A COMPLETE COUNTABLE $L^0_{\omega_1}$ THEORY WITH MAXIMAL MODELS OF MANY CARDINALITIES

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A COMPLETE COUNTABLE $L_{\omega_1}^\infty$ THEORY WITH MAXIMAL MODELS OF MANY CARDINALITIES

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Because of the compactness of first order logic, every structure has a proper elementarily equivalent extension. However, in the countably compact language $L_{\omega_1}^\infty$ obtained from first order logic by adding a new quantifier $Q$ and interpreting $Qx$ as "there are at least $\omega_1$ $x$'s such that . . .," the situation is radically different. Indeed there are structures of countable type which are maximal in the sense of having no proper $L_{\omega_1}^\infty$-extensions, and the class $S$ of cardinals admitting such maximal structures is known to be large. Here it is shown that there is a countable complete $L_{\omega_1}^\infty$ theory $T$ having maximal models of cardinality $\kappa$ for each $\kappa \geq 2_1$ which is in $S$. The problem of giving a complete characterization of the maximal model spectra of $L_{\omega_1}^\infty$ theories $T$ remains open: what classes of cardinals have the form $Sp(T) = \{\kappa: \text{there is a maximal model of } T \text{ of cardinality } \kappa\}$ for $T$ a (complete, countable) $L_{\omega_1}^\infty$ theory.

That $S$ is large is shown in [4]. Assuming the $GCH$, it is particularly simple to describe: $S$ is the set of uncountable cardinals which are less than the first uncountable measurable cardinal and not weakly compact. Here we will need the fact that $2_1 \in S$; this is proved in [4] without assuming the $GCH$. The countable compactness of $L_{\omega_1}^\infty$ is shown in Fuhrken [2]. For additional results and references on the model theory of $L_{\omega_1}^\infty$ see Kiesler [3].

1. Notation and preliminaries.

1.1. Relatively common notation. We identify cardinals with initial ordinals, and each ordinal with the set of smaller ordinals. We use $\alpha, \beta, \gamma$ for ordinals, $\kappa, \lambda, \mu$ for cardinals, and $m, n$ for finite cardinals. $S(X) = \{t: t \subseteq X\}$; $cX$ is the cardinality of $X$; $2_1$ is the cardinality of the continuum; $\omega_1$ the first uncountable cardinal; $\Pi_{i \in Y} X_i$ the cartesian product; $Y^X$ the set of all functions on $Y$ into $X$, $f | x$ the restriction of the function $f$ to $x$.

The type $\tau \Sigma$ of a set $\Sigma$ of formulas is the set of non-logical symbols occurring $\Sigma$.

In this paper all structures will be relational structures. Capital german letters are used for structures, and the corresponding roman letters for their universes. Alternatively we may write $|\mathfrak{A}|$ for the universe of $\mathfrak{A}$. The type $\tau \mathfrak{A}$ of $\mathfrak{A}$ is the set of non-logical symbols.
having denotations in $\mathfrak{A}$, so that $\mathfrak{A} = \langle A, S^a \rangle_{a \in \mathfrak{A}}$. We use sans serif letters for non-logical symbols, and if $\mathfrak{A}$ is understood we may use roman letters for the corresponding denotations, so $S = S^a$. If $S$ is a relation with rank $n + 1$, and the last argument is a function of the first $n$ places, we call $S$ a function. If $R_i (i \in I)$ are relations, then $(\mathfrak{A}, R_i)_{i \in I}$ is a structure $\mathfrak{B}$ which results from $\mathfrak{A}$ by extending the type of $\mathfrak{A}$ to include new relation symbols $R_i (i \in I)$, where $R_i^\mathfrak{B}$ is the relation $R_i$ (appropriately restricted to $A$).

The phrase “$\kappa$ admits a structure such that...” means “there is a structure $\mathfrak{A}$ such that $c | \mathfrak{A} | = \kappa$ and...”

1.2. Less common notation, special sums and products. As usual $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$ mean respectively that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$, $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$. Similarly $\mathfrak{A} \equiv_{w_1} \mathfrak{B}$ means that $\mathfrak{A}, \mathfrak{B}$ are $L_{w_1}^\mathfrak{A}$-equivalent, i.e. that $\mathfrak{A}, \mathfrak{B}$ have the same true $L_{w_1}^\mathfrak{A}$-sentences, and $\mathfrak{A} \prec_{w_1} \mathfrak{B}$ means that $\mathfrak{A}$ is an $L_{w_1}^\mathfrak{A}$-substructure of $\mathfrak{B}$, i.e. that $\mathfrak{A} \subseteq \mathfrak{B}$ and for every $L_{w_1}^\mathfrak{A}$ formula $\theta$, and every assignment $z$ in $\mathfrak{A}$, $\mathfrak{A} \models \theta[z]$ iff $\mathfrak{B} \models \theta[z]$. If $K$ is a class of structures, $Th_{w_1} K$ is the set of $L_{w_1}^\mathfrak{A}$-sentence true in every $\mathfrak{A} \in K$. If $\Sigma$ is a set of sentences, $Mod \Sigma$ is the class of structures (of some fixed type) such that $\Sigma \subseteq Th_{w_1} \mathfrak{A}$.

Let $t \subseteq \tau \mathfrak{A}$ and let $\phi \neq V \subseteq |\mathfrak{A}|$. Then $\mathfrak{A} | (V, t)$ is the $t$-reduct of the substructure of $\mathfrak{A}$ determined by $V$, i.e. if $\mathfrak{B}$ is the substructure of $\mathfrak{A}$ determined by $V$, $\mathfrak{A} | (V, t)$ is the structure $\mathfrak{B}$ with universe $|\mathfrak{B}|$ and type $t$ determined by $R^\mathfrak{A} = R^\mathfrak{B}$ for $R$ in $t$. We write $\mathfrak{A} | t$ for $\mathfrak{A} | (|\mathfrak{A}|, t)$. If $V$ is a unary relation symbol, then we will write $\mathfrak{A} | (V, t)$ for (the relativized reduct) $\mathfrak{A} | (V^\mathfrak{A}, t)$.

If $t$ is a relational type, we can find a relational type $t^* \supseteq t$, and a set $Sk(t)$ of first order sentences of type $t^*$ with the following properties: (i) if $\tau \mathfrak{A} = t$, then there is an expansion $\tau \mathfrak{A}^*$ of $\mathfrak{A}$ with $\tau \mathfrak{A}^* = t^*$ and $\mathfrak{A}^* \in Mod Sk(t)$ (ii) if $\mathfrak{A}, \mathfrak{B} \in Mod Sk(t)$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$. In fact we may take $Sk(t)$ to be the set of sentences which assert that the Skolem relations satisfy their defining sentences, e.g.

$$\forall z[\forall y(R_\theta(x, y) \longrightarrow \theta(x, y)) \land (\exists y\theta(x, y) \longrightarrow \exists yR_\theta(x, y))]$$

If $\langle \mathfrak{A}_i : i \in I \rangle$ is a family of relational structures all of type $t$, and having pairwise disjoint universes, then $\Sigma_{t \in I} \mathfrak{A}_i$ is the structure $\mathfrak{B}$ of type $t$ such that $B = \bigcup_{i \in I} A_i$, and $R^\mathfrak{B} = \bigcup_{i \in I} R_i^\mathfrak{A}_i$ for each $R \in t$. If the universes of the $\mathfrak{A}_i$ are not disjoint, then $\Sigma_{t \in I} \mathfrak{A}_i$ is $\Sigma_{t \in I} \mathfrak{A}_i$, where $\mathfrak{A}_i$ is some isomorphic copy of $\mathfrak{A}_i$, and the universes of the $\mathfrak{A}_i$ are pairwise disjoint. If $\mathfrak{A}_i$ and $\mathfrak{A}_k$ have different types, $\mathfrak{A}_i \oplus \mathfrak{A}_k$ is defined as follows. First expand each to a structure of type $\tau \mathfrak{A}_i \cup \tau \mathfrak{A}_k$ by adding empty relations, to obtain $\mathfrak{A}_i', \mathfrak{A}_k'$ respectively. Then
Let \( \langle \mathcal{A}_i : i \in I \rangle \) be a family of structures, with \( \tau\mathcal{A}_i = t_i \). Choose \( t'_i = \{ R^i : R \in t_i \} \) pairwise disjoint copies of the \( t_i \) (i.e. \( R \mapsto R' \) is \( 1 \to 1 \) and \( R, R' \) have the same rank). Let \( \mathcal{A}_i (i \in I) \) be new unary relation symbols. Define \( \mathcal{B} = \mathcal{F}(\mathcal{A}_i : i \in I) \) of type \( t = \{ A_i : i \in I \} \cup \bigcup \{ t'_i : i \in I \} \) as follows: \( |\mathcal{B}| = \bigcup_{i \in I} A_i, A^*_i = A_i, \) and \( (R_i)^* = R'^i \).

Define \( \mathcal{P}_{i \in D} (\mathcal{A}_i, \mathcal{D}) = (\mathcal{F}(\mathcal{D}, \sum_{i \in D} \mathcal{A}_i), K) \), where \( K = \{ (x, i) : i \in D \) and \( x \in [\mathcal{A}_i] \}\).

**Definition 1.** (a) \( \mathcal{A} \) is maximal iff wherever \( \mathcal{A} \models \mathcal{B} \) and \( \mathcal{A} \equiv_{\omega_1} \mathcal{B} \) then \( \mathcal{A} = \mathcal{B} \).

(b) \( \mathcal{A} \) is strongly maximal iff \( \mathcal{A} = (\mathcal{A}', \theta^\mathcal{A}) \), where \( \mathcal{U} \) is unary, and whenever \( \mathcal{A} \models \mathcal{B}, \mathcal{A} \equiv \mathcal{B}, \) and \( \psi^\mathcal{U} = \mathcal{Y}^\mathcal{A} \), then \( \mathcal{A} = \mathcal{B} \).

(c) \( S \) is the set of cardinals \( \kappa \) which admit a maximal model of countable type; \( S' = \{ \kappa \in S : \kappa \geq 2_1 \} \).

(d) \( Sp(T) = \{ \kappa : \kappa \text{ admits a maximal model of } T \} \)

**Remark.** This notion of strongly maximal is weaker than the notion of strongly maximal introduced in [4], but is all that is needed in this paper.

2. Products and preservation of \( L^\omega_{\omega_1} \)-equivalence. We will need to know that \( L^\omega_{\omega_1} \)-equivalence is preserved under the operations \( \Sigma \) and \( \wp \) defined above. The results we need follow from Wojciechowska’s generalizations of the Feferman-Vaught theorems on generalized products [5]. The following corollary of Wojciechowska’s main theorem will suffice for our purpose. In this corollary, \( \mathcal{D} \) is an expansion of \( \langle S(I), \cup, \sim \rangle \), \( \mathcal{A} = \langle \mathcal{A}_i : i \in I \rangle \) is a family of structures (of fixed type) indexed on \( I \), and \( \mathcal{P}(\mathcal{A}, \mathcal{D}) \) is the Feferman-Vaught generalized product [1].

**Corollary 2.1.** Suppose that \( \mathcal{A}_i \equiv_{\omega_1} \mathcal{B}_i, i \in I \). Then \( \mathcal{P}(\langle \mathcal{A}_i \rangle_{i \in I}, \mathcal{D}) \equiv_{\omega_1} \mathcal{P}(\langle \mathcal{B}_i \rangle_{i \in I}, \mathcal{D}). \) Similarly if \( \mathcal{A}_i \prec_{\omega_1} \mathcal{B}_i, i \in I \) then \( \mathcal{P}(\langle \mathcal{A}_i \rangle_{i \in I}, \mathcal{D}) \prec_{\omega_1} \mathcal{P}(\langle \mathcal{B}_i \rangle_{i \in I}, \mathcal{D}). \)

From this corollary we prove

**Corollary 2.2.** (a) If \( \mathcal{A}_i \equiv_{\omega_1} \mathcal{B}_i \) then \( \sum_{i \in I} \mathcal{A}_i \equiv_{\omega_1} \sum_{i \in I} \mathcal{B}_i \), and if \( \mathcal{A}_i \prec_{\omega_1} \mathcal{B}_i \) then \( \sum_{i \in I} \mathcal{A}_i \prec_{\omega_1} \sum_{i \in I} \mathcal{B}_i \).
(b) If \( \mathcal{A}_i \equiv_{\omega_1} \mathcal{B}_i \) then \( \mathcal{P}_{i \in D} (\mathcal{A}_i, \mathcal{D}) \equiv_{\omega_1} \mathcal{P}_{i \in D} (\mathcal{B}_i, \mathcal{D}). \)

**Proof of (a).** If \( c \in [\mathcal{A}], \) and \( U \) is a unary predicate not in \( \tau\mathcal{A} \), we define \( \mathcal{A}' \) of type \( \tau\mathcal{A} \cup \{ U \} \) by

\[ \mathcal{A}' = (|\mathcal{A}| \cup \{ c \}, A, R^\mathcal{A})_{R \in \tau} \]
In Feferman-Vaught [1] it is shown that the cardinal sum \( \sum_{i \in I} \mathcal{A}_i \) is a (relativized reduct of) a generalized product \( \mathcal{P}(\langle \mathcal{A}_i \rangle_{i \in I}, \mathcal{E}) \). Thus we can obtain Corollary 2.2a from Corollary 2.1 and the following simple modification of Lemma 4.7 of Feferman-Vaught [1].

**Lemma 2.3.** (a) For every formula \( \theta \) of \( L_{2t}^\omega \) of type \( t \cup \{u\} \) there is a formula \( \varphi \) of type \( t \) such that \( \theta \) and \( \varphi \) have the same free variables and for all \( \mathcal{A} \) of type \( t \),

\[
\mathcal{A} \models \theta \iff \varphi, 
\]

(where \( \varphi' \) is obtained from \( \varphi \) by relativizing all quantifiers to \( \mathcal{U} \)).

(b) Hence \( \mathcal{A} \models \varphi \) iff \( \mathcal{A}' \models \varphi' \), and \( \mathcal{A} \prec \mathcal{A}' \) iff \( \mathcal{A} \prec \mathcal{A}' \).

**Proof.** The proof of (a) is an easy induction on \( \theta \) based on the following fact: If \( \varphi \) is any formula of type \( \tau \mathcal{A} \), and \( \varphi' \) is obtained from \( \varphi \) by replacing each atomic subformula in which the variable \( x \) occurs by \( \exists x(\forall x \land \neg(x = x)) \), then \( \mathcal{A} \models \exists x(\neg \forall x \land \varphi) \iff \varphi' \). Part (b) follows easily from part (a) using the fact that \( c \) is definable in \( \mathcal{A}' \). This proves the lemma.

**Proof of Corollary 2.2b.** We now consider the product \( \mathcal{P}_{i \in D}(\mathcal{A}_i, \mathcal{D}) \). We may assume that \( 0 \in D \) and that \( i \notin |\mathcal{A}_i|, \ i \in D \). Then we can form \( \mathcal{A}' \) as in Lemma 2.3a with \( |\mathcal{A}'| = |\mathcal{A}_i| \cup \{i\} \), and \( \mathcal{A}' \) with \( |\mathcal{A}'| = |\mathcal{A}_i| \cup \{i\} \cup \{0\} \). Let \( \mathcal{E} = \langle SD, \cup, \sim, R^\mathcal{E} \rangle_{R \in \tau \mathcal{D}} \) where \( R^\mathcal{E} = \langle \langle \{a\}, \ldots, \{x_n\} \rangle; \langle \{a\}, \ldots, \{x_n\} \rangle \in R^\mathcal{E} \rangle \). We show that \( \mathcal{P}_{i \in D}(\mathcal{A}_i, \mathcal{D}) \) is isomorphic to a relativized reduct of the generalized product \( \mathcal{P}_{i \in D}(\mathcal{A}_i, \mathcal{E}) \). Now \( \mathcal{C} = \mathcal{P}_{i \in D}(\mathcal{A}_i, \mathcal{D}) = (\mathcal{P}(\mathcal{D}, \sum_{i \in D} \mathcal{A}_i), K) \) has type \( t = (\tau \mathcal{D})' \cup (\tau \mathcal{A}_i)' \cup \{D, A, K\} \), where \( D \) denotes \( |\mathcal{D}| \) and \( A \) denotes \( |\sum_{i \in D} \mathcal{A}_i| \) and \( K = \langle \{x, y\}; \ x \in |\mathcal{A}_i| \text{ and } y = \iota \rangle \). (Thus \( C = A \cup D \).) We define \( \eta_i: |\mathcal{C}| \to \Pi_{i \in D} A'_i \) as follows: For \( i \in D, \eta_i \) is the function which is 0 except at \( i \), where \( \eta_i(i) = i \). For \( a \in |\mathcal{A}_i| \), \( \eta_a \) is the function which is 0 except at \( i \), where \( \eta_a(i) = a \). Clearly \( \eta_i \) is 1-1. For \( r \in t \) we write \( R_r \) for the relation induced on \( \Pi_{i \in D} A'_i \) by \( r \) via \( \eta_i \), i.e., \( \mathcal{E} = \langle \langle \{a\}, \ldots, \{x_n\} \rangle_{R \in \tau \mathcal{D}} \rangle \). We show that for each \( r \in t, R_r \) is definable in \( \mathcal{P}(\langle \mathcal{A}_i \rangle_{i \in D}, \mathcal{E}) \). For \( r \in t \) we define an acceptable sequence \( \xi_r \) such that \( R_r \) is easily defined using \( Q_{i \in D} \) (for the definition of acceptable sequence \( \xi \), and of \( Q_i \), see Feferman-Vaught [1]). To describe the sequence \( \xi_r \) we suppose that \( I(x), Z(x) \) are formulas of type \( \tau(\mathcal{A}_i') \) which define \( i \) and \( 0 \) respectively, and that \( \text{Sing}(x) \) is a formula of type \( \tau \mathcal{D} \) which asserts that \( X \subseteq D \) is a singleton.

Note that \( f \in D \) iff \( X_0 = \{i; f(i) = 0\} \) is a singleton, and \( X_0 \subseteq X_1 = \{i; f(i) = i\} \). Thus \( D_0 = Q_{i \in D} \), where \( \xi_D \) is the sequence which asserts
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\[
\text{Sing} (X_0) \land X_0 \subseteq X_1.
\]

\[
X_0 = \{ i : \mathcal{M}' \models -Z(v_0) \left[ f(i) \right] \}
\]

\[
X_1 = \{ i : \mathcal{M}' \models I(v_0) \left[ f(i) \right] \}
\]

(i.e., $\xi_D = \langle \text{Sing} (X_0) \land X_0 \subseteq X_1, \neg Z(v_0), I(v_0) \rangle$). Similarly $A_0$ is given by

\[
\text{Sing} (X_0) < X_0 \subseteq X_1
\]

\[
X_0: -Z(v_0)
\]

\[
X_1: -I(v_0).
\]

Now $\langle f, g \rangle \in K_0$ iff $f \in A_0$, $g \in D_0$, and $f(i) \neq 0$ exactly when $g(i) = i$.

Thus $K_0$ is definable using the sequences for $A, D$ and the sequence given by

\[
X_0 = X_1
\]

\[
X_0: -Z(v_0)
\]

\[
X_1: I(v_0).
\]

For $R \in \tau D$, use

\[
RX_0X_1
\]

\[
X_0: I(v_0)
\]

\[
X_1: I(v_1)
\]

and for $R \in \tau M$, use

\[
X_0 \neq 0
\]

\[
K_0: Rv_0v_1
\]

3. Main result.

3.1. Some maximal structures with many automorphisms.

Let $\mathcal{I} = \langle \ast2 \cup \ast2, \ast2, \subseteq, \ast2, F \rangle_{n \in \omega}$, where $F$ is a four place relation: $Fabxy$ iff $a, b \in \ast2$ and $x \subseteq a, y \subseteq b$ and $x, y \in \ast2$ for some $n$.

The structure $\langle \ast2, \subseteq \rangle$ is the full binary tree, $\ast2$ is the set of branches, $\ast2$ the set of nodes at the $n$th level, and for each pair of branches $b, b'$ the set $\{(x, y) : Fbb'xy\}$ is an order preserving function on the nodes contained in $b$ onto the nodes contained in $b'$. In [4], $\mathcal{I}$ was shown to be maximal.

We now construct two structures $\mathcal{I}_R$ and $\mathcal{I}_S$, both of type $\tau(\mathcal{I}) \cup \{ \mathcal{B} \}$; in $\mathcal{I}_R$, $\mathcal{B}$ denotes the set $R$ of eventually right turning branches; in $\mathcal{I}_S$, $\mathcal{B}$ denotes $R \cup \{ c \}$, where $c$ always turns left. More precisely,
\[ \mathcal{I}_R = (\mathcal{I}, R) \] where \[ R = \left\{ b \in \omega \{0, 1\} : \lim_{n \to \omega} b_n = 1 \right\}, \]
and
\[ \mathcal{I}_S = (\mathcal{I}, S) \] where \[ S = R \cup \{c\} \] and \[ c \in \omega \{0\}. \]

**Lemma 3.1.** Let \( f : \omega \to \omega \). Then there is a unique automorphism \( g \) of \( \mathcal{I}_R \) such that for all \( n \) and \( x \in |\mathcal{I}| \),
\[
(gx)_n = \begin{cases} x_n & \text{if } f(x|n) = 0 \\ 1 - x_n & \text{if } f(x|n) = 1 \end{cases}
\]
(i.e., twist when \( f = 1 \)).

**Proof.** Clearly, \( g \) is 1–1 and onto; it is also an automorphism since \( x \leq y \) iff \( g(x) \leq g(y) \), and any automorphism of \( (\omega \cup \omega, \leq) \) is an automorphism of \( \mathcal{I} \).

**Lemma 3.2.** If \( D \subseteq |\mathcal{I}| \sim \{c\} \) and \( D \) is finite, then there is an isomorphism \( g \) on \( \mathcal{I}_R \) onto \( \mathcal{I}_S \) such that for all \( b \in D \), \( g(b) = b \).

**Proof.** Clearly we may assume that \( D \subseteq \omega \). Let \( n \) be chosen so that if \( b \in D \) then \( b(n) = 1 \) for some \( m < n \). Let \( e \) be the branch such that \( e(m) = 0 \) for \( m < n \) and \( e(m) = 1 \) when \( m \geq n \). Define \( f : \omega \to \omega \) by \( f(e| m) = 1 \) if \( m \geq n \), \( f(x) = 0 \) in all other cases. Let \( g \) be the automorphism of \( \mathcal{I} \) induced by \( f \) as in Lemma 3.1. Clearly, if \( b \in R \) and \( b \neq e \) then \( g(b) \in R \) since \( g(b)_p = (b)_p \) except for finitely many \( p \). Similarly, if \( b \in R \) and \( b \neq e \), then \( f(b) \notin R \). Finally \( f(e) = e \), so \( f \) takes \( R \) to \( R \cup \{c\} \).

**3.2. Main lemma.** Next we show that for every \( \kappa \in S, \kappa \geq \omega \), we can find \( T \) with \( (\omega, \kappa) \subseteq \text{Sp}(T) \). In fact what we need is the following

**Lemma 3.3.** For each \( \kappa \in S, \kappa \geq \omega \), there are structures \( \mathfrak{A}_\kappa, \mathfrak{B}_\kappa \) such that
(i) \( c\mathfrak{A}_\kappa = \mathfrak{Z}_1 \) and \( c\mathfrak{B}_\kappa = \kappa \),
(ii) \( \tau\mathfrak{A}_\kappa = \tau\mathfrak{B}_\kappa \) is countable and the same for all \( \kappa \), and \( \mathfrak{A}_\kappa \equiv_{\omega_1} \mathfrak{B}_\kappa \).
Also, if \( \Sigma = \bigcap_{\kappa \in S} \text{Th}_{\omega_1} \mathfrak{A}_\kappa \) then
(iii) \( C \in \text{Mod} \Sigma \) and \( \mathfrak{B}_\kappa \subseteq C \) implies \( \mathfrak{B}_\kappa = C \),
(iv) \( C \in \text{Mod} \Sigma \) and \( \mathfrak{A}_\kappa \subseteq C \) implies \( \mathfrak{A}_\kappa = C \).

**Proof.** We construct \( \mathfrak{A}_\kappa, \mathfrak{B}_\kappa \) from the structures \( \mathcal{I}_R, \mathcal{I}_S \) defined above, and \( M_\kappa \) which we now describe.

In [4] it was shown that for each \( \kappa \in S \) there is a strongly maximal structure \( M_\kappa \) of power \( \kappa \) and countable type. Since any expansion of a strongly maximal model is strongly maximal, we may assume without loss of generality that all \( M_\kappa \) have the same type \( t = \tau \text{Sk}(t), \)
and that $\mathcal{M}_\kappa \in \text{ModSk}(t)$. Thus for all $\kappa$, if $\mathcal{M}_\kappa \subseteq \mathcal{M}' \in \text{ModSk}(t)$ then $\mathcal{M}_\kappa \prec \mathcal{M}'$. Hence there is a $\mathcal{U} \in \tau \text{Sk}(t)$ such that for all $\kappa$, $\mathcal{M}_\kappa \subseteq \mathcal{M}' \in \text{ModSk}(t)$ and $\mathcal{U}^{\mathcal{M}_\kappa} = \omega$ implies that $\mathcal{M}_\kappa \neq \mathcal{M}'$.

We now fix $\kappa$ and construct $\mathfrak{A}_\kappa$, $\mathfrak{B}_\kappa$; to simplify notation we drop the subscript $\kappa$. By the downward Lowenheim-Skolem theorem for $L_\kappa^\omega$ there is $\mathfrak{A} \prec_{\omega_1} \mathfrak{M}$ with $c\mathfrak{A} = \mathfrak{D}$. Let $\mathfrak{A}_\alpha$, $\alpha \in R$, be pairwise disjoint copies of $\mathfrak{A}$, each disjoint from $\mathcal{I}$ and $\mathfrak{M}$, and let $\mathfrak{M}_\kappa = \mathfrak{M}$. Let $\mathfrak{A}_i = \sum_{b \in R} \mathfrak{A}_b$, $\mathfrak{A}_b' = \sum_{b \in R} \mathfrak{A}_b + \mathfrak{M} = \sum_{s \in S} \mathfrak{A}_s$, and $\mathfrak{B}_i = \sum_{b \in R} \mathfrak{B}_b + \mathfrak{M}$.

Let $H$ be the function on $\mathfrak{B}_\alpha$ into $R \cup \{c\}$ defined by

$$H(x) = \begin{cases} b & \text{if } x \in \mathfrak{A}_b, \\ c & \text{if } x \in \mathfrak{M}. \end{cases}$$

Let $\mathcal{I}_0$ be a copy of $\mathcal{I}$ disjoint from the structures so far mentioned. For each $b \in R$, let $G_b$ be a function on $\mathcal{I}_0$ onto $\mathfrak{A}_b$.

Now we define

$$\mathfrak{A} = (\mathcal{I}(\mathcal{I}_0, \mathfrak{A}_i, \mathcal{I}_0), H, G_b)_{b \in R}$$
$$\mathfrak{B}' = (\mathcal{I}(\mathcal{I}_0, \mathfrak{B}_i, \mathcal{I}_0), H, G_b)_{b \in R}$$
$$\mathfrak{B} = (\mathcal{I}(\mathcal{I}_0, \mathfrak{B}, \mathcal{I}_0), H, G_b)_{b \in R}.$$

It is evident that $c\mathfrak{A} = \mathfrak{B}$ and $c\mathfrak{B} = \kappa$, and that $\tau\mathfrak{A} = \tau\mathfrak{B}$ is countable. Moreover this type is independent of $\kappa$ because all the $\mathcal{M}_\kappa$ have the same type. To establish $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$, we prove that $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}'$ and $\mathfrak{B} \prec_{\omega_1} \mathfrak{B}$.

We now show that $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}'$. In fact, we show that if $t$ is a finite subset of $\tau\mathfrak{A}$, then $\mathfrak{A} | t \equiv \mathfrak{B}' | t$. Given the finite type $t$, let $D = \{ b \in R : \mathfrak{G}_b \in t \}$. By Lemma 3.1, there is an isomorphism $f$ on $\mathcal{I}_0$ onto $\mathcal{I}_0$ such that for all $b \in D$, $f(b) = b$. For each $b, b' \in S$ choose an isomorphism $g_{b, b'}$ on $\mathfrak{A}_b$ onto $\mathfrak{A}_{b'}$, with $g_{b, b'}$ the identity when $b = b'$.

Now it is easily seen that we can extend $f$ to an isomorphism on $\mathfrak{A} | t$ onto $\mathfrak{B} | t$ by defining $f(x) = g_{b,f(b)}(x)$ for all $x \in \mathfrak{A}_b$ and $f(x) = x$ for $x \in \mathcal{I}_0$.

We complete the proof that $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$ by showing that $\mathfrak{B} \prec_{\omega_1} \mathfrak{B}$. Let $\mathfrak{C} = (\mathcal{I}(\mathcal{I}_0, \mathfrak{A}_i, \mathcal{I}_0), H, G_b, c)_{b \in R}$ (treat $c$ as the unary relation $\{c\}$). Now let $\mathfrak{D} = \mathfrak{C} \uplus (\mathcal{M}, W^\mathfrak{C})$, $\mathfrak{D}' = \mathfrak{C} \uplus (\mathcal{M}, W^\mathfrak{C})$ where $W^\mathfrak{C} = |\mathfrak{M}|$ and $W^\mathfrak{D} = |\mathfrak{M}|$. By Corollary 2.2a and the definition of $\uplus$, we have $\mathfrak{D} \prec_{\omega_1} \mathfrak{D}'$. It is enough to show that to every formula $\varphi$ of type $\tau\mathfrak{D}'$, there is a formula $\varphi^\mathfrak{D}$ of type $\tau\mathfrak{D}$ such that for all assignments $z$ to $\mathfrak{D}'$, $\mathfrak{D}' \models \varphi[z]$ iff $\mathfrak{D} \models \varphi^\mathfrak{D}[z]$, and $\mathfrak{B} \models \varphi[z]$ iff $\mathfrak{D}' \models \varphi^\mathfrak{D}[z]$. We define $\varphi^\mathfrak{D}$ inductively as follows:

$$R^\mathfrak{D}u_0 \cdots u_{n-1} = R^\mathfrak{D}u_0 \cdots u_{n-1} \quad \text{for all } R \in \tau\mathfrak{D}'$$
$$H^\mathfrak{D}u_0 u_1 = H^\mathfrak{D}u_0 u_1 \lor [W^\mathfrak{D}u_0 \land u_1 \equiv c]$$
$$(-\varphi)^\mathfrak{D} = -\varphi^\mathfrak{D}$$
An easy induction on $\varphi$ shows that the function taking $\varphi$ into $\varphi^*$ is as required. This completes the proof that $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$.

Now we prove (iii). Suppose that $\mathcal{C} \in \text{Mod } \Sigma$ and $\mathcal{B} \subseteq \mathcal{C}$. We must show that $\mathcal{B} = \mathcal{C}$. Since $T$ is maximal it is easy to see that $\mathcal{C}$ has the form $(\mathcal{S} (\mathcal{F}_i, \mathcal{C}_i, \mathcal{F}_0), H^x, G^x_{b \in R})$, for some $\mathcal{C}_i \equiv \mathcal{B}$. Thus for each $b \in R$, domain of $G^x_b = \tau^x_b$, since there is a sentence true in all $\mathcal{A}$'s which asserts that $G_b$ is a function with domain $\tau_b$. Thus $G^x_b = G^x_b$. It follows that in $\mathcal{C}$, range $G_b$ meets $H^{-1}(b)$. But in all $\mathcal{A}$'s, if range $G_b$ meets $H^{-1}(x)$, then $H^{-1}(x) \subseteq \text{range } G_b$, and this is expressible by the sentence

$$\forall z[\exists x \exists y (H(x, y) \land G_b(x, y)) \rightarrow \forall y (H(y, z) \rightarrow \exists x G(x, y))]$$.

Thus for each $b \in R$, $(H^x)^{-1}(b) \subseteq \text{range } G_b$. Now in all $\mathcal{A}$, $|\mathcal{A}| \subseteq \bigcup_{b \in R} H^{-1}(b)$. Since there are unary predicate symbols $A_i$, $B$ such $(A_i)^x = |\mathcal{A}|$, $b^x = R$, this is expressible by a first order sentence. Now $|A_i|^x = |\mathcal{C}_i|$, and $b^x = S = R \cup \{c\}$, so we have

$$|\mathcal{C}_i| \subseteq \bigcup_{b \in R} (H^x)^{-1}(b) \cup (H^x)^{-1}(c)$$.

Since we already have $(H^x)^{-1}(b) \subseteq |\mathcal{B}|$ for $b \in R$, it remains only to show that $(H^x)^{-1}(c) \subseteq \mathcal{M}$. Now each $\mathcal{M}$, and hence each $\mathcal{N}_i$, is a model of $\text{Sk } (t)$. It follows that if $\sigma \in \text{Sk } (t)$, then for each $\mathcal{A}$ we have

$$\forall z (b(x) \rightarrow \sigma^z)$$

where $\sigma^z$ is obtained from $\sigma$ by relativizing all quantifiers to $H(x, z)$ (treating $z$ as a constant). In particular then,

$$\mathcal{C}_i = \mathcal{C}_i | ((H^x)^{-1}(c), \tau^x_M) \in \text{Mod } \text{Sk } (t)$$.

Evidently, we also have $\mathcal{M} \subseteq \mathcal{C}_e$. Also since in each $\mathcal{A}$, $U^x = U^x \cap (H^x)^{-1}(z)$ is countable for each $z \in \mathcal{B}^n$, there is an $(L^x_{\omega_1})$ sentence in $\Sigma$ which asserts this. It follows that $U^x = U^x \cap (H^x)^{-1}(c)$ is countable. Thus since $\mathcal{M}$ is strongly maximal, it follows that $(H^x)^{-1}(c) \subseteq |\mathcal{M}|$. This completes the proof of (iii); the proof of (iv) is exactly the same; replacing $\mathcal{B}$ by $\mathcal{A}$ and deleting reference to $\mathcal{M}$ and $c$. This completes the proof of Lemma 3.3.

3.3. Main theorem.

**Theorem 3.4.** There is a complete countable $L^x_{\omega_1}$-theory $T$ such
that for every $\kappa \geq \aleph_1$, $T$ has a maximal model of power $\kappa$ if there is a maximal structure of power $\kappa$, i.e., $\Sp(T) = S \cap \{\kappa : \kappa \geq \aleph_1\}$.

Proof. Let $\mathfrak{A}_\kappa$, $\mathfrak{B}_\kappa$ be the structures given by Lemma 3.3. Let $\{T_d : d \in \mathcal{D}\} = \{Th_{\aleph_1} \mathfrak{A}_\kappa : \kappa \in S\}$. We now construct $\mathcal{L}_\aleph_1$-equivalent maximal structures $\mathfrak{C}_\kappa$ for each $\kappa \in S$, with $\mathfrak{C}_\kappa$ of power $\kappa$. Taking $T = Th_{\aleph_1} \mathfrak{C}_\kappa$ will complete the proof. First let

$$
\mathfrak{C}_{\kappa,d} = \begin{cases} 
\mathfrak{B}_\kappa & \text{if } T_d = Th_{\aleph_1} \mathfrak{B}_\kappa \\
\mathfrak{A}_\kappa & \text{otherwise, where } Th_{\aleph_1} \mathfrak{A} = T_d.
\end{cases}
$$

Let $\mathfrak{D}$ be any maximal structure with $|\mathfrak{D}| = \aleph_1$, and let

$$
\mathfrak{C}_\kappa = \prod_{d \in D} (\mathfrak{C}_{\kappa,d}, \mathfrak{D}) .
$$

Evidently $\mathfrak{C}_\kappa$ is of power $\kappa$. By Corollary 2.2 for $\kappa, \lambda \in S$, and $\kappa, \lambda \geq \aleph_1$, $\mathfrak{C}_\kappa \equiv_{\aleph_1} \mathfrak{C}_\lambda$.

It remains to show that each $\mathfrak{C}_\kappa$ ($\kappa \in S$) is maximal. To simplify notation we omit the subscript $\kappa$ from $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ in the remainder of the proof (thus we write $\mathfrak{C}_d$ for $\mathfrak{C}_{\kappa,d}$). Suppose $\mathfrak{C} \equiv_{\aleph_1} \mathfrak{C}'$ and $\mathfrak{C} \subseteq \mathfrak{C}'$. We must show $\mathfrak{C} = \mathfrak{C}'$. Clearly $\mathfrak{D} = \mathfrak{C} \vdash (D, t)$ for some type $t$. It is easy to see that if $\mathfrak{D}' = \mathfrak{C}' \vdash (D, t)$ then $\mathfrak{D} \equiv_{\aleph_1} \mathfrak{D}'$ and $\mathfrak{D} \subseteq \mathfrak{D}'$. Since $\mathfrak{D}$ is maximal it follows that $\mathfrak{D}' = \mathfrak{D}$. Notice that for $d \in D$, $\mathfrak{C}_d = \mathfrak{C} \vdash (K^{-1}(d), t)$, where $t$ is the type of $\mathfrak{A}$. Clearly $\forall x (\exists y (x \land Kxy))$ is true in $\mathfrak{C}$ and hence in $\mathfrak{C}'$. Thus, putting $\mathfrak{C}_d = \mathfrak{C}' \vdash ((K\kappa)^{-1}(d), t)$ we have $|\mathfrak{C}'| = D \cup \bigcup_{d \in D} |\mathfrak{C}'|$. To see $\mathfrak{C} = \mathfrak{C}'$ it suffices to show that $\mathfrak{C}_d = \mathfrak{C}_d'$ for each $d \in D$.

It is evident that $\mathfrak{C}_d \subseteq \mathfrak{C}_d'$. Although $\mathfrak{C} \equiv_{\aleph_1} \mathfrak{C}'$, we cannot immediately conclude that $\mathfrak{C}_d \equiv_{\aleph_1} \mathfrak{C}_d'$ (and hence by the maximality of $\mathfrak{C}_d$ that $\mathfrak{C}_d = \mathfrak{C}_d'$) because $d$ may not be definable in $\mathfrak{C}$. However, to conclude that $\mathfrak{C}_d = \mathfrak{C}_d'$, it suffices to show, by parts (iii) and (iv) of Lemma 3.3, that $\mathfrak{C}_d' \in \Mod(\Sigma)$ where $\Sigma = \bigcap_{\kappa \in S} Th_{\aleph_1} \mathfrak{A}_\kappa$. Now in $\mathfrak{C}$ we have, for each $\sigma \in \Sigma$,

$$
\forall d (D(d) \rightarrow \sigma^d)
$$

where $\sigma^d$ is obtained from $\sigma$ by relativizing all quantifiers to $K(x, d)$ (treating $d$ as a constant). Thus, since $\mathfrak{C} \equiv_{\aleph_1} \mathfrak{C}'$, we have for each $d \in D$, $\mathfrak{C}_d \in \Mod \Sigma$. Thus $|\mathfrak{C}_d'| = |\mathfrak{C}_d|$, and hence $\mathfrak{C} = \mathfrak{C}'$, as was to be shown. This completes the proof of Theorem 3.4.

4. Problems.

1. Is there a set $\Gamma'$ ($\Gamma$ countable, $\Gamma$ complete) of $\mathcal{L}_\aleph_1$-sentences such that both $S \cap \Sp(\Gamma)$ and $S \sim \Sp(\Gamma)$ are cofinal with the first
measurable cardinal? I.e. is there a cardinal $\kappa$ less than the first measurable such that whenever $\bigcup(\kappa \cap \text{Sp}(I)) = \kappa$ we have $\text{Sp}(I) \supseteq S \sim K$?

(2) Is Theorem 3.4 true if we replace $\exists_i$ by $\omega_i$?

(3) What is the least $\kappa$ such that whenever $\bigcup(\kappa \cap \text{Sp}(I)) = \kappa$ we have $\bigcup \text{Sp}(I') \supseteq S \sim \kappa$.

(4) More generally, we would like a characterization of those classes of cardinals of the form $\text{Sp}(I)$ ($I$ countable, $I'$ complete).

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