_p > 0$  such that  $\gamma_p < \beta_p$ , no asymptote to  $A$  intersects  $\bar{S}(p, \gamma_p)$  and  $\bar{S}(p, \gamma_p)$  is homeomorphic to the closed unit disk in  $E^2$ . Denote by  $B_i, 1 \leq i \leq m$ , the maximal co-rays to  $A$  from  $p$  and by  $x_i$  the intersection of  $B_i$  with  $K(p, \gamma_p) = \{x \mid px = \gamma_p\}$ . Let the indexing be such that  $x_{i+1}$  follows  $x_i$  where  $x_{m+1} = x_1$ . The points  $x_i$  partition  $K(p, \gamma_p)$  into sub-arcs  $K_i, 1 \leq i \leq m$ , where  $K_i$  has end-points  $x_i$  and  $x_{i+1}$ . These arcs with the co-rays  $B_i$  partition  $\bar{S}(p, \gamma_p)$  into closed simply connected regions  $D_1, \dots, D_m$  with non-empty mutually disjoint interiors such that each  $D_i$  is bounded by  $B_i \cap \bar{S}(p, \gamma_p), K_i$  and  $B_{i+1} \cap \bar{S}(p, \gamma_p)$ .

Choose  $\bar{p} \in \bar{R}$  over  $p$ . Since  $\gamma_p < \beta_p \leq \rho(p)/2$ , the covering map sends  $\bar{S}(\bar{p}, \gamma_p)$  isometrically onto  $\bar{S}(p, \gamma_p)$ . Let  $\bar{B}_i$  with initial point  $\bar{p}$  lie over  $B_i$ .  $\bar{B}_i$  is then a co-ray to a ray  $\bar{A}_i$  over  $A$ .  $\bar{S}(\bar{p}, \gamma_p)$  is partitioned into closed simply connected regions  $\bar{D}_i$  over  $D_i$  where  $\bar{D}_i$  is bounded by  $\bar{B}_i \cap \bar{S}(\bar{p}, \gamma_p), \bar{K}_i$  over  $K_i$  and  $\bar{B}_{i+1} \cap \bar{S}(\bar{p}, \gamma_p)$ .

For each  $x \in S(p, \gamma_p)$  a co-ray from  $x$  to  $A$  is the image of a co-ray from  $\bar{x} \in \bar{S}(\bar{p}, \gamma_p)$  over  $x$  to one of the rays  $\bar{A}_i, 1 \leq i \leq m$ . Since  $\bar{R}$  has a transitive and hence symmetric asymptote relation, we can say that a co-ray from  $x$  to  $A$  is the image of the co-ray from  $\bar{x}$  to one of the rays  $\bar{B}_i, 1 \leq i \leq m$ .

Consider  $x \in D_i, x \neq p$ . We assert that if  $\gamma_p$  is sufficiently small then any co-ray from  $x$  to  $A$  is the image of the co-ray from  $\bar{x} \in \bar{D}_i$

over  $x$  to one of the rays  $\bar{B}_i$  or  $\bar{B}_{i+1}$ . Assume otherwise and fix  $\gamma_p < \beta_p$ . There is then a sequence of points  $x_n \rightarrow p$  in the interior of  $D_i$  such that each  $x_n$  is the origin of a co-ray  $H_n$  to  $A$  where  $H_n$  is the image of  $\bar{H}_n$ , the co-ray from  $\bar{x}_n \in D$  over  $x_n$  to some  $\bar{B}_j, j \neq i, i + 1$  (we can assume without loss of generality that each  $\bar{H}_n$  is a co-ray to the same  $\bar{B}_j$ ). A sub-sequence of the  $\bar{H}_n$  then converges to  $\bar{B}_j$  which is impossible since  $\bar{B}_i \cup \bar{B}_{i+1}$  separates  $\bar{B}_j$  from each  $\bar{H}_n$ . The assertion thus follows and hence the lemma.

Continuing in this manner, we prove the following result.

**THEOREM 10.** *Let  $R$  be a  $G$ -surface and  $A$  a ray in  $R$ . If  $\bar{R}$  is straight and has a transitive asymptote relation, and if  $A$  ultimately lies in a tube, then for each  $p \in C(A)$  there is a closed region  $V$  containing  $p$  in its interior that is homeomorphic to a closed disk  $D$  in such a way that  $p$  corresponds to the center of  $D$  and  $C(A) \cap V$  to the union of a number of radii of  $D$  equal to the number of co-rays from  $p$  to  $A$ .*

*Proof.* We begin where the proof of (9) ends. Each  $x \in K_i$  determines a unique co-point  $\phi(x)$  to  $A$ . It follows from (4) that the map  $\phi: K_i \rightarrow \phi(K_i)$  is continuous.

On  $K_i$  choose  $y_i$  so close to  $x_i$  that no point of the sub-arc  $K(x_i, y_i)$  of  $K_i$  joining  $x_i$  and  $y_i$  is a co-point to  $A$  and so that  $L_i = \phi[K(x_i, y_i)]$  is, with the exception of  $p = \phi(x_i)$ , interior to  $D_i$ . This is possible since  $\phi$  is continuous and  $C(A)$  is closed.

Let  $\bar{y}_i \in D_i$  lie over  $y_i$  and  $\bar{K}(\bar{x}_i, \bar{y}_i)$  over  $K(x_i, y_i)$  be the subarc of  $\bar{K}_i$  joining  $\bar{x}_i$  and  $\bar{y}_i$ . By (7) if  $y_i$  is chosen sufficiently close to  $x_i$  then the co-ray from each  $\bar{x} \in \bar{K}(\bar{x}_i, \bar{y}_i)$  to  $\bar{B}_i$  lies over a co-ray to  $A$ . Let  $\bar{H}_i$  be the co-ray from  $\bar{\phi}(\bar{y}_i)$  over  $\phi(y_i)$  to  $\bar{B}_i$ . Then  $\bar{H}_i$  lies over a co-ray to  $A$  from  $\phi(y_i)$ .

$\phi(y_i)$  is the origin of exactly two maximal co-rays to  $A$ . Let  $U_i$  denote the remaining maximal co-ray to  $A$ . Since  $\bar{H}_i$  is the co-ray from  $\bar{\phi}(\bar{y}_i)$  to  $\bar{B}_i$ , the ray  $\bar{U}_i$  over  $U_i$  from  $\bar{\phi}(\bar{y}_i)$  is a co-ray to  $\bar{B}_{i+1}$ .

Denote by  $z_i$  the intersection of  $U_i$  with  $K_i$ . The choice of  $y_i$  guarantees that  $z_i \notin K(x_i, y_i)$ . Let  $K(z_i, x_{i+1})$  be the sub-arc of  $K_i$  joining  $z_i$  and  $x_{i+1}$ . It follows that  $K(x_i, y_i)$  and  $K(z_i, x_{i+1})$  have no points in common. Let  $x$  be an interior point of  $K(x_i, y_i)$ .  $\phi(x)$  is the origin of exactly two maximal co-rays to  $A$ . If  $\bar{\phi}(x) \in \bar{D}_i$  lies over  $\phi(x)$  then the co-ray  $\bar{H}_x$  from  $\bar{\phi}(x)$  to  $\bar{B}_i$  and the co-ray  $\bar{U}_x$  from  $\bar{\phi}(x)$  to  $\bar{B}_{i+1}$  lie over the maximal co-rays to  $A$  from  $\phi(x)$ .  $y_i$  was chosen so that  $\bar{U}_x$  cannot intersect  $\bar{K}(\bar{x}_i, \bar{y}_i)$ . Neither can  $\bar{U}_x$  intersect  $\bar{B}_i, \bar{B}_{i+1}, \bar{H}_i$  or  $\bar{U}_i$ .  $\bar{U}_x$  must then intersect  $\bar{K}(\bar{z}_i, \bar{x}_{i+1})$  over  $K(z_i, x_{i+1})$  and  $U_x$  intersects  $K(z_i, x_{i+1})$ . It follows that  $\phi$  restricted to  $K(x_i, y_i)$  is one-to-one and  $L_i = \phi[K(x_i, y_i)]$  is an arc joining  $p$  and  $\phi(y_i)$ .



We know that each  $x \in L_i$  is the origin of exactly two maximal co-rays to  $A$ . One of these,  $H_x$ , intersects  $K(x_i, y_i)$  and the other,  $U_x$ , intersects  $K(z_i, x_{i+1})$ . With  $x \in L_i$  associate  $\sigma(x) = U_x \cap K(z_i, x_{i+1})$ . The continuity of the map  $\sigma: L_i \rightarrow K(z_i, x_{i+1})$  can be shown by a standard argument.  $\sigma(L_i)$  is then a connected sub-set of  $K(z_i, x_{i+1})$  that contains both  $z_i$  and  $x_{i+1}$ . Thus  $\sigma(L_i) = K(z_i, x_{i+1})$  and  $L_i = \phi[K(z_i, x_{i+1})]$ .

Consider the closed region  $V_i$  bounded by  $B_i \cap \bar{S}(p, \gamma_p)$ ,  $K(x_i, y_i)$ ,  $H_i \cap \bar{S}(p, \gamma_p)$ ,  $U_i \cap \bar{S}(p, \gamma_p)$ ,  $K(z_i, x_{i+1})$  and  $B_{i+1} \cap \bar{S}(p, \gamma_p)$ .  $V_i \cap C(A) = L_i$  and  $V = V_1 \cup \dots \cup V_m$  is then the desired closed region.

We note that since  $\gamma_p > 0$  can be arbitrarily small we can find such a  $V$  contained in any neighborhood of  $p$ . This implies that  $C(A)$  is locally arc-wise connected and that the arc-wise connected components of  $C(A)$  are closed in  $C(A)$  and hence are closed in  $R$ .

We conclude this section with some remarks on the applicability of the preceding results.

A  $G$ -surface  $R$  of finite connectivity can be regarded as a subspace of a compact manifold  $M$  of finite genus  $\gamma$ . As such, it is obtained from  $M$  by excluding a finite number of points  $a_i$ . There are simple closed pairwise disjoint geodesic polygons  $P_i$  in  $R$  each of which bounds a closed region  $M_i$  in  $M$  homeomorphic to a disk and containing  $a_i$ , but no other  $a_j$ , in its interior. The set  $T_i = M_i - \{a_i\}$  is then a tube in  $R$ .

Each ray in  $R$  must ultimately lie in one of the tubes  $T_i$  and the preceding results apply to the extent that the universal covering surface has the appropriate properties. For example, if  $R$  has convex capsules then the universal covering surface  $\bar{R}$  is straight. If in addition  $\bar{R}$  has the divergence property then the asymptote relation is transitive. In particular this is the case when  $R$  has nonpositive curvature (see *GG*, pp. 249-50).

**5. The covering map and the co-ray relation.** In view of (7) it is natural to ask when a co-ray to  $\bar{A}$  in  $\bar{R}$  over  $A$  lies over a co-ray to  $A$  in  $R$ . In this section we present some partial answers to this question primarily for use in establishing our principal results in the following section.

Let  $R$  be a  $G$ -surface whose universal covering surface  $\bar{R}$  is straight and let  $A$  with origin  $q$  be a ray in  $R$  which ultimately lies in a tube  $T$ . Fix  $p$  in  $R$  and  $\bar{A}$  in  $\bar{R}$  over  $A$  with origin  $\bar{q}$ . It can be seen from the proof of (6), by applying covering motions of  $\bar{R}$  to the rays  $\bar{A}_i$  if necessary, that if  $B$  is a co-ray from  $p$  to  $A$  then there is a sequence  $x_n \rightarrow p$ , a sequence  $z_n$  on  $A$  with  $qz_n \rightarrow \infty$  and a sequence of segments  $T(x_n, z_n) \rightarrow B$  such that if  $\bar{z}_n$  on  $\bar{A}$  lies over  $z_n$  then the segments  $T(\bar{x}_n, \bar{z}_n)$  over  $T(x_n, z_n)$  converge to a co-ray  $\bar{B}$  to  $\bar{A}$  over  $B$ .

If  $\bar{p}$  is the origin of  $\bar{B}$  then  $\bar{p}$  lies over  $p$  and since  $\bar{x}_n\bar{z}_n = x_nz_n$  we have  $\alpha(\bar{A}, \bar{p}) = \alpha(A, p)$  (see *GG*, p. 31).

**LEMMA 11.** *Let  $A$  be a ray in a  $G$ -surface  $R$ . If the universal covering surface  $\bar{R}$  is straight and if  $A$  ultimately lies in a tube then for any  $p \in R$  and any ray  $\bar{A}$  in  $\bar{R}$  over  $A$  there is no point  $\bar{p}_1 \in \bar{R}$  over  $p$  with  $\alpha(\bar{A}, \bar{p}_1) < \alpha(A, p)$ .*

*Proof.* Assume otherwise. Let  $\bar{p}, x_n, \bar{x}_n$  and  $\bar{z}_n$  be as above and let  $\bar{t}_n \rightarrow \bar{p}_1$  where  $\bar{t}_n$  lies over  $x_n$ . For sufficiently large  $n$ ,  $\bar{t}_n\bar{z}_n < \bar{x}_n\bar{z}_n$  which contradicts the assumption that  $T(\bar{x}_n, \bar{z}_n)$  lies over a segment.

We present now a sufficient condition for a co-ray to  $\bar{A}$  in  $\bar{R}$  to lie over a co-ray to  $A$  in  $R$ .

**THEOREM 12.** *Let  $R$  be a  $G$ -surface whose universal covering surface is straight and let the ray  $A$  ultimately lie in a tube. For any  $p \in R$  and any ray  $\bar{A}$  in  $\bar{R}$  over  $A$ , if  $\bar{p} \in \bar{R}$  lies over  $p$  and  $\alpha(\bar{A}, \bar{p}) = \alpha(A, p)$  then the co-ray  $\bar{B}$  from  $\bar{p}$  to  $\bar{A}$  lies over a co-ray  $B$  to  $A$ .*

*Proof.* We show first that  $\bar{B}$  lies over a ray  $B$ . If  $\bar{x} \in \bar{B}$  then  $\bar{p}$  is a foot of  $\bar{x}$  on the limit sphere  $K_\infty(\bar{A}, \bar{p})$  (see *GG*, p. 135). Since no points over  $p$  are interior to  $K_\infty(\bar{A}, \bar{p})$ ,  $T(\bar{p}, \bar{x})$  lies over a segment. Since  $\bar{x} \in \bar{B}$  is arbitrary,  $\bar{B}$  lies over a ray.

Let  $x \neq p$  be any point of  $B$  and let  $\bar{x}$  on  $\bar{B}$  lie over  $x$ . Choose  $z_n$  on  $A$  with  $qz_n \rightarrow \infty$  and let  $\bar{z}_n$  on  $\bar{A}$  lie over  $z_n$ . Since  $\bar{R}$  is straight,  $T(\bar{p}, \bar{z}_n)$  converges to  $\bar{B}$ . For sufficiently large  $n$  we can choose  $\bar{x}_n$  in  $T(\bar{p}, \bar{z}_n)$  such that  $\bar{p}\bar{x}_n = \bar{p}\bar{x} = px$ . Then  $\bar{x}_n \rightarrow \bar{x}$  and, letting  $\bar{x}_n$  lie over  $x_n, x_n \rightarrow x$ .

We then have the following:

(a)  $\text{Limit } (\bar{p}\bar{z}_n - pz_n) = 0$  since  $\bar{z}_n\bar{q} = z_nq$  and  $\text{limit } (\bar{p}\bar{z}_n - \bar{z}_n\bar{q}) = \alpha(\bar{A}, \bar{p}) = \alpha(A, p) = \text{limit } (pz_n - z_nq)$ .

(b)  $\text{Limit } (\bar{p}\bar{x}_n - px_n) = 0$  since  $\bar{p}\bar{x}_n \rightarrow \bar{p}\bar{x} = px$  and  $px_n \rightarrow px$ .

(c)  $\bar{x}_n\bar{z}_n = \bar{p}\bar{z}_n - \bar{p}\bar{x}_n$  and  $x_nz_n \geq pz_n - px_n$ .

From (c) we have  $0 \leq \bar{x}_n\bar{z}_n - x_nz_n = (\bar{p}\bar{z}_n - \bar{p}\bar{x}_n - x_nz_n) \leq \bar{p}\bar{z}_n - \bar{p}\bar{x}_n - pz_n + px_n = (\bar{p}\bar{z}_n - pz_n) - (\bar{p}\bar{x}_n - px_n)$ .

This inequality in conjunction with (a) and (b) yields  $\text{limit } (\bar{x}_n\bar{z}_n - x_nz_n) = 0$ . We then have  $\alpha(A, x) = \text{limit } (x_nz_n - z_nq) = \text{limit } [(\bar{x}_n\bar{z}_n - z_nq) + (x_nz_n - \bar{x}_n\bar{z}_n)] = \text{limit } (\bar{x}_n\bar{z}_n - z_nq) = \alpha(\bar{A}, \bar{x}) = \alpha(\bar{A}, \bar{p}) - \bar{p}\bar{x} = \alpha(A, p) - px$ . The assertion then follows from a result of Busemann (see *GG*, p. 136).

In his thesis (University of Southern California, 1970) the author believed he had carried the above line of reasoning further and obtained, under the hypothesis of (12), a negative answer to a still unsolved problem of Busemann: can a maximal co-ray be a proper sub-ray of another ray? Unfortunately this assertion with its implica-

tion of transitive co-rays in a certain class of  $G$ -surfaces was reported in Busemann [2, p. 89 (13) and p. 90 (15)] before an error in the proof was discovered by the author.

Let  $P$  be a geodesic polygon that bounds  $T$  the tube containing  $A$ . We may assume without loss of generality that  $P$  contain  $q$  but no other points of  $A$ . Given  $\bar{q}$  in  $\bar{R}$  over  $q$ , there is exactly one ray  $\bar{A}$  over  $A$  with origin  $\bar{q}$  and exactly one geodesic polygon  $\bar{P}$  over  $P$  with initial point  $\bar{q}$ . The end-point  $\bar{q}'$  of  $\bar{P}$  also lies over  $q$  and is the origin of exactly one ray  $\bar{A}'$  over  $A$ .  $\bar{A}$ ,  $\bar{P}$  and  $\bar{A}'$  bound a simply connected region  $\bar{T}$  over  $T$  on the interior of which the covering map is one-to-one.

**PROPOSITION 13.** *Let  $\lambda$  denote the length of  $P$  and hence of  $P$ . If  $\bar{p}$  lies in the interior of  $\bar{T}$  with  $\bar{p}\bar{A} < \bar{p}\bar{P} - \lambda$  then the co-ray  $\bar{B}$  from  $\bar{p}$  to  $\bar{A}$ , the co-ray  $\bar{B}'$  from  $\bar{p}$  to  $\bar{A}'$  or both lie over a co-ray to  $A$ .*

*Proof.* Let  $z_n$  be a sequence on  $A$  with  $qz_n \rightarrow \infty$  and let  $\bar{z}_n$  on  $\bar{A}$  lie over  $z_n$ . Assume that  $\bar{z}_n'$  exterior to  $\bar{T}$  also lies over  $z_n$  and that  $T(\bar{p}, \bar{z}_n')$  lies over a segment.  $T(\bar{p}, \bar{z}_n')$  can intersect neither  $\bar{A}$  nor  $\bar{A}'$  and so must intersect  $\bar{P}$ . Since  $T(\bar{z}_n, \bar{q})$  lies over the unique segment  $T(z_n, q)$ ,  $\bar{p}\bar{z}_n' \geq \bar{z}_n'\bar{P} + \bar{p}\bar{P} \geq \bar{z}_n'\bar{q} - \lambda + \bar{p}\bar{P} > \bar{z}_n\bar{q} - \lambda + \bar{p}\bar{P}$  for all  $n$ . On the other hand for sufficiently large  $n$ ,  $\bar{p}\bar{z}_n \leq \bar{p}\bar{A} + \bar{z}_n\bar{q} < \bar{p}\bar{P} - \lambda + \bar{z}_n\bar{q}$ . Thus for sufficiently large  $n$ ,  $T(\bar{p}, \bar{z}_n')$  does not lie over a segment.

Let  $\bar{z}_n$  on  $\bar{A}'$  lie over  $z_n$ . For sufficiently large  $n$  either  $T(\bar{p}, \bar{z}_n)$ ,  $T(\bar{p}, \bar{z}_n')$  or both lie over a segment. It follows that  $\alpha(A, p) = \alpha(\bar{A}, \bar{p})$ ,  $\alpha(A, p) = \alpha(\bar{A}', \bar{p})$  or both. The proposition then follows from (12).

Observe that the conflicting inequalities arise because  $T(\bar{p}, \bar{z}_n')$  intersects  $\bar{P}$ . This means that if  $T(\bar{p}, \bar{z}_n')$  lies over a segment then, for sufficiently large  $n$ , it does not intersect  $\bar{P}$ . Thus if  $\bar{B}'$  lies over a co-ray to  $A$ , then  $\bar{B}'$  does not intersect  $\bar{P}$ . Likewise if  $\bar{B}$  lies over a co-ray to  $A$  then  $\bar{B}$  lies in  $\bar{T}$ .

**DEFINITION 14.** The distance from co-ray to ray is weakly bounded if for a co-ray  $B$  from  $p$  to  $A$  there is a sequence  $x_n$  on  $B$  with  $x_n p \rightarrow \infty$  such that  $x_n A$  is bounded.

In particular the distance from co-ray to ray is weakly bounded in a straight space with convex capsules where, in fact, both  $xA$  and  $yB$  are bounded for  $x \in B$  and  $y \in A$ . An example of Busemann ( $GG$ , p. 137) shows that the latter do not necessarily follow from the distance from co-ray to ray being weakly bounded.

**PROPOSITION 15.** *Let  $R$  be a  $G$ -surface whose universal covering surface  $\bar{R}$  is straight and has the distance from co-ray to ray weakly bounded. Let  $A, \bar{A}, q, \bar{q}, T$  and  $\bar{T}$  be as in (13). Let  $\bar{B}$  be a co-ray to  $\bar{A}$  in  $\bar{R}$  such that a sub-ray of  $\bar{B}$  lies in  $\bar{T}$ . If  $\bar{B}$  lies over a co-ray*

to  $A$  then there is a point  $\bar{x}_0$  on  $\bar{B}$  and a point  $\bar{z}_0$  on  $\bar{A}$  such that the sub-ray of  $\bar{B}$  from  $\bar{x}_0$ , the sub-ray of  $\bar{A}$  from  $\bar{z}_0$  and  $T(\bar{x}_0, \bar{z}_0)$  bound a sub-region of  $\bar{T}$  the co-ray from each point of which to  $\bar{A}$  lies over a co-ray to  $A$ .

*Proof.* We may assume without loss of generality that the origin  $\bar{p}$  of  $\bar{B}$  is exterior to  $\bar{T}$ . There is a sequence  $\bar{x}_n$  in  $\bar{B}$  with  $\bar{x}_n\bar{p} \rightarrow \infty$  and a constant  $M > 0$  such that  $\bar{x}_n\bar{A} < M$  for all  $n$ . Let  $\bar{z}_n$  be a foot of  $\bar{x}_n$  on  $\bar{A}$ . Then  $\bar{z}_n\bar{q} \rightarrow \infty$  and  $\bar{q}\bar{T}(\bar{x}_n, \bar{z}_n) \rightarrow \infty$ .

Choose  $N$  such that for  $n \geq N$  if  $\bar{y} \in T(\bar{x}_n, \bar{z}_n)$  then  $M < \bar{y}\bar{p} - \lambda$ . For each  $n \geq N$ , if  $\bar{y} \in T(\bar{x}_n, \bar{z}_n)$  then the co-ray  $\bar{H}$  from  $\bar{y}$  to  $\bar{A}$  lies over a co-ray to  $A$ . Otherwise the co-ray  $\bar{H}'$  from  $\bar{y}$  to  $\bar{A}'$  lies over a co-ray to  $A$  in which case  $\bar{H}'$  would either co-incide with  $\bar{H}$  or intersect  $\bar{B}$ , which is impossible.

Consider the sub-region of  $\bar{T}$  bounded by  $T(\bar{x}_N, \bar{z}_N)$ , the sub-ray of  $\bar{B}$  from  $\bar{x}_N$  and the sub-ray of  $\bar{A}$  from  $\bar{z}_N$ . Let  $\bar{y}$  be any point in the interior of this sub-region. For sufficiently large  $n > N$ ,  $\bar{y}$  is in the interior of the region bounded by  $T(\bar{x}_N, \bar{z}_N)$ ,  $T(\bar{x}_N, \bar{x}_n)$ ,  $T(\bar{z}_N, \bar{z}_n)$  and  $T(\bar{x}_n, \bar{z}_n)$ . Any ray  $\bar{H}$  from  $\bar{y}$  that lies over a co-ray  $A$  must intersect one of these segments and so must be a co-ray to  $\bar{A}$ .

Slight modification of the preceding proof yields the following:

**PROPOSITION 16.** *Under the assumptions of (15), if  $\bar{R}$  has a transitive asymptote relation and  $\bar{A}$  and  $\bar{A}'$  are co-rays to each other then there are points  $\bar{z}$  and  $\bar{z}'$  on  $\bar{A}$  and  $\bar{A}'$  respectively such that the sub-ray of  $\bar{A}$  from  $\bar{z}$ , the sub-ray of  $\bar{A}'$  from  $\bar{z}'$  and  $T(\bar{z}, \bar{z}')$  bound a sub-region of  $\bar{T}$  the co-ray from any point of which to  $\bar{A}$  lies over a co-ray to  $A$ .*

We also obtain a result of Nasu [5].

**COROLLARY 17.** *Under the assumptions of (16), if  $\bar{A}$  and  $\bar{A}'$  are co-rays to each other then there is a sub-tube of  $T$  disjoint from  $C(A)$ . If  $\bar{A}$  and  $\bar{A}'$  are not co-rays to each other then no sub-tube of  $T$  is disjoint from  $C(A)$ .*

*Proof.* The first assertion is a direct consequence of (8) and (16). To prove the second assertion let  $\bar{x}(t)$ ,  $0 \leq t \leq 1$ , be any curve in  $\bar{T}$  with  $\bar{x}(0)$  in  $\bar{A}$  and  $\bar{x}(1)$  in  $\bar{A}'$ . There is then a largest value of  $t$ ,  $t_0$ , such that for  $0 \leq t \leq t_0$  the co-ray from  $\bar{x}(t)$  to  $\bar{A}$  lies over a co-ray to  $A$ . Since  $\bar{A}$  and  $\bar{A}'$  are not co-rays to each other  $0 < t_0 < 1$ . Then  $\bar{x}(t_0)$  is the origin of two rays lying over co-rays to  $A$ .

6. The structure of  $C(A)$  in a class  $G$ -surfaces. In this section

we analyze  $C(A)$  in case  $R$  is a  $G$ -surface of finite connectivity with a straight universal covering surface with a transitive asymptote relation and the distance from co-ray to ray weakly bounded. We have mentioned previously that this includes all  $G$ -surfaces of finite connectivity whose universal covering surface has transitive asymptotes, and hence includes all  $G$ -surfaces of finite connectivity with non-positive curvature.

The following consequence of (16) is basic to our analysis.

**PROPOSITION 18.** *Let  $R$  be a  $G$ -surface of the above type. If  $A$  is a ray in  $R$  then  $C(A)$  does not separate  $R$ .*

*Proof.* Since  $R$  has finite connectivity,  $A$  lies ultimately in a tube  $T$ . Assume that the proposition is false.  $C(A)$  then separates  $R$  into at least two components. We consider two cases.

(i) The tube  $T$  or a sub-tube is contained in one of the components. Consider a point  $x$  in a different component. A co-ray  $B$  from  $x$  to  $A$  has a sub-ray contained in  $T$ . Thus  $B$  intersects  $C(A)$  which is impossible.

(ii) None of the components of  $R - C(A)$  contains a sub-tube of  $T$ . Assume without loss of generality that the initial point  $q$  of  $A$  is not in  $C(A)$ . Then  $A$  lies in one of the components. Choose  $x \in T$  in a component that does not contain  $A$ . Then the co-ray  $B$  from  $x$  to  $A$  has a sub-ray contained in  $T$ .

Let  $P$  be the simple closed geodesic polygon bounding  $T$ . We may assume that  $q$  is the initial point of  $P$ . Choose  $\bar{q} \in \bar{R}$  over  $q$  and let  $\bar{A}$  and  $\bar{P}$ , each with initial point  $\bar{q}$ , lie over  $A$  and  $P$ . The end-point  $\bar{q}'$  of  $\bar{P}$  lies over  $q$ . Let  $\bar{A}'$  with initial point  $\bar{q}'$  lie over  $A$ . Then  $\bar{A}$ ,  $\bar{P}$  and  $\bar{A}'$  bound a simply connected region  $\bar{T}$  over  $T$ .

Let  $\bar{B}$  with a sub-ray in  $\bar{T}$  lie over  $B$ .  $\bar{B}$  is a co-ray to  $\bar{A}'$  or  $\bar{A}$ , say  $\bar{A}$  to be definite. It follows from (16) that there is a segment  $T(\bar{x}_0, \bar{y}_0)$  in  $\bar{T}$  joining  $\bar{B}$  to  $\bar{A}$  no point of which lies over a co-ray to  $A$ . Thus  $C(A)$  does not separate  $B$  and  $A$  which is a contradiction.

It was mentioned at the end of §4 that a  $G$ -surface of finite connectivity can be regarded as a sub-space of a compact manifold  $M$  of finite genus  $\gamma$ . As such it is obtained from  $M$  by the removal of a finite number of points  $a_i, 1 \leq i \leq N$ , each of which corresponds to a tube  $T_i$  bounded by  $P_i$ , a simple geodesic polygon in  $R$ .

**DEFINITION 19.** Given a ray  $A$  in  $R$ , a  $G$ -surface of finite connectivity, denote by  $C^*(A)$  the closure of  $C(A)$  relative to  $M$ .  $C(A)$  is said to occupy a tube  $T_j$  if the point  $a_j$  in  $M$  that determines  $T_j$  is in  $C^*(A)$ . Similarly a component  $C_i^*(A)$  of  $C^*(A)$  occupies  $T_j$  if  $a_j$  is in  $C_i^*(A)$ .

We note that  $C^*(A)$  is obtained from  $C(A)$  by adjoining those  $a_j$  that corresponds to tubes occupied by  $C(A)$ .

A sufficiently small deleted neighborhood of  $a_j$  in  $C^*(A)$  can be thought of as a sub-tube of  $T_j$ . The next theorem extends (10) to include those  $a_j$  in  $C^*(A)$ . Ultimately this allows us to assert that  $C^*(A)$  is triangulable as a one dimensional simplicial complex, a fact from which we derive the principal results of this section.

**THEOREM 20.** *Let  $R$  be a  $G$ -surface of finite connectivity with a straight universal covering surface  $\bar{R}$  having a transitive asymptote relation and the distance from co-ray to ray weakly bounded. Let  $A$  be a ray in  $R$  and let  $T$  be a tube occupied by  $C(A)$ . The geodesic polygon  $P$  bounding  $T$  can be chosen so that  $C(A) \cap T$  consists of a finite number of disjoint unbounded arcs emanating from  $P$ .*

*Proof.* We consider the case that  $T$  is not the tube that contains  $A$  or a sub-ray thereof. Let  $y_n$  be an unbounded sequence in  $T$ . Let  $B_n$  be a co-ray from  $y_n$  to  $A$ . Each  $B_n$  intersects  $P$  in a first point  $x_n$ . Since  $P$  is compact, the sequence  $x_n$  is bounded and a sub-sequence of  $B_n$  converges to an asymptote  $L^+$  to  $A$ . Let  $q$  be the first point in which  $L^+$  intersects  $P$  and let  $H$  be the negative sub-ray of  $L^+$  with origin  $q$ . We note that  $H$  is contained in  $T$  and has only  $q$  in common with  $P$ .

In  $\bar{R}$ , the universe covering surface, choose a simply connected region  $\bar{T}$  over  $T$ , bounded by  $\bar{H}$  and  $\bar{H}'$  over  $H$  with initial points  $\bar{q}$  and  $\bar{q}'$  respectively, and by  $\bar{P}$  over  $P$  with  $\bar{q}$  and  $\bar{q}'$  as initial and final points. We note that the covering map is one-to-one on the interior of  $\bar{T}$ .

Assume without loss of generality that  $\bar{P}$  is a segment. Then any ray that lies over a co-ray to  $A$  can intersect  $\bar{P}$  at most once. Furthermore any ray lying over a co-ray to  $A$  that intersects  $\bar{P}$  must originate from  $\bar{T}$ . This, with (10), implies that  $\bar{P}$  contains at most a finite number of points that lie over co-points to  $A$ .

We show that  $\bar{P}$  can be replaced by a geodesic polygon  $\bar{P}'$  bounding a sub-region  $\bar{T}'$  of  $\bar{T}$  that lies over a sub-tube  $T'$  of  $T$  and is such that those points of  $\bar{T}'$  that lie over  $C(A) \cap T'$  form a finite number of unbounded disjoint arcs emanating from  $\bar{P}'$ .

Since  $\bar{P}$  is compact it can be covered by a finite number of neighborhoods of the type in (10). It follows that there are a finite number of rays  $\bar{A}_i$ ,  $1 \leq i \leq m$ , over  $A$ , none a co-ray to any other, to one of which any ray from a point of  $\bar{P}$  that lies over a co-ray to  $A$  must be a co-ray. Furthermore, if  $\bar{x} \in \bar{P}$  does not lie over a co-point to  $A$ , and if the co-ray from  $\bar{x}$  to  $\bar{A}_j$  lies over a co-ray to  $A$  then by

(10) there is a sub-segment of  $\bar{P}$  the co-ray from any point of which to  $\bar{A}_j$  lies over a co-ray to  $A$ .

Hence  $\bar{P}$  can be partitioned into a finite number of non-overlapping segments  $I_1, \dots, I_k$  whose end-points are either end-points of  $\bar{P}$  or lie over co-points to  $A$ . For each  $I_i$  there is an  $\bar{A}_{j(i)}$  the co-ray to which from any point of  $I_i$  lies over a co-ray to  $A$ .

In  $I_i$  suppose two points  $\bar{x}$  and  $\bar{y}$  the asymptotes through which to  $\bar{A}_{j(i)}$  lie over asymptotes to  $A$  and are denoted by  $\bar{B}(\bar{x})$  and  $\bar{B}(\bar{y})$  respectively. A ray lying over a co-ray to  $A$  in the strip bounded by  $\bar{B}(\bar{x})$  and  $\bar{B}(\bar{y})$  cannot intersect either  $\bar{B}(\bar{x})$  or  $\bar{B}(\bar{y})$ . Such a ray must be a co-ray to  $\bar{A}_{j(i)}$ . Thus the asymptotes to  $\bar{A}_{j(i)}$  through points of  $I_i$  between  $\bar{x}$  and  $\bar{y}$  lie over asymptotes to  $A$ . This implies that those points of  $I_i$  the asymptotes through which to  $A_{j(i)}$  lie over asymptotes to  $A$  form a sub-segment of  $I_i$ . We note that this sub-segment might consist of a single point or be empty.

It follows that  $\bar{P}$  contains a finite number of disjoint segments  $K_0, \dots, K_r$  whose points are the points of  $\bar{P}$  that lie on straight lines that lie over asymptotes to  $A$ . Let  $\bar{x}_i$  and  $\bar{y}_{i+1}$ ,  $0 \leq i \leq r$ , denote the end-points of  $K_i$  indexed so that  $\bar{q} = \bar{x}_0$ ,  $\bar{q}' = \bar{y}_{r+1}$  and  $K_{i+1}$  follows  $K_i$  on  $\bar{P}$ . Let  $J_i$ ,  $1 \leq i \leq r$ , denote the sub-segment of  $\bar{P}$  joining  $\bar{y}_i$  and  $\bar{x}_i$ . Then  $\bar{P} = K_0 \cup J_1 \cup K_1 \cup \dots \cup J_r \cup K_r$ .

We will alter  $\bar{P}$  by altering the segments  $J_i$ . Each point  $\bar{z}$  in  $J_i$  determines a unique point  $\phi(\bar{z})$  in  $\bar{T}$  that lies over a co-point to  $A$ . By (4)  $\phi$  is continuous. Consider in  $J_i$  a sequence  $\bar{z}_n \rightarrow \bar{x}_i$ . For sufficiently large  $n$  the co-ray  $\bar{B}_n$  from  $\phi(\bar{z}_n)$  to  $\bar{A}_{j(i)}$  lies over a co-ray to  $A$ , and the sequence  $\bar{B}_n$  converges to the asymptote to  $\bar{A}_{j(i)}$  through  $\bar{x}_i$ . Furthermore, since  $C(A)$  is closed in  $R$ ,  $\phi(\bar{z}_n)\bar{x}_i \rightarrow \infty$ .

$\phi(\bar{z}_n)$  is the origin of at least one other ray  $\bar{B}'_n$  that lies over a co-ray to  $A$ . Since  $\phi(\bar{z}_n)\bar{x}_i \rightarrow \infty$ , a sub-sequence of  $\bar{B}'_n$  converges to an oriented straight line lying over an asymptote to  $A$ . The only possibility for the latter is the asymptote to  $\bar{A}_{j(i-1)}$  through  $\bar{y}_i$ .

Thus if  $\bar{z}_i$  in  $J_i$  is chosen sufficiently close to  $\bar{x}_i$  then  $\phi(\bar{z}_i)$  is the origin of exactly two rays that lie over co-rays to  $A$ : the co-ray  $\bar{B}_i$  to  $\bar{A}_{j(i)}$  and the co-ray  $\bar{B}'_i$  to  $\bar{A}_{j(i-1)}$ . Let  $\bar{z}'_i$  denote the intersection of  $\bar{B}'_i$  with  $J_i$ . If  $\bar{z}_i$  is sufficiently close to  $\bar{x}_i$  then the image of  $T(\bar{z}_i, \bar{x}_i) - \{\bar{x}_i\}$  under  $\phi$  coincides with that of  $T(\bar{y}_i, \bar{z}'_i) - \{\bar{y}_i\}$  under  $\phi$ , and their common image is an unbounded arc emanating from  $\phi(\bar{z}_i)$ .

We replace  $J_i$  with  $J'_i = T(\bar{y}_i, \bar{z}'_i) \cup T(\bar{z}'_i, \phi(\bar{z}_i)) \cup T(\phi(\bar{z}_i), \bar{z}_i) \cup T(\bar{z}_i, \bar{x}_i)$ . When this is done for each  $i$ ,  $1 \leq i \leq r$ , we have the desired geodesic polygon  $\bar{P}'$ .

The case in which  $T$  contains  $A$  or a sub-ray thereof is treated in a similar manner (although it involves a few more details).

Let  $T_i = M_i - \{a_i\}$  where  $M_i$  is homeomorphic to a closed disk

and contains  $a_i$  in its interior. If  $T_i$  is occupied by  $C(A)$  it is clear from (20) that  $P_i$  may be chosen so that  $M_i$  is homeomorphic to a closed disk in such a way that  $a_i$  corresponds to the center and  $C^*(A) \cap M$  to a finite number of radii. On the other hand, if  $T_i$  is not occupied by  $C(A)$  then, by (17),  $P_i$  may be chosen so that  $T_i$  is disjoint from  $C(A)$ . If  $\text{Int } T_i$  denotes the interior of  $T_i$  then  $R - \cup \text{Int } T_i$  is compact and  $C(A) \cap (R - \cup \text{Int } T_i)$  can be covered by a finite number of neighborhoods of the type in (10). The following then holds.

**COROLLARY 21.** *Let  $A$  be a ray in  $R$ .  $C^*(A)$  is triangulable as a one dimensional simplicial complex.*

**DEFINITION 22.** Given a ray  $A$  in  $R$  and  $p$  a co-point to  $A$  denote by  $m(p)$  the number of co-rays from  $p$  to  $A$  minus two. If  $m(p) > 0$  then  $p$  is called a multiple co-point to  $A$ .

It is clear that in any triangulation of  $C^*(A)$  the multiple co-points will be vertices. In particular (20) implies that  $C(A)$  contains only a finite number of such points. We now state the principal result of this section.

**THEOREM 23.** *Let  $R$  be an orientable (non-orientable)  $G$ -surface of finite connectivity with a straight universal covering surface  $\bar{R}$  having a transitive asymptote relation and the distance from co-ray to ray weakly bounded. If  $A$  is a ray in  $R$ , let  $\pi(A)$  denote the number of components of  $C(A)$ ,  $\mu(A)$  the number of multiple co-points to  $A$ ,  $N$  the number of tubes in  $R$  and  $\gamma$  the genus of  $M$ , the compact surface of which  $R$  is a subspace. It then follows that  $\mu(A) \leq N - 2 + 2\gamma$  ( $\mu(A) \leq N - 2 + \gamma$ ),  $\pi(A) \leq N - 1 + 2\gamma$  ( $\pi(A) \leq N - 1 + \gamma$ ) and no co-point to  $A$  is the origin of more than  $N + 2\gamma(N + \gamma)$  co-rays to  $A$ .*

*Proof.* We assume that  $C(A) \neq \emptyset$ . Let  $\beta_0$  and  $\beta_1$  be the first two Betti numbers of  $C^*(A)$ . The Euler-Poincaré characteristic of  $C^*(A)$  is  $\chi(C^*(A)) = \beta_0 - \beta_1$ .

$\beta_0 = \sigma(A)$ , the number of components of  $C^*(A)$ .  $C^*(A)$  can be regarded as a subcomplex of  $M$  which is likewise triangulable. Since  $C(A)$  does not separate  $R$ ,  $C^*(A)$  does not separate  $M$  and no one cycle in  $C^*(A)$  bounds in  $M$ . Thus  $\beta_1 \leq 2\gamma$ . In the non-orientable case a bounding one cycle in  $C^*(A)$  would correspond to a torsion element in  $H_1(M)$  and the inequality is  $\beta_1 \leq \gamma$ .

Consider a component  $C_i^*(A)$  of  $C^*(A)$ . Let  $\pi_i$  be the number of components of  $C(A)$  included in  $C_i^*(A)$ , let  $\Delta_i$  be the number of tubes occupied by  $C_i^*(A)$  and let  $p(i, j)$ ,  $1 \leq j \leq \delta_i$ , be the multiple co-points in  $C_i^*(A)$ . The Euler-Poincaré characteristic of  $C^*(A)$  is  $\chi(C_i^*(A)) = (\Delta_i + \delta_i) - (\delta_i + \pi_i + \sum_j m(p(i, j)))$ ,  $1 \leq j \leq \delta_i$ .



Summing over  $i = 1, \dots, \sigma(A)$  we obtain  $\chi(C^*(A)) = \sum_i \Delta_i - (\pi(A) + \sum_i \sum_j m(p(i, j))) = \beta_0 - \beta_1 \geq \sigma(A) - 2\gamma$ .

Then  $2\gamma + N \geq 2\gamma + \sum \Delta_i \geq \sigma(A) + \pi(A) + \sum_i \sum_j m(p(i, j))$ . This yields two inequalities:  $2\gamma + N \geq \sigma(A) + \pi(A) + \mu(A)$  and  $2\gamma + N \geq \sigma(A) + \pi(A) + \mu(A) - 1 + \max m(p(i, j))$ .

Finally we obtain  $2\gamma + N \geq 1 + \pi(A)$ ,  $2\gamma + N \geq 2 + \mu(A)$  and  $2\gamma + N \geq 2 + \max m(p(i, j))$  which yield the desired results. In the non-orientable case  $2\gamma$  is replaced by  $\gamma$  in the preceding inequalities.

## REFERENCES

1. H. Busemann, *The Geometry of Geodesics*, Academic Press, New York, 1955.
2. ———, *Recent Synthetic Differential Geometry*, Springer-Verlag, Berlin, 1970.
3. P. Hilton and S. Wylie, *Homology Theory*, Cambridge University Press, London and New York, 1960.
4. Y. Nasu, *On asymptotic conjugate points*, Tohoku Math. J., **7** (1956), 157-165.
5. ———, *On asymptotes in a metric space with non-positive curvature*, Tohoku Math. J., **9** (1957), 68-95.
6. ———, *On asymptotes in a 2-dimensional metric space*, *Tensor* **7** (1957), 173-184.

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